Mathematical Logic. An Introduction

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1 Introduction

Mathematics models real world phenomena like space, time, number, probability, games, etc. It proceeds from initial assumptions to conclusions by rigorous arguments. Its results are "universal" and "logically valid", in that they do not depend on external or implicit conditions which may change with time, nature or society.

It is remarkable that mathematics is also able to model itself: mathematical logic defines rigorously what mathematical statements and rigorous arguments are. The mathematical enquiry into the mathematical method leads to deep insights into mathematics, applications to classical field of mathematics, and to new mathematical theories. The study of mathematical language has also influenced the theory of formal and natural languages in computer science, linguistics and philosophy.

1.1 A simple proof

We want to indicate that rigorous mathematical proofs can be generated by applying simple text manipulations to mathematical statements. Let us consider a fragment of the elementary theory of functions which expresses that the composition of two surjective maps is surjective as well:

> Let f and g be *surjective*, i.e., for all y there is x such that $y = f(x)$, and for all y there is x such that $y = g(x)$.
> *Theorem.* $g \circ f$ is surjective, i.e., for all y there is x such that $y = g(f(x))$.
> *Proof*. Consider any y. Choose z such that $y = g(z)$. Choose x such that $z = f(x)$. Then $y = g(f(x))$. Thus there is x such that $y = g(f(x))$. Thus for all y there is x such that $y = g(f(x))$.
> *Qed.*

These statements and arguments are expressed in an austere and systematic language, which can be normalized further. Logical symbols like $\forall$ and $\exists$ abbreviate figures of language like "for all" or "there exists":

> Let $\forall y \exists x\, y = f(x)$.
> Let $\forall y \exists x\, y = g(x)$.
> Theorem. $\forall y \exists x\, y = g(f(x))$.
> Proof. Consider y.
> $\exists x\, y = g(x)$.
> Let $y = g(z)$.
> $\exists x\, z = f(x)$.
> Let $z = f(x)$.
> $y = g(f(x))$.
> Thus $\exists x\, y = g(f(x))$.
> Thus $\exists x\, y = g(f(x))$.
> Thus $\forall y \exists x\, y = g(f(x))$.
> Qed.

These lines can be considered as formal sequences of symbols. Certain sequences of symbols are acceptable as mathematical formulas. There are rules for the formation of formulas which are acceptable in a proof. These rules have a purely formal character and they can be applied irrespectively of the "meaning" of the symbols and formulas.

1.2 Formal proofs

In the example, $\exists x\, y = g(f(x))$ is inferred from $y = g(f(x))$. The rule of *existential quantification*: "put $\exists x$ in front of a formula" can usually be applied. It has the character of a left-multiplication by $\exists x$.

$$\exists x\,, \varphi \mapsto \exists x\, \varphi.$$

Logical rules satisfy certain algebraic laws like associativity. Another interesting operation is *substitution*: From $y = g(z)$ and $z = f(x)$ infer $y = g(f(x))$ by a "find-and-replace"-substitution of z by $f(x)$.

Given a sufficient collection of rules, the above sequence of formulas, involving "keywords" like "let" and "thus" is a *deduction* or *derivation* in which every line is generated from earlier ones by syntactical rules. Mathematical results may be provable simply by the application of formal rules. In analogy with the formal rules of the infinitesimal calculus one calls a system of rules a *calculus*.

1.3 Syntax and semantics

Obviously we do not just want to describe a formal derivation as a kind of domino but we want to *interpret* the occuring symbols as mathematical objects. Thus we let variables x, y, ... range over some domain like the real numbers $\mathbb{R}$ and let f and g stand for functions $F, G: \mathbb{R} \to \mathbb{R}$. Observe that the symbol or "name" f is not identical to the function F, and indeed f might also be interpretated as another function F'. To emphasize the distinction between names and objects, we classify symbols, formulas and derivations as *syntax* whereas the interpretations of symbols belong to the realm of *semantics*.

By interpreting $x, y, \ldots$ and $f, g, \ldots$ in a structure like $(\mathbb{R}, F, G)$ we can define straightforwardly whether a formula like $\exists x\, g(f(x))$ is *satisfied* in the structure. A formula is *logically valid* if it is satisfied under *all* interpretations. The fundamental theorem of mathematical logic and the central result of this course is GÖDEL's completeness theorem:

Theorem. *There is a calculus with finitely many rules such that a formula is derivable in the calculus iff it is logically valid.*

1.4 Set theory

In modern mathematics notions can usually be reduced to set theory: non-negative integers correspond to cardinalities of finite sets, integers can be obtained via pairs of non-negative integers, rational numbers via pairs of integers, and real numbers via subsets of the rationals, etc. Geometric notions can be defined from real numbers using analytic geometry: a point is a pair of real numbers, a line is a set of points, etc. It is remarkable that the basic set theoretical axioms can be formulated in the logical language indicated above. So mathematics may be understood abstractly as

Mathematics = (first-order) logic + set theory.

Note that we only propose this as a reasonable abstract viewpoint corresponding to the logical analysis of mathematics. This perspective leaves out many important aspects like the applicability, intuitiveness and beauty of mathematics.

1.5 Circularity

We shall use *sets* as symbols which can then be used to formulate the axioms of *set* theory. We shall *prove* theorems about *proofs*. This kind of circularity seems to be unavoidable in comprehensive foundational science: linguistics has to *talk* about *language*, *brain research* has to be carried out by brains. Circularity can lead to paradoxes like the liar's paradox: "I am a liar", or "this sentence is false". Circularity poses many problems and seems to undermine the value of foundational theories. We suggest that the reader takes a *naive* standpoint in these matters: there are sets and proofs which are just as obvious as natural numbers. Then theories are formed which abstractly describe the naive objects.

A closer analysis of circularity in logic leads to the famous *incompleteness theorems* of GÖDEL's:

Theorem. *Formal theories which are strong enough to "formalize themselves" are not complete, i.e., there are statements such that neither it nor its negation can be proved in that theory. Moreover such theories cannot prove their own consistency.*

It is no surprise that these results, besides their initial mathematical meaning had a tremendous impact on the theory of knowledge outside mathematics, e.g., in philosophy, psychology, linguistics.

2 Set theoretic preliminaries

To model the mathematical method, we have to formalize mathematical language and general structures by mathematical objects. The most basic mathematical objects seem to be *sets*. We briefly present some facts from set theory which are used in the sequel.

In line with our introductory remarks on circularity we initially treat set theory *naively*, i.e., we view sets and set theoretic operations as concrete mental constructs. We shall later introduce a powerful axiom system for sets. From an axiomatic standpoint most of our arguments can be carried out under weak set theoretical hypotheses. In particular it will not be necessary to use sets of high cardinality.

The theory of *finite* sets is based on the *empty set* $\emptyset = \{\}$ and operations like

$$x \mapsto \{x\};\; x, y \mapsto \{x, y\};\; x, y \mapsto x \cup y;\; x, y \mapsto x \cap y;\; x, y \mapsto x \setminus y.$$

The operation $x, y \mapsto \{\{x\}, \{x, y\}\}$ defines the *ordered pair* of x and y. Its crucial property is that

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\} \text{ if and only if } x = x' \text{ and } y = y'.$$

The ordered pair $\{\{x\}, \{x, y\}\}$ is denoted by (x, y). Ordered pairs allow to formalize (binary) relations and functions:

⟨rigid| − ⟩ a *relation* is a set R of ordered pairs;

⟨rigid| − ⟩ a *function* is a relation f such that for all x, y, y' holds: if $(x, y) \in f$ and $(x, y') \in f$ then $y = y'$. Then $f(x)$ denotes the unique y such that $(x, y) \in f$.

We assume standard notions and notations from relation theory, see also Definition 2 below. For binary relations R we can use the *infix* notation $a R b$ instead of $(a, b) \in R$.

If a function maps the elements of a set a into a set b we write

$$f : a \to b.$$

In case we do not want to specify the target set b, we can also write $f : a \to V$ where V is understood to be the *universe* of all sets. We assume the usual notions of function theory like *injective*, *surjective*, *bijective*, etc.

It is natural to formalize the integer n by some set with n elements. We shall later see that the following formalization can be carried out uniformly in set theory:

$$\begin{aligned}
0 &= \emptyset \\
1 &= \{0\} \\
2 &= \{0, 1\} \\
&\vdots \\
n + 1 &= \{0, 1, \ldots, n\} = \{0, 1, \ldots, n - 1\} \cup \{n\} = n \cup \{n\} \\
&\vdots \\
\mathbb{N} = \omega &= \{0, 1, \ldots\}
\end{aligned}$$

These integers satisfy the usual laws of complete induction and recursion.

A *finite sequence* is a function $w: n \to V$ for some integer $n \in \mathbb{N}$ which is the *length* of w. We write w_i instead of $w(i)$, and the sequence w may also be denoted by $w_0 \ldots w_{n-1}$. Note that the empty set $\emptyset$ is the unique finite sequence of length 0.

For finite sequences $w = w_0 \ldots w_{m-1}$ and $w' = w'_0 \ldots w'_{n-1}$ let $w\hat{\ } w' = w_0 \ldots w_{m-1} w'_0 \ldots w'_{n-1}$ be the *concatenation* of w and w'. $w\hat{\ } w' : m + n \to V$ can be defined by

$$w\hat{\ } w'(i) = \begin{cases} w(i), \text{ if } i < m; \\ w'(i - m), \text{ if } i \geqslant m. \end{cases}$$

We also write ww' for $w\hat{\ } w'$. This operation is a *monoid* satisfying some cancellation rules:

Proposition 1. *Let w, w', w'' be finite sequences. Then*

a) $(w\hat{\ } w')\hat{\ } w'' = w\hat{\ }(w'\hat{\ } w'')$.

b) $\emptyset\hat{\ } w = w\hat{\ }\emptyset = w$.

c) $w\hat{\ } w' = w\hat{\ } w'' \to w' = w''$.

d) $w'\hat{\ } w = w''\hat{\ } w \to w' = w''$.

Proof. We only check the associative law a). Let n, n', $n'' \in \mathbb{N}$ such that $w = w_0 \ldots w_{n-1}$, $w' = w'_0 \ldots w'_{n'-1}$, $w'' = w''_0 \ldots w''_{n''-1}$. Then

$$\begin{aligned} (w\hat{\ } w')\hat{\ } w'' &= (w_0 \ldots w_{n-1} w'_0 \ldots w'_{n'-1})\hat{\ } w''_0 \ldots w''_{n''-1} \\ &= w_0 \ldots w_{n-1} w'_0 \ldots w'_{n'-1} w''_0 \ldots w''_{n''-1} \\ &= w_0 \ldots w_{n-1}\hat{\ }(w'_0 \ldots w'_{n'-1} w''_0 \ldots w''_{n''-1}) \\ &= w_0 \ldots w_{n-1}\hat{\ }(w'_0 \ldots w'_{n'-1}\hat{\ } w''_0 \ldots w''_{n''-1}) \\ &= w\hat{\ }(w'\hat{\ } w''). \end{aligned}$$

The trouble with this argument is the intuitive but vague use of the *ellipses* "...". In mathematical logic we have to ultimately eliminate such vaguenesses. So we show that for all $i < n + n' + n''$

$$((w\hat{\ } w')\hat{\ } w'')(i) = (w\hat{\ }(w'\hat{\ } w''))(i).$$

Case 1: $i < n$. Then

$$\begin{aligned} ((w\hat{\ } w')\hat{\ } w'')(i) &= (w\hat{\ } w')(i) \\ &= w(i) \\ &= (w\hat{\ }(w'\hat{\ } w''))(i). \end{aligned}$$

Case 2: $n \leqslant i < n + n'$. Then

$$\begin{aligned} ((w\hat{\ } w')\hat{\ } w'')(i) &= (w\hat{\ } w')(i) \\ &= w'(i - n) \\ &= (w'\hat{\ } w'')(i - n) \\ &= (w\hat{\ }(w'\hat{\ } w''))(i). \end{aligned}$$

Case 3: $n + n' \leqslant i < n + n' + n''$. Then

$$\begin{aligned} ((w\hat{\ } w')\hat{\ } w'')(i) &= w''(i - (n + n')) \\ &= w'\hat{\ } w''(i - (n + n') + n') = w'\hat{\ } w''(i - n) \\ &= (w\hat{\ }(w'\hat{\ } w''))(i - n + n) \\ &= (w\hat{\ }(w'\hat{\ } w''))(i). \end{aligned}$$

$\square$

A set x is *finite*, if there is an integer $n \in \mathbb{N}$ and a surjective function $f: n \to x$. The smallest such n is called the *cardinality* of the finite set x and denoted by $n = \operatorname{card}(x)$. The usual cardinality properties for finite sets follow from properties of finite sequences.

A set x is *denumerable* or *countable* if there is a surjective function $f\colon \mathbb{N} \to x$. If the set is not finite, it is *countably infinite*. Its cardinality is ω, written as $\omega = \operatorname{card}(x)$. Under sufficient set theoretical assumptions, the union

$$\bigcup_{n\in\omega} x_n$$

where each x_n is countable is again countable.

If a set x is not countable, it is *uncountable*. Within set theory one can develop an efficient notion of cardinality for uncountable sets.

The theory of infinite sets usually requires the *axiom of choice* which is equivalent to ZORN's lemma.

Definition 2. *Let A be a set and $\leqslant$ be a binary relation. Define*

a)$(A, \leqslant)$ is transitive *if for all $a, b, c \in A$*

$$a \leqslant b \text{ and } b \leqslant c \text{ implies } a \leqslant c.$$

b)$(A, \leqslant)$ is reflexive *if for all $a \in A$ holds $a \leqslant a$.*

c)$(A, \leqslant)$ is a partial order *if $(A, \leqslant)$ is transitive and reflexive and $A \neq \emptyset$.*

So let $(A, \leqslant)$ is be a partial order.

a)$z \in A$ is a maximal element *of A if there is no $a \in A$ with $z \leqslant a$ and $z \neq a$.*

b)If $X \subseteq A$ then u is an upper bound *for X if for all $x \in X$ holds $x \leqslant u$.*

c)$I \subseteq A$ is linear *if for all $a, b \in I$*

$$a \leqslant b \text{ or } b \leqslant a.$$

d)$(A, \leqslant)$ is inductive *if every linear subset of A has an upper bound.*

ZORN*'s lemma* states

Theorem 3. *Every inductive partial order has a maximal element.*

3 Symbols and words

Intuitively and also in our theory a word is a finite sequence of symbols. A symbol has some basic information about its role within words. E.g., the symbol $\leqslant$ is usually used to stand for a binary relation. So we let symbols include such type information. We provide us with a sufficient collection of symbols.

Definition 4. *The* basic symbols *of first-order logic are*

a)$\equiv$ for equality*,*

b)$\neg, \rightarrow, \bot$ for the logical operations of negation*,* implication *and the truth value* false*,*

c)$\forall$ for universal quantification*,*

d)(and) for auxiliary bracketing.

e)variables v_n for $n \in \mathbb{N}$.

Let $\mathrm{Var} = \{v_n | n \in \mathbb{N}\}$ *be the set of variables and let S_0 be the set of basic symbols.*

An n-ary relation symbol*, for $n \in \mathbb{N}$, is (a set) of the form $R = (x, 0, n)$; here 0 indicates that the values of a relation will be* truth values*. 0-ary relation symbols are also called* propositional constant symbols*. An n-ary* function symbol*, for $n \in \mathbb{N}$, is (a set) of the form $f = (x, 1, n)$ where 1 indicates that the values of a function will be elements of a structure. 0-ary function symbols are also called* constant symbols*.*

A symbol set *or a* language *is a set of relation symbols and function symbols.*

We assume that the basic symbols are pairwise distinct and are distinct from any relation or function symbol. For concreteness one could for example set $\equiv = 0$, $\neg = 1$, $\rightarrow = 2$, $\perp = 3$, $(= 4$, $) = 5$, *and* $v_n = (1, n)$ *for* $n \in \mathbb{N}$.

An n-ary relation symbol is intended to denote an n-ary relation; an n-ary function symbol is intended to denote an n-ary function. A symbol set is sometimes called a *type* because it describes the type of structures which will later interpret the symbols. We shall denote variables by letters like $x, y, z, \ldots$, relation symbols by $P, Q, R, \ldots$, functions symbols by $f, g, h, \ldots$ and constant symbols by $c, c_0, c_1, \ldots$ We shall also use other typographical symbols in line with standard mathematical practice. A symbol like $<$, e.g., usually denotes a binary relation, and we could assume for definiteness that there is some fixed set theoretic formalization of $<$ like $< = (999, 0, 2)$. Instead of the arbitrary 999 one could also take the number of $<$ in some typographical font.

Example 5. The *language of group theory* is the language

$$S_{\mathrm{Gr}} = \{\circ, e\},$$

where $\circ$ is a binary (= 2-ary) function symbol and e is a constant symbol. Again one could be definite about the coding of symbols and set $S_{\mathrm{Gr}} = \{(80, 1, 2), (87, 1, 0)\}$, e.g., but we shall not care much about such details. As usual in algebra, one also uses an *extended language of group theory*

$$S_{\mathrm{Gr}} = \{\circ, ^{-1}, e\}$$

to describe groups, where $^{-1}$ is a unary (= 1-ary) function symbol.

Definition 6. *Let S be a language. A* word *over S is a finite sequence*

$$w: n \rightarrow S_0 \cup S.$$

Let S^* be the set of all words over S. The empty set $\emptyset$ is also called the empty word.

Let S be a symbol set. We want to formalize how a word like $\exists x\, y = g(f(x))$ can be produced from a word like $y = g(f(x))$.

Definition 7. *A relation $R \subseteq (S^*)^n \times S^*$ is called a* rule *(over S). A* calculus *(over S) is a set $\mathcal{C}$ of rules (over S).*

We work with rules which *produce* words out of given words. A rule

$$\{(\text{arguments, production})|\ldots\}$$

is usually written as a *production rule* of the form

$$\frac{\text{arguments}}{\text{production}} \quad \text{or} \quad \frac{\text{preconditions}}{\text{conclusion}}.$$

For the existential quantification mentioned in the introduction we may for example write

$$\frac{\varphi}{\exists x \varphi}$$

where the production is the concatenation of $\exists x$ and φ.

Definition 8. *Let $\mathcal{C}$ be a calculus over S. Let $R \subseteq (S^*)^n \times S^*$ be a rule of $\mathcal{C}$. For $X \subseteq S^*$ set*

$$R[X] = \{w \in S^* \mid \text{there are words } u_0, \ldots, u_{n-1} \in X \text{ such that } R(u_0, \ldots, u_{n-1}, w) \text{ holds}\}.$$

Then the product *of $\mathcal{C}$ is the smallest subset of S^* closed under the rules of $\mathcal{C}$:*

$$\mathrm{Prod}(\mathcal{C}) = \bigcap \{X \subseteq S^* \mid \text{for all rules } R \in \mathcal{C} \text{ holds } R[X] \subseteq X\}.$$

The product of a calculus can also be described "from below" by:

Definition 9. *Let $\mathcal{C}$ be a calculus over S. A sequence $w^{(0)}, \ldots, w^{(k-1)} \in S^*$ is called a* derivation *in $\mathcal{C}$ if for every $l < k$ there exists a rule $R \in \mathcal{C}$, $R \subseteq (S^*)^n \times S^*$ and $l_0, \ldots, l_{n-1} < l$ such that*

$$R(w^{(l_0)}, \ldots, w^{(l_{n-1})}, w^{(l)}).$$

This means that every word of the derivation can be derived from earlier words of the derivation by application of one of the rules of the calculus. We shall later define a calculus such that the sequence of sentences

Let $\forall y \exists x\, y = f(x)$.
Let $\forall y \exists x\, y = g(x)$.
Consider y.
$\exists x\, y = g(x)$.
Let $y = g(z)$.
$\exists x\, z = f(x)$.
Let $z = f(x)$.
$y = g(f(x))$.
Thus $\exists x\, y = g(f(x))$.
Thus $\exists x\, y = g(f(x))$.
Thus $\forall y \exists x\, y = g(f(x))$.
Qed.

is basically a derivation in that calculus.

Everything in the product of a calculus can be obtained by a derivation.

Proposition 10. *Let $\mathcal{C}$ be a calculus over S. Then*

$$\mathrm{Prod}(\mathcal{C}) = \{w \,|\, \textit{there is a derivation } w^{(0)}, \ldots, w^{(k-1)} = w \textit{ in } \mathcal{C}\}.$$

Proof. The equality of sets can be proved by two inclusions.
($\subseteq$) The set

$$X = \{w \,|\, \text{there is a derivation } w^{(0)}, \ldots, w^{(k-1)} = w \text{ in } \mathcal{C}\}$$

satisfies the closure property $R[X] \subseteq X$ for all rules $R \in \mathcal{C}$. Since $\mathrm{Prod}(\mathcal{C})$ is the intersection of all such sets, $\mathrm{Prod}(\mathcal{C}) \subseteq X$.
($\supseteq$) Consider $w \in X$. Consider a derivation $w^{(0)}, \ldots, w^{(k-1)} = w$ in $\mathcal{C}$. We show by induction on $l < k$ that $w^{(l)} \in \mathrm{Prod}(\mathcal{C})$. Let $l < k$ and assume that for all $i < l$ holds $w^{(i)} \in \mathrm{Prod}(\mathcal{C})$. Take a rule $R \in \mathcal{C}$, $R \subseteq (\mathbb{A}^*)^n \times \mathbb{A}^*$ and $l_0, \ldots, l_{n-1} < l$ such that $R(w^{(l_0)}, \ldots, w^{(l_{n-1})}, w^{(l)})$. Since $\mathrm{Prod}(\mathcal{C})$ is closed under application of R we get $w^{(l)} \in \mathrm{Prod}(\mathcal{C})$. Thus $w = w^{(k-1)} \in \mathrm{Prod}(\mathcal{C})$. □

Exercise 1. (Natural numbers 1) Consider the symbol set $S = \{|\}$. The set $S^* = \{\emptyset, |, ||, |||, \ldots\}$ of words may be identified with the set $\mathbb{N}$ of natural numbers. Formulate a calculus $\mathcal{C}$ such that $\mathrm{Prod}(\mathcal{C}) = S^*$.

4 Induction and recursion on calculi

Derivations in a calculus have finite length so that one can carry out inductions and recursions along the lengths of derivations. We formulate appropriate induction and recursion theorems which generalize *complete induction* and *recursion* for natural numbers. Note the recursion is linked to induction but requires stronger hypothesis.

Theorem 11. *Let $\mathcal{C}$ be a calculus over S and let $\varphi(-)$ be a property which is inherited along the rules of $\mathcal{C}$:*

$$\forall R \in \mathcal{C}, R \subseteq (S^*)^k \times S^* \, \forall w^{(1)}, \ldots, w^{(k)}, w \in S^*, R(w^{(1)}, \ldots, w^{(k)}, w)\, (\varphi(w^{(1)}) \wedge \ldots \wedge \varphi(w^{(k)}) \rightarrow \varphi(w)).$$

Then

$$\forall w \in \mathrm{Prod}(\mathcal{C})\, \varphi(w).$$

Proof. By assumption, $\{w \in S^* | \varphi(w)\}$ is closed under the rules of $\mathcal{C}$. Since $\mathrm{Prod}(\mathcal{C})$ is the intersection of all sets which are closed under $\mathcal{C}$,

$$\mathrm{Prod}(\mathcal{C}) \subseteq \{w \in S^* | \varphi(w)\}. \qquad \square$$

Definition 12. *A calculus $\mathcal{C}$ over S is* uniquely readable *if for every $w \in \mathrm{Prod}(\mathcal{C})$ there are a unique rule $R \in \mathcal{C}, R \subseteq (S^*)^k \times S^*$ and unique $w^{(1)}, \ldots, w^{(k)} \in S^*$ such that*

$$R(w^{(1)}, \ldots, w^{(k)}, w).$$

Theorem 13. *Let $\mathcal{C}$ be a calculus over S which is uniquely readable and let $(G_R | R \in \mathcal{C})$ be a sequence of* recursion rules, *i.e., for $R \in \mathcal{C}, R \subseteq (S^*)^k \times S^*$ let $G_R: V^k \to V$ where V is the universe of all sets. Then there is a uniquely determined function $F: \mathrm{Prod}(\mathcal{C}) \to V$ such that the following* recursion equation *is satisfied for all $R \in \mathcal{C}, R \subseteq (S^*)^k \times S^*$ and $w^{(1)}, \ldots, w^{(k)}, w \in \mathrm{Prod}(\mathcal{C})$, $R(w^{(1)}, \ldots, w^{(k)}, w)$:*

$$F(w) = G_R(F(w^{(1)}), \ldots, F(w^{(k)})).$$

We say that F is defined by recursion along $\mathcal{C}$ *by the recursion rules $(G_R | R \in \mathcal{C})$.*

Proof. We define $F(w)$ by complete recursion on the length of the shortest derivation of w in $\mathcal{C}$. Assume that $F(u)$ is already uniquely defined for all $u \in \mathrm{Prod}(\mathcal{C})$ with shorter derivation length. Let w have shortest derivation $w^{(0)}, \ldots, w^{(l-1)}$. By the unique readability of $\mathcal{C}$ there are $R \in \mathcal{C}$, $R \subseteq (S^*)^k \times S^*$ and $w^{(i_0)}, \ldots, w^{(i_{k-1})}$ with $i_0, \ldots, i_{k-1} < l-1$ such that

$$R(w^{(i_0)}, \ldots, w^{(i_{k-1})}, w).$$

Then we can uniquely define

$$F(w) = G_R(F(w^{(i_0)}), \ldots, F(w^{(i_{k-1})})). \qquad \square$$

5 Terms and formulas

Fix a symbol set S for the remainder of this section. We generate the *terms* and *formulas* of the corresponding language L^S by calculi.

Definition 14. *The* term calculus *(for S) consists of the following rules:*

a) $\frac{}{x}$ *for all variables x;*

b) $\frac{}{c}$ *for all constant symbols $c \in S$;*

c) $\frac{t_0 \, t_1 \ldots t_{n-1}}{f t_0 \ldots t_{n-1}}$ *for all n-ary function symbols $f \in S$.*

*Let T^S be the product of the term calculus. T^S is the set of all S-*terms.

Definition 15. *The* formula calculus *(for S) consists of the following rules:*

a) $\frac{}{\bot}$ *produces* falsity*;*

b) $\frac{}{t_0 \equiv t_1}$ *for all S-terms $t_0, t_1 \in T^S$ produces* equations*;*

c) $\frac{}{R t_0 \ldots t_{n-1}}$ *for all n-ary relation symbols $R \in S$ and all S-terms $t_0, \ldots, t_{n-1} \in T^S$ produces* relational formulas*;*

d) $\frac{\varphi}{\neg\varphi}$ *produces* negations *of formulas;*

e) $\frac{\varphi \quad \psi}{(\varphi \to \psi)}$ *produces* implications*;*

f) $\frac{\varphi}{\forall x \varphi}$ *for all variables x produces* universalizations.

Let L^S be the product of the formula calculus. L^S is the set of all S-formulas, and it is also called the first-order language *for the symbol set S. Formulas produced by rules a-c) are called* atomic formulas *since they constitute the initial steps of the formula calculus.*

Example 16. S-terms and S-formulas formalize the naive concept of a "mathematical formula". The standard axioms of *group theory* can be written as in the extended language of group theory as $S_{\mathrm{Gr}'}$-formulas:

a)$\forall v_0 \forall v_1 \forall v_2 \circ v_0 \circ v_1 v_2 \equiv \circ \circ v_0 v_1 v_2$;

b)$\forall v_0 \circ v_0 e \equiv v_0$;

c)$\forall v_0 \circ v_0{}^{-1} v_0 \equiv e$.

Note that in c) the $^{-1}$-operator is "applied" to the variable v_0. The term calculus uses the bracket-free *polish notation* which writes operators before the arguments (*prefix* operators). In line with standard notations one also writes operators in *infix* and *postfix* notation, using bracket, to formulate, e.g., associativity:

$$\forall v_0 \forall v_1 \forall v_2\, v_0 \circ (v_1 \circ v_2) \equiv (v_0 \circ v_1) \circ v_2 .$$

Since the particular choice of variables should in general be irrelevant they may be denoted by letters $x, y, z, \ldots$ instead. Thus the group axioms read:

a)$\forall x \forall y \forall z\; x \circ (y \circ z) \equiv (x \circ y) \circ z$;

b)$\forall x\, x \circ e \equiv x$;

c)$\forall x\, x \circ x^{-1} \equiv e$.

Let $\Phi_{\mathrm{Gr}'} = \{\forall x \forall y \forall z\; x \circ (y \circ z) \equiv (x \circ y) \circ z, \forall x\, x \circ e \equiv x, \forall x\, x \circ x^{-1} \equiv e\}$ be the *axioms of group theory* in the extended language.

To work with terms and formulas, it is crucial that the term and formula calculi are uniquely readable. We leave the proof of these facts as exercises.

Although the language introduced will be theoretically sufficient for all mathematical purposes it is often convenient to further extend its expressiveness. We view some additional language constructs as *abbreviations* for formulas in L^S.

Definition 17. *For S-formulas φ and ψ and a variable x write*

$\langle \mathrm{rigid} | - \rangle$ $\top$ *("true") instead of* $\neg\bot$;

$\langle \mathrm{rigid} | - \rangle$ $(\varphi \vee \psi)$ *("φ or ψ") instead of* $(\neg\varphi \rightarrow \psi)$ *is the* disjunction *of* φ, ψ ;

$\langle \mathrm{rigid} | - \rangle$ $(\varphi \wedge \psi)$ *("φ and ψ") instead of* $\neg(\varphi \rightarrow \neg\psi)$ *is the* conjunction *of* φ, ψ ;

$\langle \mathrm{rigid} | - \rangle$ $(\varphi \leftrightarrow \psi)$ *("φ iff ψ") instead of* $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$ *is the* equivalence *of* φ, ψ ;

$\langle \mathrm{rigid} | - \rangle$ $\exists x \varphi$ *("for all x holds φ") instead of* $\neg \forall x \neg \varphi$.

For the sake of simplicity one often omits redundant brackets, in particular outer brackets. So we usually write $\varphi \vee \psi$ instead of $(\varphi \vee \psi)$.

6 Structures and models

We shall *interpret* formulas like $\forall y \exists x\, y = g(f(x))$ in adequate *structures*. This interaction between language and structures is usually called *semantics*. Fix a symbol set S.

Definition 18. *An S-*structure *is a function $\mathfrak{A}\colon \{\forall\} \cup S \rightarrow V$ such that*

a)$\mathfrak{A}(\forall) \neq \emptyset$*;* $\mathfrak{A}(\forall)$ *is the* underlying set *of $\mathfrak{A}$ and is usually denoted by A or $|\mathfrak{A}|$;*

b)for every n-ary relation symbol $R \in S$, $\mathfrak{A}(R)$ is an n-ary relation *on A, i.e., $a(r) \subseteq A^n$;*

c)for every n-ary function symbol $f \in S$, $\mathfrak{A}(f)$ is an n-ary function *on A, i.e., $a(r)\colon A^n \rightarrow A$.*

Again we use customary or convenient notations for the *components* of the structure $\mathfrak{A}$, i.e., the values of $\mathfrak{A}$. One often writes $R^{\mathfrak{A}}$, $f^{\mathfrak{A}}$, or $c^{\mathfrak{A}}$ instead of $\mathfrak{A}(r)$, $\mathfrak{A}(f)$, or $\mathfrak{A}(c)$ resp. In simple cases, one may simply list the components of the structure and write, e.g.,

$$\mathfrak{A} = (A, R_0^{\mathfrak{A}}, R_1^{\mathfrak{A}}, f^{\mathfrak{A}}).$$

Example 19. Formalize the *ordered field of reals* $\mathbb{R}$ as follows. Define the language of ordered fields

$$S_{\text{oF}} = \{<, +, \cdot, 0, 1\}.$$

Then define the structure $\mathbb{R}$: $\{\forall\} \cup S_{\text{oF}} \to V$ by

$$\begin{aligned}
\mathbb{R}(\forall) &= \mathbb{R}\\
\mathbb{R}(<) = <^{\mathbb{R}} &= \{(u, v) \in \mathbb{R}^2 \,|\, u < v\}\\
\mathbb{R}(+) = +^{\mathbb{R}} &= \{(u, v, w) \in \mathbb{R}^3 \,|\, u + v = w\}\\
\mathbb{R}(\cdot) = \cdot^{\mathbb{R}} &= \{(u, v, w) \in \mathbb{R}^3 \,|\, u \cdot v = w\}\\
\mathbb{R}(0) = 0^{\mathbb{R}} &= 0 \in \mathbb{R}\\
\mathbb{R}(1) = 1^{\mathbb{R}} &= a \in \mathbb{R}
\end{aligned}$$

This defines the standard structure $\mathbb{R} = (\mathbb{R}, <, +, \cdot, 0, 1)$. The multiple use of the letter $\mathbb{R}$ corresponds to standard usage and should not lead to confusion.

Observe that the symbols could in principle be interpreted in completely different, counterintuitive ways like

$$\begin{aligned}
\mathbb{R}'(\forall) &= \mathbb{N}\\
\mathbb{R}'(<) &= \{(u, v) \in \mathbb{N}^2 \,|\, u > v\}\\
\mathbb{R}'(+) &= \{(u, v, w) \in \mathbb{N}^3 \,|\, u \cdot v = w\}\\
\mathbb{R}'(\cdot) &= \{(u, v, w) \in \mathbb{N}^3 \,|\, u + v = w\}\\
\mathbb{R}'(0) &= 1\\
\mathbb{R}'(1) &= 0
\end{aligned}$$

Example 20. Define the language of *Boolean algebras* by

$$S_{\text{BA}} = \{\wedge, \vee, -, 0, 1\}$$

where $\wedge$ and $\vee$ are binary function symbols for "and" and "or", $-$ is a unary function symbol for "not", and 0 and 1 are constant symbols. A Boolean algebra of particular importance in logic is the algebra $\mathbb{B}$ of *truth values*. Let $B = |\mathbb{B}| = \{0, 1\}$ with $0 = \mathbb{B}(0)$ and $1 = \mathbb{B}(1)$. Define the operations and $= \mathbb{B}(\wedge)$, or $= \mathbb{B}(\vee)$, and not $= \mathbb{B}(-)$ by *operation tables* in analogy to standard multiplication tables:

and	0	1
0	0	0
1	0	1

,

or	0	1
0	0	1
1	1	1

, and

not	
0	1
1	0

.

Note that we use the non-exclusive "or" instead of the exclusive "either - or".

The notion of structure leads to some related definitions.

Definition 21. *Let $\mathfrak{A}$ be an S-structure and $\mathfrak{A}'$ be an S'-structure. Then $\mathfrak{A}$ is a* reduct *of $\mathfrak{A}'$, or $\mathfrak{A}'$ is an* expansion *of $\mathfrak{A}$, if $S \subseteq S'$ and $\mathfrak{A}' \restriction (\{\forall\} \cup S) = \mathfrak{A}$.*

According to this definition, the additive group $(\mathbb{R}, +, 0)$ of reals is a reduct of the field $(\mathbb{R}, +, \cdot, 0, 1)$.

Definition 22. *Let $\mathfrak{A}$, $\mathfrak{B}$ be S-structures. Then $\mathfrak{A}$ is a* substructure *of $\mathfrak{B}$, $\mathfrak{A} \subseteq \mathfrak{B}$, if $\mathfrak{B}$ is a pointwise extension of $\mathfrak{A}$, i.e.,*

a) $A = |\mathfrak{A}| \subseteq |\mathfrak{B}|$;

b) *for every n-ary relation symbol $R \in S$ holds $R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap A^n$;*

c) *for every n-ary function symbol $f \in S$ holds $f^{\mathfrak{A}} = f^{\mathfrak{B}} \upharpoonright A^n$.*

Definition 23. *Let $\mathfrak{A}, \mathfrak{B}$ be S-structures and $h: |\mathfrak{A}| \to |\mathfrak{B}|$. Then h is a* homomorphism *from $\mathfrak{A}$ into $\mathfrak{B}$, $h: \mathfrak{A} \to \mathfrak{B}$, if*

a) *for every n-ary relation symbol $R \in S$ and for every $a_0, \ldots, a_{n-1} \in A$*

$$R^{\mathfrak{A}}(a_0, \ldots, a_{n-1}) \text{ implies } R^{\mathfrak{B}}(h(a_0), \ldots, h(a_{n-1}));$$

b) *for every n-ary function symbol $f \in S$ and for every $a_0, \ldots, a_{n-1} \in A$*

$$f^{\mathfrak{B}}(h(a_0), \ldots, h(a_{n-1})) = h(f^{\mathfrak{A}}(a_0, \ldots, a_{n-1})).$$

h is an embedding *of $\mathfrak{A}$ into $\mathfrak{B}$, $h: \mathfrak{A} \hookrightarrow \mathfrak{B}$, if moreover*

a) *h is injective;*

b) *for every n-ary relation symbol $R \in S$ and for every $a_0, \ldots, a_{n-1} \in A$*

$$R^{\mathfrak{A}}(a_0, \ldots, a_{n-1}) \text{ iff } R^{\mathfrak{B}}(h(a_0), \ldots, h(a_{n-1})).$$

If h is also bijective, it is called an isomorphism.

An S-structure interprets the symbols in S. To interpret a formula in a structure one also has to interpret the (occuring) variables.

Definition 24. *Let S be a symbol set. An S-*model *is a function*

$$\mathfrak{M}: \{\forall\} \cup S \cup \mathrm{Var} \to V$$

such that $\mathfrak{M} \upharpoonright \{\forall\} \cup S$ is an S-structure and for all $n \in \mathbb{N}$ holds $\mathfrak{M}(v_n) \in |\mathfrak{M}|$. $\mathfrak{M}(v_n)$ is the interpretation of the variable v_n in $\mathfrak{M}$.

It will sometimes be important to modify a model $\mathfrak{M}$ at specific variables. For pairwise distinct variables $x_0, \ldots, x_{r-1}$ and $a_0, \ldots, a_{r-1} \in |\mathfrak{M}|$ define

$$\mathfrak{M}\frac{a_0 \ldots a_{r-1}}{x_0 \ldots x_{r-1}} = (\mathfrak{M} \setminus \{(x_0, \mathfrak{A}(x_0)), \ldots, (x_{r-1}, \mathfrak{A}(x_{r-1}))\}) \cup \{(x_0, a_0), \ldots, (x_{r-1}, a_{r-1})\}.$$

7 The satisfaction relation

We now define the semantics of the first-order language by interpreting terms and formulas in models.

Definition 25. *Let $\mathfrak{M}$ be an S-model. Define the* interpretation *$\mathfrak{M}(t) \in |\mathfrak{M}|$ of a term $t \in T^S$ by recursion on the term calculus:*

a) *for t a variable, $\mathfrak{M}(t)$ is already defined;*

b) *for an n-ary function symbol and terms $t_0, \ldots, t_{n-1} \in T^S$, let*

$$\mathfrak{M}(f t_0 \ldots t_{n-1}) = f^{\mathfrak{A}}(\mathfrak{M}(t_0), \ldots, \mathfrak{M}(t_{n-1})).$$

This explains the interpretation of a term like $v_3^2 + v_{200}^3$ in the reals.

Definition 26. *Let $\mathfrak{M}$ be an S-model. Define the* interpretation *$\mathfrak{M}(\varphi) \in \mathbb{B}$ of a formula $\varphi \in L^S$, where $\mathbb{B} = \{0, 1\}$ is the Boolean algebra of truth values, by recursion on the formula calculus:*

a) *$\mathfrak{M}(\perp) = 0$;*

b) *for terms $t_0, t_1 \in T^S$: $\mathfrak{M}(t_0 \equiv t_1) = 1$ iff $\mathfrak{M}(t_0) = \mathfrak{M}(t_1)$;*

c) for every n-ary relation symbol $R \in S$ and terms $t_0, \ldots, t_1 \in T^S$

$$\mathfrak{M}(R t_0 \ldots t_{n-1}) = 1 \text{ iff } R^{\mathfrak{M}}(\mathfrak{M}(t_0), \ldots, \mathfrak{M}(t_{n-1}));$$

d) $\mathfrak{M}(\neg\varphi) = 1$ *iff* $\mathfrak{M}(\varphi) = 0$ *;*

e) $\mathfrak{M}(\varphi \rightarrow \psi) = 1$ *iff* $\mathfrak{M}(\varphi) = 1$ *implies* $\mathfrak{M}(\psi) = 1$*;*

f) $\mathfrak{M}(\forall v_n \varphi) = 1$ *iff for all* $a \in |\mathfrak{M}|$ *holds* $\mathfrak{M}\frac{a}{v_n}(\varphi) = 1$.

We write $\mathfrak{M} \vDash \varphi$ *instead of* $\mathfrak{M}(\varphi) = 1$*. We also say that* $\mathfrak{M}$ satisfies φ *or that* φ holds *in* $\mathfrak{M}$*. For* $\Phi \subseteq L^S$ *write* $\mathfrak{M} \vDash \Phi$ *iff* $\mathfrak{M} \vDash \varphi$ *for every* $\varphi \in \Phi$*.*

Definition 27. *Let S be a language and $\Phi \subseteq L^S$. Φ is* universally valid *if Φ holds in every S-model. Φ is* satisfiable *if there is an S-model $\mathfrak{M}$ such that $\mathfrak{M} \vDash \Phi$.*

The language extensions by the symbols $\vee, \wedge, \leftrightarrow, \exists$ is consistent with the expected meanings of the additional symbols:

Exercise 2. Prove:

a) $\mathfrak{M} \vDash (\varphi \vee \psi)$ iff $\mathfrak{M} \vDash \varphi$ *or* $\mathfrak{M} \vDash \psi$;

b) $\mathfrak{M} \vDash (\varphi \vee \psi)$ iff $\mathfrak{M} \vDash \varphi$ *and* $\mathfrak{M} \vDash \psi$;

c) $\mathfrak{M} \vDash (\varphi \leftrightarrow \psi)$ iff $\mathfrak{M} \vDash \varphi$ *is equivalent to* $\mathfrak{M} \vDash \psi$;

d) $\mathfrak{M} \vDash \exists v_n \varphi$ iff *there exists* $a \in |\mathfrak{M}|$ such that $\mathfrak{M}\frac{a}{v_n} \vDash \varphi$.

With the notion of $\vDash$ we can now formally define what it means for a structure to be a group or for a function to be differentiable. Before considering examples we make some auxiliary definitions and simplifications.

It is intuitively obvious that the interpretation of a term only depends on the occuring variables, and that satisfaction for a formula only depends on its free, non-bound variables.

Definition 28. *For $t \in T^S$ define* $\text{var}(t) \subseteq \{v_n | n \in \mathbb{N}\}$ *by recursion on the term calculus:*

$\langle$rigid$|-\rangle$ $\text{var}(x) = \{x\}$;

$\langle$rigid$|-\rangle$ $\text{var}(c) = \emptyset$;

$\langle$rigid$|-\rangle$ $\text{var}(f t_0 \ldots t_{n-1}) = \bigcup_{i<n} \text{var}(t_i)$.

Definition 29. *Für $\varphi \in L^S$ define the set of* free variables $\text{free}(\varphi) \subseteq \{v_n | n \in \mathbb{N}\}$ *by recursion on the formula calculus:*

$\langle$rigid$|-\rangle$ $\text{free}(t_0 \equiv t_1) = \text{var}(t_0) \cup \text{var}(t_1)$;

$\langle$rigid$|-\rangle$ $\text{free}(R t_0 \ldots t_{n-1}) = \text{var}(t_0) \cup \ldots \cup \text{var}(t_{n-1})$;

$\langle$rigid$|-\rangle$ $\text{free}(\neg\varphi) = \text{free}(\varphi)$*;*

$\langle$rigid$|-\rangle$ $\text{free}(\varphi \rightarrow \psi) = \text{free}(\varphi) \cup \text{free}(\psi)$.

$\langle$rigid$|-\rangle$ $\text{free}(\forall x \varphi) = \text{free}(\varphi) \setminus \{x\}$.

For $\Phi \subseteq L^S$ *define the set* $\text{free}(\Phi)$ *of* free variables *as*

$$\text{free}(\Phi) = \bigcup_{\varphi \in \Phi} \text{free}(\varphi).$$

Example 30.

$$\begin{aligned}
\text{free}(Ryx \rightarrow \forall y \neg y = z) &= \text{free}(Ryx) \cup \text{free}(\forall y \neg y = z) \\
&= \text{free}(Ryx) \cup (\text{free}(\neg y = z) \setminus \{y\}) \\
&= \text{free}(Ryx) \cup (\text{free}(y = z) \setminus \{y\}) \\
&= \{y, x\} \cup (\{y, z\} \setminus \{y\}) \\
&= \{y, x\} \cup \{z\} \\
&= \{x, y, z\}.
\end{aligned}$$

Definition 31.

a) *For* $n \in \mathbb{N}$ *let* $L_n^S = \{\varphi \in L^S \mid \mathrm{free}(\varphi) \subseteq \{v_0, \ldots, v_{n-1}\}\}$.

b) $\varphi \in L^S$ *is an* S-sentence *if* $\mathrm{free}(\varphi) = \emptyset$; L_0^S *is the set of* S*-sentences.*

Theorem 32. *Let* t *be an* S*-term and let* $\mathfrak{I}$ *and* $\mathfrak{I}'$ *be* S*-models with the same structure* $\mathfrak{I} \upharpoonright \{\forall\} \cup S = \mathfrak{I}' \upharpoonright \{\forall\} \cup S$ *and* $\mathfrak{I} \upharpoonright \mathrm{var}(t) = \mathfrak{I}' \upharpoonright \mathrm{var}(t)$. *Then* $\mathfrak{I}(t) = \mathfrak{I}'(t)$.

Theorem 33. *Let* φ *be an* S*-formula, and let* $\mathfrak{I}$ *and* $\mathfrak{I}'$ *be* S*-models with the same structure* $\mathfrak{I} \upharpoonright \{\forall\} \cup S = \mathfrak{I}' \upharpoonright \{\forall\} \cup S$ *and* $\mathfrak{I} \upharpoonright \mathrm{free}(\varphi) = \mathfrak{I}' \upharpoonright \mathrm{free}(\varphi)$. *Then*

$$\mathfrak{I} \vDash \varphi \quad \textit{iff} \quad \mathfrak{I}' \vDash \varphi.$$

Proof. By induction on the formula calculus.
$\varphi = t_0 \equiv t_1$: Then $\mathrm{var}(t_0) \cup \mathrm{var}(t_1) = \mathrm{frei}(\varphi)$ and

$$\begin{aligned} \mathfrak{I} \vDash \varphi & \text{ iff } \mathfrak{I}(t_0) = \mathfrak{I}(t_1) \\ & \text{ iff } \mathfrak{I}'(t_0) = \mathfrak{I}'(t_1) \text{ by the previous Theorem,} \\ & \text{ iff } \mathfrak{I}' \vDash \varphi. \end{aligned}$$

$\varphi = \psi \rightarrow \chi$ and assume the claim to be true for ψ and χ. Then

$$\begin{aligned} \mathfrak{I} \vDash \varphi & \text{ iff } \mathfrak{I} \vDash \psi \text{ implies } \mathfrak{I} \vDash \chi \\ & \text{ iff } \mathfrak{I}' \vDash \psi \text{ implies } \mathfrak{I}' \vDash \chi \text{ by the inductive assumption,} \\ & \text{ iff } \mathfrak{I}' \vDash \varphi. \end{aligned}$$

$\varphi = \forall v_n \psi$ and assume the claim to be true for ψ. Then $\mathrm{free}(\psi) \subseteq \mathrm{free}(\varphi) \cup \{v_n\}$. For all $a \in A = |\mathfrak{I}|$: $(\beta \frac{a}{v_n}) \upharpoonright \mathrm{free}(\psi) = (\beta' \frac{a}{v_n}) \upharpoonright \mathrm{free}(\psi)$ and so

$$\begin{aligned} \mathfrak{I} \vDash \varphi & \text{ iff for all } a \in A \text{ holds } (\mathfrak{A}, \beta \frac{a}{v_n}) \vDash \psi \\ & \text{ iff for all } a \in A \text{ holds } (\mathfrak{A}, \beta' \frac{a}{v_n}) \vDash \psi \text{ by the inductive assumption,} \\ & \text{ iff } \mathfrak{I}' \vDash \varphi. \end{aligned}$$

□

This allows further simplifications in notations for $\vDash$:

Definition 34. *Let* $\mathfrak{A}$ *be an* S*-structure and let* $(a_0, \ldots, a_{n-1})$ *be a sequence of elements of* A. *Let* t *be an* S*-term with* $\mathrm{var}(t) \subseteq \{v_0, \ldots, v_{n-1}\}$. *Then define*

$$t^{\mathfrak{A}}[a_0, \ldots, a_{n-1}] = \mathfrak{I}(t),$$

where $\mathfrak{I} \supseteq \mathfrak{A}$ *is an* S*-model with* $\mathfrak{I}(v_0) = a_0, \ldots, \mathfrak{I}(v_{n-1}) = a_{n-1}$.
Let φ *be an* S*-formula with* $\mathrm{free}(t) \subseteq \{v_0, \ldots, v_{n-1}\}$. *Then define*

$$\mathfrak{A} \vDash \varphi[a_0, \ldots, a_{n-1}] \quad \textit{iff} \quad \mathfrak{I} \vDash \varphi,$$

where $\mathfrak{I} \supseteq \mathfrak{A}$ *is an* S*-model with* $\mathfrak{I}(v_0) = a_0, \ldots, \mathfrak{I}(v_{n-1}) = a_{n-1}$.

In case $n = 0$ *also write* $t^{\mathfrak{A}}$ *instead of* $t^{\mathfrak{A}}[a_0, \ldots, a_{n-1}]$ *and* $\mathfrak{A} \vDash \varphi$ *instead of* $\mathfrak{A} \vDash \varphi[a_0, \ldots, a_{n-1}]$. *In this case we also say:* $\mathfrak{A}$ *is a* model *of* φ, $\mathfrak{A}$ satisfies φ *or* φ is true *in* $\mathfrak{A}$.

For $\Phi \subseteq L_0^S$ *a set of sentences also write*

$$\mathfrak{A} \vDash \Phi \quad \textit{iff for all} \quad \varphi \in \Phi \textit{ holds}: \mathfrak{A} \vDash \varphi.$$

Example 35. *Groups*. $S_{\langle \mathrm{syntax} | Gr | x \rangle} := \{\circ, e\}$ with a binary function symbol $\circ$ and a constant symbol e is the *language of groups theory*. The group axioms are

a) $\forall v_0 \forall v_1 \forall v_2 \circ v_0 \circ v_1 v_2 \equiv \circ \circ v_0 v_1 v_2$;

b)$\forall v_0 \circ v_0\, e \equiv v_0$;

c)$\forall v_0 \exists v_1 \circ v_0 v_1 \equiv e$.

This define the axiom set

$$\Phi_{\mathrm{Gr}} = \{\forall v_0\, \forall v_1\, \forall v_2 \circ v_0 \circ v_1 v_2 \equiv \circ \circ v_0 v_1 v_2,\ \forall v_0 \circ v_0\, e \equiv v_0,\ \forall v_0 \exists v_1 \circ v_0 v_1 \equiv e\}.$$

An S-structure $\mathfrak{G} = (G, *, k)$ satisfies Φ_{Gr} iff it is a group in the ordinary sense.

Definition 36. *Let S be a language and let $\Phi \subseteq L_0^S$ be a set of S-sentences. Then*

$$\mathrm{Mod}^S \Phi = \{\mathfrak{A} \,|\, \mathfrak{A} \textit{ is an } S\textit{-structure and } \mathfrak{A} \vDash \Phi\}$$

is the model class *of* Φ. *In case* $\Phi = \{\Phi\}$ *we also write* $\mathrm{Mod}^S \varphi$ *instead of* $\mathrm{Mod}^S \Phi$. *We also say that* Φ *is an* axiom system *for* $\mathrm{Mod}^S \Phi$*, or that* Φ axiomatizes *the class* $\mathrm{Mod}^S \Phi$ *.*

Thus $\mathrm{Mod}^{S_{\mathrm{Gr}}} \Phi_{\mathrm{Gr}}$ is the model class of all groups. Model classes are studied in generality within *model theory* which is a branch of mathematical logic. For specific Φ the model class $\mathrm{Mod}^S \Phi$ is examined in subfields of mathematics: group theory, ring theory, graph theory, etc. Some typical questions questions are: Is $\mathrm{Mod}^S \Phi \neq \emptyset$, i.e., is Φ satisfiable? Can we extend $\mathrm{Mod}^S \Phi$ by adequate morphisms between models?

8 Logical implication and propositional connectives

Definition 37. *For a symbol set S and $\Phi \subseteq L^S$ and $\varphi \in L^S$ define that Φ* (logically) implies *φ ($\Phi \vDash \varphi$) iff every S-model $\mathfrak{I} \vDash \Phi$ is also a model of φ.*

Note that logical implication $\vDash$ is a relation between syntactical entities which is defined using the semantic notion of interpretation. We show that $\vDash$ satisfies certain syntactical laws. These laws correspond to the rules of a logical proof calculus.

Theorem 38. *Let S be a symbol set, $t \in T^S$, $\varphi, \psi \in L^S$, and $\Gamma, \Phi \subseteq L^S$. Then*

a) (Monotonicity) If $\Gamma \subseteq \Phi$ and $\Gamma \vDash \varphi$ then $\Phi \vDash \varphi$.

b) (Assumption property) If $\varphi \in \Gamma$ then $\Phi \vDash \varphi$.

c) ($\rightarrow$ -Introduction) If $\Gamma \cup \varphi \vDash \psi$ then $\Gamma \vDash \varphi \rightarrow \psi$.

d) ($\rightarrow$ -Elimination) If $\Gamma \vDash \varphi$ and $\Gamma \vDash \varphi \rightarrow \psi$ then $\Gamma \vDash \psi$.

e) ($\bot$-Introduction) If $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$ then $\Gamma \vDash \bot$.

f) ($\bot$-Elimination) If $\Gamma \cup \{\neg\varphi\} \vDash \bot$ then $\Gamma \vDash \varphi$.

g) ($\equiv$ -Introduction) $\Gamma \vDash t \equiv t$.

Proof. f) Assume $\Gamma \cup \{\neg\varphi\} \vDash \bot$. Consider an S-model with $\mathfrak{I} \vDash \Gamma$. Assume that $\mathfrak{I} \nvDash \varphi$. Then $\mathfrak{I} \vDash \neg\varphi$. $\mathfrak{I} \vDash \Gamma \cup \{\neg\varphi\}$, and by assumption, $\mathfrak{I} \vDash \bot$. But by the definition of the satisfaction relation, this is false. Thus $\mathfrak{I} \vDash \varphi$. Thus $\Gamma \vDash \varphi$. □

9 Substitution and quantification rules

To prove further rules for equalities and quantification, we first have to formalize *substitution.*

Definition 39. *For a term $s \in T^S$, pairwise distinct variables $x_0, \ldots, x_{r-1}$ and terms $t_0, \ldots, t_{r-1} \in T^S$ define the* (simultaneous) substitution

$$s\,\frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$$

of $t_0, \ldots, t_{r-1}$ *for* $x_0, \ldots, x_{r-1}$ *by recursion:*

a) $x \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = \begin{cases} x, \text{ if } x \neq x_0, \ldots, x \neq x_{r-1} \\ t_i, \text{ if } x = x_i \end{cases}$ *for all variables* x*;*

b) $c \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = c$ *for all constant symbols* c*;*

c) $(f s_0 \ldots s_{n-1}) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = f s_0 \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} \ldots s_{n-1} \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$ *for all* n*-ary function symbols* f*.*

Note that the simultaneous substitution

$$s \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$$

is in general different from a succesive substitution

$$s \frac{t_0}{x_0} \frac{t_1}{x_1} \ldots \frac{t_{r-1}}{x_{r-1}}$$

which depends on the order of substitution. E.g., $x \frac{y x}{x y} = y$, $x \frac{y}{x} \frac{x}{y} = y \frac{x}{y} = x$ and $x \frac{x}{y} \frac{y}{x} = x \frac{y}{x} = y$.

Definition 40. *For a formula* $\varphi \in L^S$*, pairwise distinct variables* $x_0, \ldots, x_{r-1}$ *and terms* $t_0, \ldots, t_{r-1} \in T^S$ *define the* (simultaneous) substitution

$$\varphi \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$$

of $t_0, \ldots, t_{r-1}$ *for* $x_0, \ldots, x_{r-1}$ *by recursion:*

a) $(s_0 \equiv s_1) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = s_0 \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} \equiv s_1 \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$ *for all terms* $s_0, s_1 \in T^S$*;*

b) $(R s_0 \ldots s_{n-1}) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = R s_0 \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} \ldots s_{n-1} \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$ *for all* n*-ary relation symbols* R *and terms* $s_0, \ldots, s_{n-1} \in T^S$*;*

c) $(\neg \varphi) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = \neg(\varphi \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}})$*;*

d) $(\varphi \to \psi) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = (\varphi \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} \to \psi \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}})$*;*

e) for $(\forall x \varphi) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$ *distinguish two cases:*

⟨rigid| − ⟩ *if* $x \in \{x_0, \ldots, x_{r-1}\}$*, assume that* $x = x_0$*. Choose* $i \in \mathbb{N}$ *minimal such that* $u = v_i$ *does not occur in* $\forall x \varphi$*,* $t_0, \ldots, t_{r-1}$ *and* $x_0, \ldots, x_{r-1}$*. Then set*

$$(\forall x \varphi) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = \forall u (\varphi \frac{t_1 \ldots t_{r-1} \, u}{x_1 \ldots x_{r-1} \, x}).$$

⟨rigid| − ⟩ *if* $x \notin \{x_0, \ldots, x_{r-1}\}$*, choose* $i \in \mathbb{N}$ *minimal such that* $u = v_i$ *does not occur in* $\forall x \varphi$*,* $t_0, \ldots, t_{r-1}$ *and* $x_0, \ldots, x_{r-1}$ *and set*

$$(\forall x \varphi) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = \forall u (\varphi \frac{t_0 \ldots t_{r-1} \, u}{x_0 \ldots x_{r-1} \, x}).$$

The following substitution theorem shows that syntactic substitution corresponds semantically to a (simultaneous) modification of assignments by interpreted terms.

Theorem 41. *Consider an* S*-model* $\mathfrak{M}$*, pairwise distinct variables* $x_0, \ldots, x_{r-1}$ *and terms* $t_0, \ldots, t_{r-1} \in T^S$*.*

a) If $s \in T^S$ *is a term,*

$$\mathfrak{M}(s \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}) = \mathfrak{M} \frac{\mathfrak{M}(t_0) \ldots \mathfrak{M}(t_{r-1})}{x_0 \ldots x_{r-1}}(s).$$

b) If $\varphi \in L^S$ *is a formula,*

$$\mathfrak{M} \vDash \varphi \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} \text{ iff } \mathfrak{M} \frac{\mathfrak{M}(t_0) \ldots \mathfrak{M}(t_{r-1})}{x_0 \ldots x_{r-1}} \vDash \varphi.$$

Proof. By induction on the complexities of s and φ.
a) *Case 1*: $s = x$.
Case 1.1: $x \notin \{x_0, \ldots, x_{r-1}\}$. Then

$$\mathfrak{M}(x \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}) = \mathfrak{M}(x) = \mathfrak{M} \frac{\mathfrak{M}(t_0) \ldots \mathfrak{M}(t_{r-1})}{x_0 \ldots x_{r-1}}(x).$$

Case 1.2: $x = x_i$. Then

$$\mathfrak{M}(x \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}) = \mathfrak{M}(t_i) = \mathfrak{M} \frac{\mathfrak{M}(t_0) \ldots \mathfrak{M}(t_{r-1})}{x_0 \ldots x_{r-1}}(x).$$

Case 2: $s = c$ is a constant symbol. Then

$$\mathfrak{M}(c \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}) = \mathfrak{M}(c) = \mathfrak{M} \frac{\mathfrak{M}(t_0) \ldots \mathfrak{M}(t_{r-1})}{x_0 \ldots x_{r-1}}(c).$$

Case 3: $s = f s_0 \ldots s_{n-1}$ where $f \in S$ is an n-ary function symbol and the terms $s_0, \ldots, s_{n-1} \in T^S$ satisfy the theorem. Then

$$\begin{aligned}
\mathfrak{M}((f s_0 \ldots s_{n-1}) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}) &= \mathfrak{M}(f s_0 \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} \ldots s_{n-1} \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}) \\
&= \mathfrak{M}(f)(\mathfrak{M}(s_0 \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}), \ldots, \mathfrak{M}(s_{n-1} \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}})) \\
&= \mathfrak{M}(f)(\mathfrak{M} \frac{\mathfrak{M}(t_0) \ldots \mathfrak{M}(t_{r-1})}{x_0 \ldots x_{r-1}}(s_0), \ldots, \mathfrak{M} \frac{\mathfrak{M}(t_0) \ldots \mathfrak{M}(t_{r-1})}{x_0 \ldots x_{r-1}}(s_{n-1})) \\
&= \mathfrak{M} \frac{\mathfrak{M}(t_0) \ldots . \mathfrak{M}(t_{r-1})}{x_0 \ldots x_{r-1}}(f s_0 \ldots s_{n-1}).
\end{aligned}$$

Assuming that the substitution theorem is proved for terms, we prove
b) *Case 4*: $\varphi = s_0 \equiv s_1$. Then

$$\begin{aligned}
\mathfrak{I} \vDash (s_0 \equiv s_1) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} \quad &\text{iff} \quad \mathfrak{I} \vDash (s_0 \frac{t_0 \ldots . t_{r-1}}{x_0 \ldots x_{r-1}} \equiv s_1 \frac{t_0 \ldots . t_{r-1}}{x_0 \ldots x_{r-1}}) \\
&\text{iff} \quad \mathfrak{I}(s_0 \frac{t_0 \ldots . t_{r-1}}{x_0 \ldots x_{r-1}}) = \mathfrak{I}(s_1 \frac{t_0 \ldots . t_{r-1}}{x_0 \ldots x_{r-1}}) \\
&\text{iff} \quad \mathfrak{I} \frac{\mathfrak{I}(t_0) \ldots . \mathfrak{I}(t_{r-1})}{x_0 \ldots x_{r-1}}(s_0) = \mathfrak{I} \frac{\mathfrak{I}(t_0) \ldots . \mathfrak{I}(t_{r-1})}{x_0 \ldots x_{r-1}}(s_1) \\
&\text{iff} \quad \mathfrak{I} \frac{\mathfrak{I}(t_0) \ldots . \mathfrak{I}(t_{r-1})}{x_0 \ldots x_{r-1}} \vDash s_0 \equiv s_1 .
\end{aligned}$$

Propositional connectives of formulas like $\neg$ and $\rightarrow$ behave similar to terms, so we only consider universal quantification:
Case 5: $\varphi = (\forall x \psi) \frac{t_0 \ldots . t_{r-1}}{x_0 \ldots x_{r-1}}$, assuming that the theorem holds for ψ.
Case 5.1: $x = x_0$. Choose $i \in \mathbb{N}$ minimal such that $u = v_i$ does not occur in $\forall x \varphi$, $t_0, \ldots ., t_{r-1}$ and $x_0, \ldots, x_{r-1}$. Then

$$(\forall x \varphi) \frac{t_0 \ldots . t_{r-1}}{x_0 \ldots x_{r-1}} = \forall u (\varphi \frac{t_1 \ldots t_{r-1} \, u}{x_1 \ldots x_{r-1} \, x}).$$

$$\begin{aligned}
\mathfrak{M} \vDash (\forall x \varphi) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} \quad &\text{iff} \quad \mathfrak{M} \vDash \forall u (\varphi \frac{t_1 \ldots t_{r-1} \, u}{x_1 \ldots x_{r-1} \, x}) \\
&\text{iff} \quad \text{for all } a \in M \text{ holds } \mathfrak{M}\frac{a}{u} \vDash \varphi \frac{t_1 \ldots t_{r-1} \, u}{x_1 \ldots x_{r-1} \, x} \\
&\qquad \text{(definition of } \vDash) \\
&\text{iff} \quad \text{for all } a \in M \text{ holds} \\
&\qquad (\mathfrak{M}\frac{a}{u}) \frac{\mathfrak{M}\frac{a}{u}(t_1) \ldots \mathfrak{M}\frac{a}{u}(t_{r-1}) \mathfrak{M}\frac{a}{u}(u)}{x_1 \ldots x_{r-1} \, x} \vDash \varphi \\
&\qquad \text{(inductive hypothesis for } \varphi) \\
&\text{iff} \quad \text{for all } a \in M \text{ holds} \\
&\qquad (\mathfrak{M}\frac{a}{u}) \frac{\mathfrak{M}(t_1) \ldots \mathfrak{M}(t_{r-1}) \, a}{x_1 \ldots x_{r-1} \, x} \vDash \varphi
\end{aligned}$$

(since u does not occur in t_i)

iff for all $a \in M$ holds
$$\mathfrak{M}\frac{\mathfrak{M}(t_1)\ldots\mathfrak{M}(t_{r-1})\,a}{x_1\ldots x_{r-1}\,x} \vDash \varphi$$
(since u does not occur in φ)

iff for all $a \in M$ holds
$$(\mathfrak{M}\frac{\mathfrak{M}(t_1)\ldots\mathfrak{M}(t_{r-1})}{x_1\ldots x_{r-1}})\frac{a}{x} \vDash \varphi$$
(by simple properties of assignments)

iff $(\mathfrak{M}\frac{\mathfrak{M}(t_1)\ldots\mathfrak{M}(t_{r-1})}{x_1\ldots x_{r-1}}) \vDash \forall x\,\varphi$

(definition of $\vDash$)

iff $(\mathfrak{M}\frac{\mathfrak{M}(t_0)\,\mathfrak{M}(t_1)\ldots\mathfrak{M}(t_{r-1})}{x_0\,x_1\ldots x_{r-1}}) \vDash \forall x\,\varphi$

(since $x = x_0$ is not free in $\forall x\,\varphi$).

Case 5.2: $x \notin \{x_0, \ldots, x_{r-1}\}$. Then proceed similarly. Choose $i \in \mathbb{N}$ minimal such that $u = v_i$ does not occur in $\forall x\varphi$, $t_0, \ldots., t_{r-1}$ and $x_0, \ldots, x_{r-1}$. Then
$$(\forall x\,\varphi)\frac{t_0\ldots t_{r-1}}{x_0\ldots x_{r-1}} = \forall u\,(\varphi\frac{t_0\ldots t_{r-1}\,u}{x_0\ldots x_{r-1}\,x}).$$

$\mathfrak{M} \vDash (\forall x\,\varphi)\frac{t_0\ldots t_{r-1}}{x_0\ldots x_{r-1}}$ iff $\mathfrak{M} \vDash \forall u\,(\varphi\frac{t_0\ldots t_{r-1}\,u}{x_0\ldots x_{r-1}\,x})$

iff for all $a \in M$ holds $\mathfrak{M}\frac{a}{u} \vDash \varphi\frac{t_0\ldots t_{r-1}\,u}{x_0\ldots x_{r-1}\,x}$

(definition of $\vDash$)

iff for all $a \in M$ holds
$$(\mathfrak{M}\frac{a}{u})\frac{\mathfrak{M}\frac{a}{u}(t_0)\ldots\mathfrak{M}\frac{a}{u}(t_{r-1})\,\mathfrak{M}\frac{a}{u}(u)}{x_0\ldots x_{r-1}\,x} \vDash \varphi$$
(inductive hypothesis for φ)

iff for all $a \in M$ holds
$$(\mathfrak{M}\frac{a}{u})\frac{\mathfrak{M}(t_0)\ldots\mathfrak{M}(t_{r-1})\,a}{x_0\ldots x_{r-1}\,x} \vDash \varphi$$
(since u does not occur in t_i)

iff for all $a \in M$ holds
$$\mathfrak{M}\frac{\mathfrak{M}(t_0)\ldots\mathfrak{M}(t_{r-1})\,a}{x_0\ldots x_{r-1}\,x} \vDash \varphi$$
(since u does not occur in φ)

iff for all $a \in M$ holds
$$(\mathfrak{M}\frac{\mathfrak{M}(t_0)\ldots\mathfrak{M}(t_{r-1})}{x_0\ldots x_{r-1}})\frac{a}{x} \vDash \varphi$$
(by simple properties of assignments)

iff $(\mathfrak{M}\frac{\mathfrak{M}(t_0)\ldots\mathfrak{M}(t_{r-1})}{x_0\ldots x_{r-1}}) \vDash \forall x\,\varphi$

(definition of $\vDash$)

$\square$

We can now formulate further properties of the $\vDash$ relation.

Theorem 42. *Let S be a language. Let x, y be variables, $t, t' \in T^S$, $\varphi \in L^S$, and $\Gamma \subseteq L^S$. Then:*

a) ($\forall$-Introduction) If $\Gamma \vDash \varphi\frac{y}{x}$ and $y \notin \mathrm{free}(\Gamma \cup \{\forall x\varphi\})$ then $\Gamma \vDash \forall x\,\varphi$.

b) ($\forall$-elimination) If $\Gamma \vDash \forall x\,\varphi$ then $\Gamma \vDash \varphi\frac{t}{x}$.

c) ($\equiv$-Elimination or substitution) If $\Gamma \vDash \varphi\frac{t}{x}$ and $\Gamma \vDash t \equiv t'$ then $\Gamma \vDash \varphi\frac{t'}{x}$.

Proof. a) Let $\Gamma \vDash \varphi \frac{y}{x}$ and $y \notin \mathrm{free}(\Gamma \cup \{\forall x \varphi\})$. Consider an S-model $\mathfrak{I}$ with $\mathfrak{I} \vDash \Gamma$. Let $a \in A = |\mathfrak{I}|$. Since $y \notin \mathrm{free}(\Gamma)$, $\mathfrak{I}\frac{a}{y} \vDash \Gamma$. By assumption, $\mathfrak{I}\frac{a}{y} \vDash \varphi \frac{y}{x}$. By the substitution theorem,

$$(\mathfrak{I}\frac{a}{y})\frac{\mathfrak{I}\frac{a}{y}(y)}{x} \vDash \varphi \text{ and so } (\mathfrak{I}\frac{a}{y})\frac{a}{x} \vDash \varphi$$

Case 1: $x = y$. Then $\mathfrak{I}\frac{a}{x} \vDash \varphi$.
Case 2: $x \neq y$. Then $\mathfrak{I}\frac{a\,a}{y\,x} \vDash \varphi$, and since $y \notin \mathrm{free}(\varphi)$ we have $\mathfrak{I}\frac{a}{x} \vDash \varphi$.

Thus $\mathfrak{I} \vDash \forall x \varphi$. Thus $\Gamma \vDash \forall x \varphi$.

b) Let $\Gamma \vDash \forall x \varphi$. Consider an Smodel $\mathfrak{I}$ with $\mathfrak{I} \vDash \Gamma$. For all $a \in A = |\mathfrak{I}|$ holds $\mathfrak{I}\frac{a}{x} \vDash \varphi$. In particular $\mathfrak{I}\frac{\mathfrak{I}(t)}{x} \vDash \varphi$. By the substitution theorem, $\mathfrak{I} \vDash \varphi \frac{t}{x}$. Thus $\Gamma \vDash \varphi \frac{t}{x}$.

c) Let $\Gamma \vDash \varphi \frac{t}{x}$ and $\Gamma \vDash t \equiv t'$. Consider an S-model $\mathfrak{I}$ mit $\mathfrak{I} \vDash \Gamma$. By assumption $\mathfrak{I} \vDash \varphi \frac{t}{x}$ and $\mathfrak{I} \vDash t \equiv t'$. By the substitution theorem

$$\mathfrak{I}\frac{\mathfrak{I}(t)}{x} \vDash \varphi.$$

Since $\mathfrak{I}(t) = \mathfrak{I}(t')$,

$$\mathfrak{I}\frac{\mathfrak{I}(t')}{x} \vDash \varphi$$

and again by the substitution theorem

$$\mathfrak{I} \vDash \varphi \frac{t'}{x}.$$

Thus $\Gamma \vDash \varphi \frac{t'}{x}$. □

Note that in proving these proof rules we have used corresponding forms of arguments in the language of our discourse. This "circularity" is a general feature in formalizations of logic.

10 A sequent calculus

We can put the rules of implication established in the previous two sections in the form of a calculus which leads from correct implications $\Phi \vDash \varphi$ to further correct implications $\Phi' \vDash \varphi'$. Our *sequent calculus* will work on finite *sequents* $(\varphi_0, \ldots, \varphi_{n-1}, \varphi_n)$ of formulas, whose intuition is that $\{\varphi_0, \ldots, \varphi_{n-1}\}$ implies φ_n. The GÖDEL completeness theorem shows that these rules actually generate the implication relation $\vDash$. Fix a language S for this section.

Definition 43. *A finite sequence* $(\varphi_0, \ldots, \varphi_{n-1}, \varphi_n)$ *is called a* sequent. *The initial segment* $\Gamma = (\varphi_0, \ldots, \varphi_{n-1})$ *is the* antecedent *and* φ_n *is the* succedent *of the sequent. We usually write* $\varphi_0 \ldots \varphi_{n-1} \varphi_n$ *or* $\Gamma \varphi_n$ *instead of* $(\varphi_0, \ldots, \varphi_{n-1}, \varphi_n)$. *To emphasize the last element of the antecedent we may also denote the sequent by* $\Gamma' \varphi_{n-1} \varphi_n$ *with* $\Gamma' = (\varphi_0, \ldots, \varphi_{n-2})$.

A sequent $\varphi_0 \ldots \varphi_{n-1} \varphi$ *is* correct *if* $\{\varphi_0 \ldots \varphi_{n-1}\} \vDash \varphi$.

Definition 44. *The* sequent calculus *consists of the following (sequent-)rules:*

⟨rigid|−⟩ *monotonicity* (MR) $\dfrac{\Gamma \quad \varphi}{\Gamma \quad \psi \quad \varphi}$

⟨rigid|−⟩ *assumption* (AR) $\dfrac{}{\Gamma \quad \varphi \quad \varphi}$

⟨rigid|−⟩ $\to$*-introduction* $(\to I)$ $\dfrac{\Gamma \quad \varphi \quad \psi}{\Gamma \quad \varphi \to \psi}$

⟨rigid|−⟩ $\to$*-elimination* $(\to E)$ $\dfrac{\begin{array}{l}\Gamma \quad \varphi \\ \Gamma \quad \varphi \to \psi\end{array}}{\Gamma \quad \psi}$

⟨rigid|−⟩ $\bot$*-introduction* $(\bot I)$ $\dfrac{\begin{array}{l}\Gamma \quad \varphi \\ \Gamma \quad \neg\varphi\end{array}}{\Gamma \quad \bot}$

$\langle \text{rigid} | - \rangle$ $\perp$-*elimination* $(\perp E)$ $\dfrac{\Gamma \;\; \neg\varphi \;\; \perp}{\Gamma \qquad\quad \varphi}$

$\langle \text{rigid} | - \rangle$ $\forall$-*introduction* $(\forall I)$ $\dfrac{\Gamma \;\; \varphi\frac{y}{x}}{\Gamma \;\; \forall x\varphi}$, *if* $y \notin \text{free}(\Gamma \cup \{\forall x\varphi\})$

$\langle \text{rigid} | - \rangle$ $\forall$-*elimination* $(\forall E)$ $\dfrac{\Gamma \;\; \forall x\varphi}{\Gamma \;\; \varphi\frac{t}{x}}$, *if* $t \in T^S$

$\langle \text{rigid} | - \rangle$ $\equiv$-*introduction* $(\equiv I)$ $\dfrac{}{\Gamma \;\; t \equiv t}$, *if* $t \in T^S$

$\langle \text{rigid} | - \rangle$ $\equiv$-*elimination* $(\equiv E)$ $\dfrac{\begin{matrix}\Gamma & \varphi\frac{t}{x} \\ \Gamma & t \equiv t'\end{matrix}}{\Gamma \;\; \varphi\frac{t'}{x}}$

The deduction relation *is the smallest subset* $\vdash \subseteq \text{Seq}(S)$ *of the set of sequents which is closed under these rules. We write* $\varphi_0 \ldots \varphi_{n-1} \vdash \varphi$ *instead of* $\varphi_0 \ldots \varphi_{n-1}\, \varphi \in \vdash$ *. For* Φ *an arbitrary set of formulas define* $\Phi \vdash \varphi$ *iff there are* $\varphi_0, \ldots, \varphi_{n-1} \in \Phi$ *such that* $\varphi_0 \ldots \varphi_{n-1} \vdash \varphi$ *. We say that* φ *can be* deduced *or* derived *from* $\varphi_0 \ldots \varphi_{n-1}$ *or* Φ*, resp. We also write* $\vdash \varphi$ *instead of* $\emptyset \vdash \varphi$ *and say that* φ *is a* tautology*.*

Theorem 45. *A formula* $\varphi \in L^S$ *is derivable from* $\Gamma = \varphi_0 \ldots \varphi_{n-1}$ *(*$\Gamma \vdash \varphi$*) iff there is a* derivation *or a* formal proof

$$(\Gamma_0\varphi_0, \Gamma_1\varphi_1, \ldots, \Gamma_{k-1}\varphi_{k-1})$$

of $\Gamma\varphi = \Gamma_{k-1}\varphi_{k-1}$ *, in which every sequent* $\Gamma_i\varphi_i$ *is generated by a sequent rule from sequents* $\Gamma_{i_0}\varphi_{i_0}, \ldots, \Gamma_{i_{n-1}}\varphi_{i_{n-1}}$ *with* $i_0, \ldots, i_{n-1} < i$ *.*

We usually write the derivation $(\Gamma_0\varphi_0, \Gamma_1\varphi_1, \ldots, \Gamma_{k-1}\varphi_{k-1})$ *as a vertical scheme*

$$\begin{array}{ll} \Gamma_0 & \varphi_0 \\ \Gamma_1 & \varphi_1 \\ \vdots & \\ \Gamma_{k-1} & \varphi_{k-1} \end{array}$$

where we may also mark rules and other remarks along the course of the derivation.

In our theorems on the laws of implication we have already shown:

Theorem 46. *The sequent calculus is* correct*, i.e., every rule of the sequent calculus leads from correct sequents to correct sequents. Thus every derivable sequent is correct. This means that*

$$\vdash \subseteq \vDash .$$

The converse inclusion corresponds to

Definition 47. *The sequent calculus is* complete *if* $\vDash \subseteq \vdash$ *.*

The Gödel completeness theorem proves the completeness of the sequent calculus. The definition of $\vdash$ immediately implies the following *finiteness* or *compactness theorem*.

Theorem 48. *Let* $\Phi \subseteq L^S$ *and* $\varphi \in \Phi$ *. Then* $\Phi \vdash \varphi$ *iff there is a finite subset* $\Phi_0 \subseteq \Phi$ *such that* $\Phi_0 \vdash \varphi$ *.*

After proving the completeness theorem, such structural properties carry over to the implication relation $\vDash$.

11 Derivable sequent rules

The composition of rules of the sequent calculus yields *derived sequent rules* which are again correct. First note:

Lemma 49. *Assume that*

$$\begin{array}{ll} \Gamma & \varphi_0 \\ \vdots & \\ \Gamma & \varphi_{k-1} \\ \hline \Gamma & \varphi_k \end{array}$$

is a derived rule of the sequent calculus. Then

$$\begin{array}{ll} \Gamma_0 & \varphi_0 \\ \vdots & \\ \Gamma_{k-1} & \varphi_{k-1} \\ \hline \Gamma_k & \varphi_k \end{array} \text{, where } \Gamma_0, \ldots, \Gamma_{k-1} \text{ are initial sequences of } \Gamma_k$$

is also a derived rule of the sequent calculus.

Proof. This follows immediately from iterated applications of the monotonicity rule. □

We now list several derived rules.

11.1 Auxiliary rules

We write the derivation of rules as proofs in the sequent calculus where the premisses of the derivation are written above the upper horizontal line and the conclusion as last row.

ex falsum libenter $\dfrac{\Gamma\ \ \bot}{\Gamma\ \ \varphi}$:

$$\begin{array}{llll} 1. & \Gamma & & \bot \\ \hline 2. & \Gamma & \neg\varphi & \bot \\ \hline 3. & \Gamma & & \varphi \end{array}$$

$\neg$-*Introduction* $\dfrac{\Gamma\ \ \varphi\ \ \bot}{\Gamma\ \ \ \ \neg\varphi}$:

$$\begin{array}{lllll} 1. & \Gamma & \varphi & & \bot \\ \hline 2. & \Gamma & & & \varphi \to \bot \\ 3. & \Gamma & \neg\neg\varphi & & \neg\neg\varphi \\ 4. & \Gamma & \neg\neg\varphi & \neg\varphi & \neg\varphi \\ 5. & \Gamma & \neg\neg\varphi & \neg\varphi & \bot \\ 6. & \Gamma & \neg\neg\varphi & & \varphi \\ 7. & \Gamma & \neg\neg\varphi & & \bot \\ \hline 8. & \Gamma & & & \neg\varphi \end{array}$$

$$\begin{array}{llll} 1. & \Gamma & & \neg\varphi \\ \hline 2. & \Gamma & \varphi & \varphi \\ 3. & \Gamma & \varphi & \bot \\ 4. & \Gamma & \varphi & \psi \\ \hline 5. & \Gamma & & \varphi \to \psi \end{array}$$

$$\begin{array}{llll} 1. & \Gamma & & \psi \\ \hline 2. & \Gamma & \varphi & \varphi \\ 3. & \Gamma & \varphi & \psi \\ \hline 4. & \Gamma & & \varphi \to \psi \end{array}$$

Cut rule

$$\begin{array}{llll} 1. & \Gamma & & \varphi \\ 2. & \Gamma & \varphi & \psi \\ \hline 3. & \Gamma & & \varphi \to \psi \\ \hline 4. & \Gamma & & \psi \end{array}$$

Contraposition

1.	Γ	φ		ψ
2.	Γ	$\neg\psi$		$\neg\psi$
3.	Γ	$\neg\psi$	φ	φ
3.	Γ	$\neg\psi$	φ	φ
4.	Γ			$\varphi \to \psi$

11.2 Introduction and elimination of $\vee$, $\wedge$, ...

$\vee$ *-Introduction*

1.	Γ		φ
2.	Γ	$\neg\varphi$	$\neg\varphi$
3.	Γ	$\neg\varphi$	$\bot$
4.	Γ	$\neg\varphi$	ψ
5.	Γ		$\neg\varphi \to \psi$
6.	Γ		$\varphi \vee \psi$

$\vee$ *-Introduction*

1.	Γ		ψ
2.	Γ	$\neg\varphi$	ψ
3.	Γ		$\neg\varphi \to \psi$
4.	Γ		$\varphi \vee \psi$

$\vee$ *-Elimination*

1.	Γ			$\varphi \vee \psi$
2.	Γ			$\varphi \to \chi$
3.	Γ			$\psi \to \chi$
4.	Γ			$\neg\varphi \to \psi$
5.	Γ	$\neg\chi$		$\neg\chi$
6.	Γ	$\neg\chi$	φ	φ
7.	Γ	$\neg\chi$	φ	χ
8.	Γ	$\neg\chi$	φ	$\bot$
9.	Γ	$\neg\chi$		$\neg\varphi$
10.	Γ	$\neg\chi$		ψ
11.	Γ	$\neg\chi$		χ
12.	Γ	$\neg\chi$		$\bot$
13.	Γ			χ

$\wedge$ *-Introduction*

1.	Γ		φ
2.	Γ		ψ
3.	Γ	$\varphi \to \neg\psi$	$\varphi \to \neg\psi$
4.	Γ	$\varphi \to \neg\psi$	$\neg\psi$
4.	Γ	$\varphi \to \neg\psi$	$\bot$
5.	Γ		$\neg(\varphi \to \neg\psi)$
6.	Γ		$\varphi \wedge \psi$

$\wedge$ *-Elimination*

1.	Γ		$\varphi \wedge \psi$
2.	Γ		$\neg(\varphi \to \neg\psi)$
3.	Γ	$\neg\varphi$	$\neg\varphi$
4.	Γ	$\neg\varphi$	$\varphi \to \neg\psi$
5.	Γ	$\neg\varphi$	$\bot$
6.	Γ		φ

$\wedge$-*Elimination*

1.	Γ		$\varphi \wedge \psi$
2.	Γ		$\neg(\varphi \rightarrow \neg\psi)$
3.	Γ	$\neg\psi$	$\neg\psi$
4.	Γ	$\neg\psi$	$\varphi \rightarrow \neg\psi$
5.	Γ	$\neg\varphi$	$\perp$
6.	Γ		φ

$\exists$-*Introduction*

1.	Γ		$\varphi\frac{t}{x}$
2.	Γ	$\forall x\neg\varphi$	$\forall x\neg\varphi$
3.	Γ	$\forall x\neg\varphi$	$\neg\varphi\frac{t}{x}$
4.	Γ	$\forall x\neg\varphi$	$\perp$
5.	Γ		$\neg\forall x\neg\varphi$
6.	Γ		$\exists x\varphi$

$\exists$-*Elimination*

1.	Γ		$\exists x\varphi$	
2.	Γ	$\varphi\frac{y}{x}$	ψ	where $y \notin \mathrm{free}(\Gamma \cup \{\exists x\varphi, \psi\})$
3.	Γ		$\neg\forall x\neg\varphi$	
4.	Γ	$\neg\psi$	$\neg\varphi\frac{y}{x}$	
5.	Γ	$\neg\psi$	$\forall x\neg\varphi$	
6.	Γ	$\neg\psi$	$\perp$	
7.	Γ		ψ	

11.3 Manipulations of antecedents

We derive rules which show that the formulas in the antecedent may be permuted arbitrarily, showing that only the *set* of antecedent formulas is relevant.

Transpositions of premisses

1.	Γ	φ	ψ	χ
2.	Γ	φ		$\psi \rightarrow \chi$
3.	Γ			$\varphi \rightarrow (\psi \rightarrow \chi)$
4.	Γ	ψ		ψ
5.	Γ	ψ	φ	φ
6.	Γ	ψ	φ	$\psi \rightarrow \chi$
7.	Γ	ψ	φ	χ

Doublication of premisses

1.	Γ	φ		ψ
2.	Γ	φ	φ	ψ

Elimination of double premisses

1.	Γ	φ	φ	ψ
2.	Γ	φ		$\varphi \rightarrow \psi$
3.	Γ			$\varphi \rightarrow (\varphi \rightarrow \psi)$
4.	Γ	φ		φ
5.	Γ	φ		ψ

Iterated applications of these rules yield:

Lemma 50. *Let $\varphi_0 \ldots \varphi_{m-1}$ and $\psi_0 \ldots \psi_{n-1}$ be antecedents such that*

$$\{\varphi_0, \ldots, \varphi_{m-1}\} = \{\psi_0, \ldots, \psi_{n-1}\}$$

and $\chi \in L^S$. Then

$$\frac{\varphi_0 \;\ldots\; \varphi_{m-1} \;\; \chi}{\psi_0 \;\ldots\; \psi_{n-1} \;\; \chi}$$

is a derived rule.

11.4 Examples of formal proofs

We give some examples of formal proofs which show that within the proof calculus $\equiv$ is an equivalence relation.

Lemma 51. *We prove the following tautologies:*

a) Reflexivity: $\vdash \forall x\, x \equiv x$

b) Symmetry: $\vdash \forall x \forall y (x \equiv y \rightarrow y \equiv x)$

c) Transitivity: $\vdash \forall x \forall y \forall z (x \equiv y \wedge y \equiv z \rightarrow x \equiv z)$

Proof. a)

$$\begin{array}{l} x \equiv x \\ \hline \forall x\, x \equiv x \end{array}$$

b)

$$\begin{array}{ll}
x \equiv y & x \equiv y \\
x \equiv y & x \equiv x \\
x \equiv y & (z \equiv x)\frac{x}{z} \\
x \equiv y & (z \equiv x)\frac{y}{x} \\
x \equiv y & y \equiv x \\
 & x \equiv y \rightarrow y \equiv x \\
 & \forall y (x \equiv y \rightarrow y \equiv x) \\ \hline
 & \forall x \forall y (x \equiv y \rightarrow y \equiv x)
\end{array}$$

c)

$$\begin{array}{ll}
x \equiv y \wedge y \equiv z & x \equiv y \wedge y \equiv z \\
x \equiv y \wedge y \equiv z & x \equiv y \\
x \equiv y \wedge y \equiv z & (x \equiv w)\frac{y}{w} \\
x \equiv y \wedge y \equiv z & y \equiv z \\
x \equiv y \wedge y \equiv z & (x \equiv w)\frac{z}{w} \\
x \equiv y \wedge y \equiv z & x \equiv z \\
 & x \equiv y \wedge y \equiv z \rightarrow x \equiv z \\
 & \forall z (x \equiv y \wedge y \equiv z \rightarrow x \equiv z) \\
 & \forall y \forall z (x \equiv y \wedge y \equiv z \rightarrow x \equiv z) \\ \hline
 & \forall x \forall y \forall z (x \equiv y \wedge y \equiv z \rightarrow x \equiv z)
\end{array}$$

□

We show moreover that $\equiv$ is a *congruence relation* from the perspective of $\vdash$.

Theorem 52. *Let φ be an S-formula and $x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}$ be pairwise distinct variables. Then*

$$\vdash \forall x_0 \ldots \forall x_{n-1} \forall y_0 \ldots \forall y_{n-1} \left(x_0 \equiv y_0 \wedge \ldots \wedge x_{n-1} \equiv y_{n-1} \rightarrow \left(\varphi \leftrightarrow \varphi \frac{y_0 \ldots y_{n-1}}{x_0 \ldots x_{n-1}}\right)\right).$$

Proof. Choose pairwise distinct "new" variables $u_0, \ldots, u_{n-1}$. Then

$$\varphi \frac{t_0 \ldots t_{n-1}}{v_0 \ldots v_{n-1}} = \varphi \frac{u_0}{v_0} \frac{u_1}{v_1} \ldots \frac{u_{n-1}}{v_{n-1}} \frac{t_0}{u_0} \frac{t_1}{u_1} \ldots \frac{t_{n-1}}{u_{n-1}}$$

and

$$\varphi\frac{t_0'...t_{n-1}'}{v_0...v_{n-1}} = \varphi\frac{u_0}{v_0}\frac{u_1}{v_1}...\frac{u_{n-1}}{v_{n-1}}\frac{t_0'}{u_0}\frac{t_1'}{u_1}...\frac{t_{n-1}'}{u_{n-1}}.$$

Thus the simultaneous substitutions can be seen as successive substitutions, and we may use the substitution rule repeatedly:

$$\begin{array}{ll}
\varphi\frac{t_0...t_{n-1}}{v_0...v_{n-1}} & \varphi\frac{t_0...t_{n-1}}{v_0...v_{n-1}} \\
\varphi\frac{u_0}{v_0}...\frac{u_{n-1}}{v_{n-1}}\frac{t_0}{u_0}...\frac{t_{n-1}}{u_{n-1}} & \varphi\frac{u_0}{v_0}...\frac{u_{n-1}}{v_{n-1}}\frac{t_0}{u_0}...\frac{t_{n-1}}{u_{n-1}} \\
\varphi\frac{u_0}{v_0}...\frac{u_{n-1}}{v_{n-1}}\frac{t_0}{u_0}...\frac{t_{n-1}}{u_{n-1}}\ t_{n-1}\equiv t_{n-1}' & \varphi\frac{u_0}{v_0}...\frac{u_{n-1}}{v_{n-1}}\frac{t_0}{u_0}...\frac{t_{n-1}'}{u_{n-1}} \\
\vdots & \\
\varphi\frac{u_0}{v_0}...\frac{u_{n-1}}{v_{n-1}}\frac{t_0}{u_0}...\frac{t_{n-1}}{u_{n-1}}\ t_{n-1}\equiv t_{n-1}'...\ t_0\equiv t_0' & \varphi\frac{u_0}{v_0}...\frac{u_{n-1}}{v_{n-1}}\frac{t_0'}{u_0}...\frac{t_{n-1}'}{u_{n-1}} \\
\varphi\frac{t_0...t_{n-1}}{v_0...v_{n-1}}\ t_0\equiv t_0'\ ...\ t_{n-1}\equiv t_{n-1}' & \varphi\frac{t_0'...t_{n-1}'}{v_0...v_{n-1}}.
\end{array}$$

□

12 Consistency

Fix a language S.

Definition 53. *A set $\Phi\subseteq L^S$ is* consistent *if $\Phi\nvdash\bot$. Φ is* inconsistent *if $\Phi\vdash\bot$.*

We prove some laws of consistency.

Lemma 54. *Let $\Phi\subseteq L^S$ and $\varphi\in L^S$. Then*

a) Φ is inconsistent iff there is $\psi\in L^S$ such that $\Phi\vdash\psi$ and $\Phi\vdash\neg\psi$.

b) $\Phi\vdash\varphi$ iff $\Phi\cup\{\neg\varphi\}$ is inconsistent.

c) If Φ is consistent, then $\Phi\cup\{\varphi\}$ is consistent or $\Phi\cup\{\neg\varphi\}$ is consistent (or both).

d) Let $\mathcal{F}$ be a family of consistent sets which is linearly ordered by inclusion, i.e., for all $\Phi,\Psi\in\mathcal{F}$ holds $\Phi\subseteq\Psi$ or $\Psi\subseteq\Phi$. Then

$$\Phi^* = \bigcup_{\Phi\in\mathcal{F}}\Phi$$

is consistent.

Proof. a) Assume $\Phi\vdash\bot$. Then by the *ex falso* rule, $\Phi\vdash\neg\bot$.

Conversely assume that $\Phi\vdash\psi$ and $\Phi\vdash\neg\psi$ for some $\psi\in L^S$. Then $\Phi\vdash\bot$ by $\bot$-introduction.

b) Assume $\Phi\vdash\varphi$. Take $\varphi_0, ..., \varphi_{n-1}\in\Phi$ such that $\varphi_0...\varphi_{n-1}\vdash\varphi$. Then we can extend a derivation of $\varphi_0...\varphi_{n-1}\vdash\varphi$ as follows

$$\begin{array}{lllll}
\varphi_0 & ... & \varphi_{n-1} & & \varphi \\
\varphi_0 & ... & \varphi_{n-1} & \neg\varphi & \neg\varphi \\
\varphi_0 & ... & \varphi_{n-1} & \neg\varphi & \bot
\end{array}$$

and $\Phi\cup\{\neg\varphi\}$ is inconsistent.

Conversely assume that $\Phi\cup\{\neg\varphi\}\vdash\bot$ and take $\varphi_0,...,\varphi_{n-1}\in\Phi$ such that $\varphi_0...\varphi_{n-1}\neg\varphi\vdash\bot$. Then $\varphi_0...\varphi_{n-1}\vdash\varphi$ and $\Phi\vdash\varphi$.

c) Assume that $\Phi\cup\{\varphi\}$ and $\Phi\cup\{\neg\varphi\}$ are inconsistent. Then there are $\varphi_0, ..., \varphi_{n-1}\in\Phi$ such that $\varphi_0...\varphi_{n-1}\vdash\varphi$ and $\varphi_0...\varphi_{n-1}\vdash\neg\varphi$. By the introduction rule for $\bot$, $\varphi_0...\varphi_{n-1}\vdash\bot$. Thus Φ is inconsistent.

d) Assume that Φ^* is inconsistent. Take $\varphi_0, ..., \varphi_{n-1}\in\Phi^*$ such that $\varphi_0 ... \varphi_{n-1}\vdash\bot$. Take $\Phi_0,...\Phi_{n-1}\in\mathcal{F}$ such that $\varphi_0\in\Phi_0, ..., \varphi_{n-1}\in\Phi_{n-1}$. Since $\mathcal{F}$ is linearly ordered by inclusion there is $\Phi\in\{\Phi_0,...\Phi_{n-1}\}$ such that $\varphi_0,...,\varphi_{n-1}\in\Phi$. Then Φ is inconsistent, contradiction. □

Note that d) implies the inductivity required for the lemma of ZORN. The proof of the completeness theorem will be based on the relation between consistency and satisfiability.

Lemma 55. *Assume that $\Phi \subseteq L^S$ is satisfiable. Then Φ is consistent.*

Proof. Assume that $\Phi \vdash \bot$. By the correctness of the sequent calculus, $\Phi \vDash \bot$. Assume that Φ is satisfiable and let $\mathfrak{I} \vDash \Phi$. Then $\mathfrak{I} \vDash \bot$. This contradicts the definition of the satisfaction relation. Thus Φ is not satisfiable. □

Lemma 56. *The sequent calculus is complete iff every consistent $\Phi \subseteq L^S$ is satisfiable.*

Proof. Assume that the sequent calculus is complete. Let $\Phi \subseteq L^S$ be consistent, i.e., $\Phi \nvdash \bot$. By completeness, $\Phi \nvDash \bot$, and we can take an S-interpretation $\mathfrak{I} \vDash \Phi$ such that $\mathfrak{I} \nvDash \bot$. Thus Φ is satisfiable.

Conversely, assume that every consistent $\Phi \subseteq L^S$ is satisfiable. Assume $\Psi \vDash \psi$. Assume for a contradiction that $\Psi \nvdash \psi$. Then $\Psi \cup \{\neg\psi\}$ is consistent. By assumption there is an S-interpretation $\mathfrak{I} \vDash \Psi \cup \{\neg\psi\}$. $\mathfrak{I} \vDash \Psi$ and $\mathfrak{I} \nvDash \psi$, which contradicts $\Psi \vDash \psi$. Thus $\Psi \vdash \psi$. □

13 Term models and HENKIN sets

In view of the previous lemma, we strive to construct interpretations for given sets $\Phi \subseteq L^S$ of S-formulas. Since we are working in great generality and abstractness, the only material available for the construction of structures is the language L^S itself. We shall build a model out of S-terms.

Definition 57. *Let S be a language and let $\Phi \subseteq L^S$ be consistent. The* term model *$\mathfrak{T}^\Phi$ of Φ is the following S-model:*

a) Define a relation $\sim$ on T^S,

$$t_0 \sim t_1 \text{ iff } \Phi \vdash t_0 \equiv t_1 .$$

$\sim$ is an equivalence relation on T^S.

b) For $t \in T^S$ let $\bar{t} = \{s \in T^S \mid s \sim t\}$ be the equivalence class of t.

c) The underlying set $T^\Phi = \mathfrak{T}^\Phi(\forall)$ of the term model is the set of $\sim$-equivalence classes

$$T^\Phi = \{\bar{t} \mid t \in T^S\}.$$

d) For an n-ary relation symbol $R \in S$ let $R^{\mathfrak{T}^\Phi}$ on T^Φ be defined by

$$(\bar{t}_0, \ldots, \bar{t}_{n-1}) \in R^{\mathfrak{T}^\Phi} \text{ iff } \Phi \vdash R t_0 \ldots t_{n-1} .$$

e) For an n-ary function symbol $f \in S$ let $f^{\mathfrak{T}^\Phi}$ on T^Φ be defined by

$$f^{\mathfrak{T}^\Phi}(\bar{t}_0, \ldots, \bar{t}_{n-1}) = \overline{f t_0 \ldots t_{n-1}} .$$

f) For $n \in \mathbb{N}$ define the variable interpretation $\mathfrak{T}^\Phi(v_n) = \bar{v_n}$.

The term model is well-defined:

Lemma 58. *In the previous construction the following holds:*

a) $\sim$ is an equivalence relation on T^S.

b) The definition of $R^{\mathfrak{T}^\Phi}$ is independent of representatives.

c) The definition of $f^{\mathfrak{T}^\Phi}$ is independent of representatives.

Proof. a) We derived the axioms of equivalence relations for $\equiv$:

$\langle \text{rigid} | - \rangle$ $\vdash \forall x\, x \equiv x$

$\langle$rigid$| - \rangle$ $\vdash \forall x \forall y (x \equiv y \rightarrow y \equiv x)$

$\langle$rigid$| - \rangle$ $\vdash \forall x \forall y \forall z (x \equiv y \wedge y \equiv z \rightarrow x \equiv z)$

Consider $t \in T^S$. Then $\vdash t \equiv t$. Thus for all $t \in T^S$ holds $t \sim t$.

Consider $t_0, t_1 \in T^S$ with $t_0 \sim t_1$. Then $\vdash t_0 \equiv t_1$. Also $\vdash t_0 \equiv t_1 \rightarrow t_1 \equiv t_0$, $\vdash t_1 \equiv t_0$, and $t_1 \sim t_0$. Thus for all $t_0, t_1 \in T^S$ with $t_0 \sim t_1$ holds $t_1 \sim t_0$.

The transitivity of $\sim$ follows similarly.

b) Let $\bar{t}_0, \ldots, \bar{t}_{n-1} \in T^\Phi$, $\bar{t}_0 = \bar{s}_0, \ldots, \bar{t}_{n-1} = \bar{s}_{n-1}$ and $\Phi \vdash R t_0 \ldots t_{n-1}$. Then $\vdash t_0 \equiv s_0$, ... , $\vdash t_{n-1} \equiv s_{n-1}$. Repeated applications of the substitution rule yield $\Phi \vdash R s_0 \ldots s_{n-1}$. Hence $\Phi \vdash R t_0 \ldots t_{n-1}$ implies $\Phi \vdash R s_0 \ldots s_{n-1}$. By the symmetry of the argument, $\Phi \vdash R t_0 \ldots t_{n-1}$ iff $\Phi \vdash R s_0 \ldots s_{n-1}$.

c) Let $\bar{t}_0, \ldots, \bar{t}_{n-1} \in T^\Phi$ and $\bar{t}_0 = \bar{s}_0, \ldots, \bar{t}_{n-1} = \bar{s}_{n-1}$. Then $\vdash t_0 \equiv s_0$, ... , $\vdash t_{n-1} \equiv s_{n-1}$. Repeated applications of the substitution rule to $\vdash f t_0 \ldots t_{n-1} \equiv f t_0 \ldots t_{n-1}$ yield

$$\vdash f t_0 \ldots t_{n-1} \equiv f s_0 \ldots s_{n-1}$$

and $\overline{f t_0 \ldots t_{n-1}} = \overline{f s_0 \ldots s_{n-1}}$. □

We aim to obtain $\mathfrak{T}^\Phi \vDash \Phi$. The initial cases of an induction over the complexity of formulas is given by

Theorem 59.

a) For terms $t \in T^S$ holds $\mathfrak{T}^\Phi(t) = \bar{t}$.

b) For atomic formulas $\varphi \in L^S$ holds

$$\mathfrak{T}^\Phi \vDash \varphi \textit{ iff } \Phi \vdash \varphi.$$

Proof. a) By induction on the term calculus. The initial cases $t = c$ where c is a constant symbol or $t = v_n$ are obvious by the definition of the term model. Now consider a term $t = f t_0 \ldots t_{n-1}$ with an n-ary function symbol $f \in S$, and assume that the claim is true for $t_0, \ldots, t_{n-1}$. Then

$$\begin{aligned} \mathfrak{T}^\Phi(f t_0 \ldots t_{n-1}) &= f^{\mathfrak{T}^\Phi}(\mathfrak{T}^\Phi(t_0), \ldots, \mathfrak{T}^\Phi(t_{n-1})) \\ &= f^{\mathfrak{T}^\Phi}(\bar{t}_0, \ldots, \overline{t_{n-1}}) \\ &= \overline{f t_0 \ldots t_{n-1}}. \end{aligned}$$

b) Let $\varphi = R t_0 \ldots t_{n-1}$ with an n-ary relation symbol $R \in S$ and $t_0, \ldots, t_{n-1} \in T^S$. Then

$$\begin{aligned} \mathfrak{T}^\Phi \vDash R t_0 \ldots t_{n-1} &\text{ iff } R^{\mathfrak{T}^\Phi}(\mathfrak{T}^\Phi(t_0), \ldots, \mathfrak{T}^\Phi(t_{n-1})) \\ &\text{ iff } R^{\mathfrak{T}^\Phi}(\bar{t}_0, \ldots, \overline{t_{n-1}}) \\ &\text{ iff } \Phi \vdash R t_0 \ldots t_{n-1}. \end{aligned}$$

Let $\varphi = t_0 \equiv t_1$ with $t_0, t_1 \in T^S$. Then

$$\begin{aligned} \mathfrak{T}^\Phi \vDash t_0 \equiv t_1 &\text{ iff } \mathfrak{T}^\Phi(t_0) = \mathfrak{T}^\Phi(t_1) \\ &\text{ iff } \bar{t}_0 = \bar{t}_1 \\ &\text{ iff } t_0 \sim t_1 \\ &\text{ iff } \Phi \vdash t_0 \equiv t_1. \end{aligned}$$

□

To extend the lemma to complex S-formulas, Φ has to satisfy some recursive properties.

Definition 60. *A set $\Phi \subseteq L^S$ of S-formulas is a* HENKIN set *if it satisfies the following properties:*

a) Φ is consistent;

b) Φ is (derivation) complete, *i.e., for all $\varphi \in L^S$*

$$\Phi \vdash \varphi \textit{ or } \Phi \vdash \neg \varphi;$$

c) Φ contains witnesses, *i.e., for all* $\forall x\varphi \in L^S$ *there is a term* $t \in T^S$ *such that*

$$\Phi \vdash \neg\forall x\varphi \rightarrow \neg\varphi\frac{t}{x}.$$

Lemma 61. *Let* $\Phi \subseteq L^S$ *be a* HENKIN *set. Then for all* $\chi, \psi \in L^S$ *and variables* x*:*

a) $\Phi \nvdash \chi$ *iff* $\Phi \vdash \neg\chi$.

b) $\Phi \vdash \chi$ *implies* $\Phi \vdash \psi$, *iff* $\Phi \vdash \chi \rightarrow \psi$.

c) For all $t \in T^S$ *holds* $\Phi \vdash \chi\frac{t}{u}$ *iff* $\Phi \vdash \forall x\chi$.

Proof. a) Assume $\Phi \nvdash \chi$. By derivation completeness, $\Phi \vdash \neg\chi$. Conversely assume $\Phi \vdash \neg\chi$. Assume for a contradiction that $\Phi \vdash \chi$. Then Φ is inconsistent. Contradiction. Thus $\Phi \nvdash \chi$.
b) Assume $\Phi \vdash \chi$ implies $\Phi \vdash \psi$.
Case 1. $\Phi \vdash \chi$. Then $\Phi \vdash \psi$ and by a previous derivation $\Phi \vdash \chi \rightarrow \psi$.
Case 2. $\Phi \nvdash \chi$. By the derivation completeness of Φ holds $\Phi \vdash \neg\chi$. And by a previous derivation $\Phi \vdash \chi \rightarrow \psi$.

Conversely assume that $\Phi \vdash \chi \rightarrow \psi$. Assume that $\Phi \vdash \chi$. By $\rightarrow$-elimination, $\Phi \vdash \psi$. Thus $\Phi \vdash \chi$ implies $\Phi \vdash \psi$.
c) Assume that for all $t \in T^S$ holds $\Phi \vdash \chi\frac{t}{u}$. Assume that $\Phi \nvdash \forall x\chi$. By a), $\Phi \vdash \neg\forall x\chi$. Since Φ contains witnesses there is a term $t \in T^S$ such that $\Phi \vdash \neg\forall x\chi \rightarrow \neg\chi\frac{t}{u}$. By $\rightarrow$-elimination, $\Phi \vdash \neg\chi\frac{t}{u}$. Contradiction. Thus $\Phi \vdash \forall x\chi$. The converse follows from the rule of $\forall$-elimination. □

Theorem 62. *Let* $\Phi \subseteq L^S$ *be a* HENKIN *set. Then*

a) For all formulas $\chi \in L^S$, *pairwise distinct variables* $\vec{x}$ *and terms* $\vec{t} \in T^S$

$$\mathfrak{T}^\Phi \vDash \chi\frac{\vec{t}}{\vec{x}} \text{ iff } \Phi \vdash \chi\frac{\vec{t}}{\vec{x}}.$$

b) $\mathfrak{T}^\Phi \vDash \Phi$.

Proof. b) follows immediately from a). a) is proved by induction on the formula calculus. The atomic case has already been proven. Consider the non-atomic cases:
i) $\chi = \bot$. Then $\bot\frac{\vec{t}}{\vec{x}} = \bot$. $\mathfrak{T}^\Phi \vDash \bot\frac{\vec{t}}{\vec{x}}$ is false by definition of the satisfaction relation $\vDash$, and $\Phi \vdash \chi\frac{\vec{t}}{\vec{x}}$ is false since Φ is consistent. Thus $\mathfrak{T}^\Phi \vDash \bot\frac{\vec{t}}{\vec{x}}$ iff $\Phi \vdash \bot\frac{\vec{t}}{\vec{x}}$.
ii.) $\chi = \neg\varphi\frac{\vec{t}}{\vec{x}}$ and assume that the claim holds for φ. Then

$$\begin{aligned}
\mathfrak{T}^\Phi \vDash \neg\varphi\frac{\vec{t}}{\vec{x}} \ &\text{iff not } \mathfrak{T}^\Phi \vDash \varphi\frac{\vec{t}}{\vec{x}}\\
&\text{iff not } \Phi \vdash \varphi\frac{\vec{t}}{\vec{x}} \text{ by the inductive assumption}\\
&\text{iff } \Phi \vdash \neg\varphi\frac{\vec{t}}{\vec{x}} \text{ by a) of the previous lemma.}
\end{aligned}$$

iii.) $\chi = (\varphi \rightarrow \psi)\frac{\vec{t}}{\vec{x}}$ and assume that the claim holds for φ and ψ. Then

$$\begin{aligned}
\mathfrak{T}^\Phi \vDash (\varphi \rightarrow \psi)\frac{\vec{t}}{\vec{x}} \ &\text{iff } \mathfrak{T}^\Phi \vDash \varphi\frac{\vec{t}}{\vec{x}} \text{ implies } \mathfrak{T}^\Phi \vDash \psi\frac{\vec{t}}{\vec{x}}\\
&\text{iff } \Phi \vdash \varphi\frac{\vec{t}}{\vec{x}} \text{ implies } \Phi \vdash \psi\frac{\vec{t}}{\vec{x}} \text{ by the inductive assumption}\\
&\text{iff } \Phi \vdash \varphi\frac{\vec{t}}{\vec{x}} \rightarrow \psi\frac{\vec{t}}{\vec{x}} \text{ by a) of the previous lemma}\\
&\text{iff } \Phi \vdash (\varphi \rightarrow \psi)\frac{\vec{t}}{\vec{x}} \text{ by the definition of substitution.}
\end{aligned}$$

iv.) $\chi = (\forall x\varphi)\frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$ and assume that the claim holds for φ. By definition of the substitution χ is of the form

$$\forall u(\varphi\frac{t_0 \ldots t_{r-1}\, u}{x_0 \ldots x_{r-1}\, x}) \text{ oder } \forall u(\varphi\frac{t_1 \ldots t_{r-1}\, u}{x_1 \ldots x_{r-1}\, x})$$

with a suitable variable u. Without loss of generality assume that χ is of the first form. Then

$$\begin{aligned}
\mathfrak{T}^{\Phi} \vDash (\forall x \varphi) \frac{\vec{t}}{\vec{x}} \ &\text{iff} \ \mathfrak{T}^{\Phi} \vDash \exists u (\varphi \frac{t_0 \ldots t_{r-1}\, u}{x_0 \ldots x_{r-1}\, x}) \\
&\text{iff} \ \text{for all } t \in T^S \text{ holds } \mathfrak{T}^{\Phi} \frac{\bar{t}}{u} \vDash \varphi \frac{t_0 \ldots t_{r-1}\, u}{x_0 \ldots x_{r-1}\, x} \\
&\text{iff} \ \text{for all } t \in T^S \text{ holds } \mathfrak{T}^{\Phi} \frac{\mathfrak{I}^{\Phi}(t)}{u} \vDash \varphi \frac{t_0 \ldots t_{r-1}\, u}{x_0 \ldots x_{r-1}\, x} \ \text{ by a previous lemma} \\
&\text{iff} \ \text{for all } t \in T^S \text{ holds } \mathfrak{T}^{\Phi} \vDash (\varphi \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}) \frac{t}{u} \ \text{ by the substitution lemma} \\
&\text{iff} \ \text{for all } t \in T^S \text{ holds } \mathfrak{T}^{\Phi} \vDash \varphi \frac{t_0 \ldots t_{r-1}\, t}{x_0 \ldots x_{r-1}\, x} \ \text{ by successive substitutions} \\
&\text{iff} \ \text{for all } t \in T^S \text{ holds } \Phi \vdash \varphi \frac{t_0 \ldots t_{r-1}\, t}{x_0 \ldots x_{r-1}\, x} \ \text{ by the inductive assumption} \\
&\text{iff} \ \text{for all } t \in T^S \text{ holds } \Phi \vdash (\varphi \frac{t_0 \ldots t_{r-1}\, u}{x_0 \ldots x_{r-1}\, x}) \frac{t}{u} \ \text{ by successive substitutions} \\
&\text{iff} \ \Phi \vdash \forall u (\varphi \frac{t_0 \ldots t_{r-1}\, u}{x_0 \ldots x_{r-1}\, x}) \ \text{ by c) of the previous lemma} \\
&\text{iff} \ \Phi \vdash (\forall x \varphi) \frac{\vec{t}}{\vec{x}}.
\end{aligned}$$

□

14 Constructing HENKIN sets

We shall show that every consistent set of formulas can be extended to a HENKIN set by "adding witnesses" and then ensuring negation completeness. We first consider witnesses.

Theorem 63. *Let $\Phi \subseteq L^S$ be consistent. Let $\varphi \in L^S$ and let z be a variable which does not occur in $\Phi \cup \{\varphi\}$. Then the set*

$$\Phi \cup \{\neg \forall x \varphi \rightarrow \neg \varphi \frac{z}{x}\}$$

is consistent.

Proof. Assume for a contradiction that $\Phi \cup \{(\neg \exists x \varphi \vee \varphi \frac{z}{x})\}$ is inconsistent. Take $\varphi_0, \ldots, \varphi_{n-1} \in \Phi$ such that

$$\varphi_0 \ldots \varphi_{n-1} \ \neg \forall x \varphi \rightarrow \neg \varphi \frac{z}{x} \ \vdash \bot.$$

Set $\Gamma = (\varphi_0, \ldots, \varphi_{n-1})$. Then continue the derivation as follows:

1.	Γ	$\neg \forall x \varphi \rightarrow \neg \varphi \frac{z}{x}$	$\bot$
2.	Γ	$\neg\neg \forall x \varphi$	$\neg\neg \forall x \varphi$
3.	Γ	$\neg\neg \forall x \varphi$	$\neg \forall x \varphi \rightarrow \neg \varphi \frac{z}{x}$
4.	Γ	$\neg\neg \forall x \varphi$	$\bot$
5.	Γ		$\neg \forall x \varphi$
6.	Γ	$\neg \varphi \frac{z}{x}$	$\neg \varphi \frac{z}{x}$
7.	Γ	$\neg \varphi \frac{z}{x}$	$\neg \forall x \varphi \rightarrow \neg \varphi \frac{z}{x}$
8.	Γ	$\neg \varphi \frac{z}{x}$	$\bot$
9.	Γ		$\varphi \frac{z}{x}$
10.	Γ		$\forall x \varphi$
11.	Γ		$\bot$

Hence Φ is inconsistent, contradiction. □

This means that "unused" variables may be used as HENKIN witnesses. Since "unused" constant symbols behave much like unused variables, we get:

Theorem 64. *Let $\Phi \subseteq L^S$ be consistent. Let $\varphi \in L^S$ and let $c \in S$ be a constant symbol which does not occur in $\Phi \cup \{\varphi\}$. Then the set*

$$\Phi \cup \{\neg \forall x \varphi \rightarrow \neg \varphi \frac{c}{x}\}$$

is consistent.

Proof. Assume that $\Phi \cup \{(\neg \exists x \varphi \vee \varphi \frac{c}{x})\}$ is inconsistent. Take a derivation

$$\begin{array}{c} \Gamma_0 \varphi_0 \\ \Gamma_1 \varphi_1 \\ \vdots \\ \Gamma_{n-1}\, \varphi_{n-1} \\ \Gamma_n\ (\neg \forall x \varphi \rightarrow \neg \varphi \frac{c}{x})\ \bot \end{array} \tag{1}$$

with $\Gamma_n \subseteq \Phi$. Choose a variable z, which does not occur in the derivation. For a formula ψ define ψ' by replacing each occurence of c by z, and for a sequence $\Gamma = (\psi_0, \ldots, \psi_{k-1})$ of formulas let $\Gamma' = (\psi_0', \ldots, \psi_{k-1}')$. Replacing each occurence of c by z in the deriavation we get

$$\begin{array}{c} \Gamma_0' \varphi_0' \\ \Gamma_1' \varphi_1' \\ \vdots \\ \Gamma_{n-1}'\, \varphi_{n-1}' \\ \Gamma_n\ (\neg \forall x \varphi \rightarrow \neg \varphi \frac{z}{x})\ \bot \end{array} \tag{2}$$

The particular form of the final sequence is due to the fact that c does not occur in $\Phi \cup \{\varphi\}$. To show that (2) is again a derivation in the sequent calculus we show that the replacement $c \mapsto z$ transforms every instance of a sequent rule in (1) into an instance of a (derivable) rule in (2). This is obvious for all rules except possibly the quantifyer rules.

So let

$$\frac{\Gamma \quad \psi \frac{y}{x}}{\Gamma \quad \forall x \psi}\ \text{, with } y \notin \mathrm{free}(\Gamma \cup \{\forall x \psi\})$$

be an $\forall$-introduction in (1). Then $(\psi \frac{y}{x})' = \psi' \frac{y}{x}$, $(\forall x \psi)' = \forall x \psi'$, and $y \notin \mathrm{free}(\Gamma' \cup \{(\forall x \psi)'\})$. Hence

$$\frac{\Gamma' \quad (\psi \frac{y}{x})'}{\Gamma' \quad (\forall x \psi)'}$$

is a justified $\forall$-introduction.

Now consider an $\forall$-elimination in (1):

$$\frac{\Gamma \quad \forall x \psi}{\Gamma \quad \psi \frac{t}{x}}$$

Then $(\forall x \psi)' = \forall x \psi'$ and $(\psi \frac{t}{x})' = \psi' \frac{t'}{x}$ where t' is obtained from t by replacing all occurences of c by z. Hence

$$\frac{\Gamma' \quad (\forall x \psi)'}{\Gamma' \quad (\psi \frac{t}{x})'}$$

is a justified $\forall$-elimination.

The derivation (2) proves that

$$\Phi \cup \{(\neg \forall x \varphi \rightarrow \neg \varphi \frac{z}{x}) \vdash \bot,$$

which contradicts the preceding lemma. $\square$

We shall now show that any consistent set of formulas can be consistently expanding to a set of formulas which contains witnesses.

Theorem 65. *Let S be a language and let $\Phi \subseteq L^S$ be consistent. Then there is a language S^ω and $\Phi^\omega \subseteq L^{S^*}$ such that*

a)S^ω extends S by constant symbols, i.e., $S \subseteq S^\omega$ and if $s \in S^\omega \setminus S$ then s is a constant symbol;

b)$\Phi^\omega \supseteq \Phi$;

c)Φ^ω is consistent;

d)Φ^ω contains witnesses;

e)if L^S is countable then so are L^{S^ω} and Φ^ω.

Proof. For every a define a "new" distinct constant symbol c_a, which does not occur in S, e.g., $c_a = ((a, S), 1, 0)$. Extend S by constant symbols c_ψ for $\psi \in L^S$:

$$S^+ = S \cup \{c_\psi | \psi \in L^S\}.$$

Then set

$$\Phi^+ = \Phi \cup \{\neg \forall x \varphi \to \neg \varphi \frac{c_{\forall x \varphi}}{x} | \forall x \varphi \in L^S\}.$$

Φ^+ contains witnesses for all universal formulas of S.
(1) $\Phi^+ \subseteq L^{S^+}$ is consistent.
Proof: Assume instead that Φ^+ is inconsistent. Choose a finite sequence $\forall x_0 \varphi_0, \ldots, \forall x_{n-1} \varphi_{n-1} \in L^S$ of pairwise distinct universal formulas such that

$$\Phi \cup \{\neg \forall x_0 \varphi_0 \to \neg \varphi_0 \frac{c_{\forall x_0 \varphi_0}}{x_0}, \ldots, \neg \forall x_{n-1} \varphi_{n-1} \to \neg \varphi_{n-1} \frac{c_{\forall x_{n-1} \varphi_{n-1}}}{x_{n-1}}\}$$

is inconsistent. But by the previous theorem one can inductively show that for all $i < n$ the set

$$\Phi \cup \{\neg \forall x_0 \varphi_0 \to \neg \varphi_0 \frac{c_{\forall x_0 \varphi_0}}{x_0}, \ldots, \neg \forall x_{n-1} \varphi_{n-1} \to \neg \varphi_{n-1} \frac{c_{\forall x_{i-1} \varphi_{\mathrm{ni}-1}}}{x_{i-1}}\}$$

is consistent. Contradiction. *qed*(1)

We iterate the $+$-operation through the integers. Define recursively

$$\begin{aligned} \Phi^0 &= \Phi \\ S^0 &= S \\ S^{n+1} &= (S^n)^+ \\ \Phi^{n+1} &= (\Phi^n)^+ \\ S^\omega &= \bigcup_{n \in \mathbb{N}} S^n \\ \Phi^\omega &= \bigcup_{n \in \mathbb{N}} \Phi^n. \end{aligned}$$

S^ω is an extension of S by constant symbols. For $n \in \mathbb{N}$, Φ^n is consistent by induction. Φ^ω is consistent by the lemma on unions of consistent sets.
(2) Φ^ω contains witnesses.
Proof. Let $\forall x \varphi \in L^{S^\omega}$. Let $n \in \mathbb{N}$ such that $\forall x \varphi \in L^{S^n}$. Then $\neg \forall x \varphi \to \neg \varphi \frac{c_{\forall x \varphi}}{x} \in \Phi^{n+1} \subseteq \Phi^\omega$.
qed(2)
(3) Let L^S be countable. Then L^{S^ω} and Φ^ω are countable.
Proof. Since L^S is countable, there can only be countably many symbols in the alphabet of $S^0 = S$. The alphabet of S^1 is obtained by adding the countable set $\{c_\psi | \psi \in L^S\}$; the alphabet of S^1 is countable as the union of two countable sets. The set of words over a countable alphabet is countable, hence L^{S^1} and $\Phi^1 \subseteq L^{S^1}$ are countable.

Inductive application of this argument show that for any $n \in \mathbb{N}$, the sets L^{S^n} and Φ^n are countable. Since countable unions of countable sets are countable, $L^{S^\omega} = \bigcup_{n \in \mathbb{N}} L^{S^n}$ and also $\Phi^\omega \subseteq L^{S^\omega}$ are countable. ☐

To get HENKIN sets we have to ensure derivation completeness.

Theorem 66. *Let S be a language and let $\Phi \subseteq L^S$ be consistent. Then there is a consistent $\Phi^* \subseteq L^S$, $\Phi^* \supseteq \Phi$ which is derivation complete.*

Proof. Define the partial order $(P, \subseteq)$ by

$$P = \{\Psi \subseteq L^S \mid \Psi \supseteq \Phi \text{ and } \Psi \text{ is consistent}\}.$$

$P \neq \emptyset$ since $\Phi \in P$. P is *inductively ordered* by a previous lemma: if $\mathcal{F} \subseteq P$ is linearly ordered by inclusion, i.e., for all $\Psi, \Psi' \in \mathcal{F}$ holds $\Psi \subseteq \Psi'$ or $\Psi' \subseteq \Psi$ then

$$\bigcup_{\Psi \in \mathcal{F}} \Psi \in P.$$

Hence $(P, \subseteq)$ satisfies the conditions of ZORN's lemma. Let Φ^* be a maximal element of $(P, \subseteq)$. By the definition of P, $\Phi^* \subseteq L^S$, $\Phi^* \supseteq \Phi$, and Φ^* is consistent. Derivation completeness follows from the following claim.
(1) For all $\varphi \in L^S$ holds $\varphi \in \Phi^*$ or $\neg\varphi \in \Phi^*$.
Proof. Φ^* is consistent. By a previous lemma, $\Phi^* \cup \{\varphi\}$ or $\Phi^* \cup \{\neg\varphi\}$ are consistent.
Case 1. $\Phi^* \cup \{\varphi\}$ is consistent. By the $\subseteq$-maximality of Φ^*, $\Phi^* \cup \{\varphi\} = \Phi^*$ and $\varphi \in \Phi^*$.
Case 2. $\Phi^* \cup \{\neg\varphi\}$ is consistent. By the $\subseteq$-maximality of Φ^*, $\Phi^* \cup \{\neg\varphi\} = \Phi^*$ and $\neg\varphi \in \Phi^*$. □

The proof uses ZORN's lemma. In case L^S is countable one can work without ZORN's lemma.

Proof. (For countable L^S) Let $L^S = \{\varphi_n | n \in \mathbb{N}\}$ be an enumeration of L^S. Define a sequence $(\Phi_n | n \in \mathbb{N})$ by recursion on n such that

i. $\Phi \subseteq \Phi_n \subseteq \Phi_{n+1} \subseteq L^S$;

ii. Φ_n is consistent.

For $n = 0$ set $\Phi_0 = \Phi$. Assume that Φ_n is defined according to i. and ii.
Case 1. $\Phi_n \cup \{\varphi_n\}$ is consistent. Then set $\Phi_{n+1} = \Phi_n \cup \{\varphi_n\}$.
Case 2. $\Phi_n \cup \{\varphi_n\}$ is inconsistent. Then $\Phi_n \cup \{\neg\varphi_n\}$ is consistent by a previous lemma, and we define $\Phi_{n+1} = \Phi_n \cup \{\neg\varphi_n\}$.

Let

$$\Phi^* = \bigcup_{n \in \mathbb{N}} \Phi_n .$$

Then Φ^* is a consistent superset of Φ. By construction, $\varphi \in \Phi^*$ or $\neg\varphi \in \Phi^*$, for all $\varphi \in L^S$. Hence Φ^* is derivation complete. □

According to Theorem 65 a given consistent set Φ can be extended to $\Phi^\omega \subseteq L^{S^\omega}$ containing witnesses. By Theorem 66 Φ^ω can be extended to a derivation complete $\Phi^* \subseteq L^{S^\omega}$. Since the latter step does not extend the language, Φ^* contains witnesses and is thus a HENKIN set:

Theorem 67. *Let S be a language and let $\Phi \subseteq L^S$ be consistent. Then there is a language S^* and $\Phi^* \subseteq L^{S^*}$ such that*

a) $S^* \supseteq S$ is an extension of S by constant symbols;

b) $\Phi^* \supseteq \Phi$ is a HENKIN set;

c) if L^S is countable then so are L^{S^*} and Φ^*.

15 The completeness theorem

We can now combine our technical preparations to show the fundamental theorems of first-order logic.

Combining Theorems 67 and 62, we obtain a general and a countable model existence theorem:

Theorem 68. (HENKIN model existence theorem) *Let* $\Phi \subseteq L^S$. *Then* Φ *is consistent iff* Φ *is satisfiable.*

Theorem 69. (Downward LÖWENHEIM-SKOLEM theorem) *Let* $\Phi \subseteq L^S$ *be a* countable *consistent set of formulas. Then* Φ *possesses a model* $\mathfrak{M} = (\mathfrak{A}, \beta) \vDash \Phi$, $\mathfrak{A} = (A, \ldots)$ *with a* countable *underlying set* A.

The word "downward" emphasises the existence of models of "small" cardinality. We shall soon also consider an upward LÖWENHEIM-SKOLEM theorem. By Lemma 56, Theorem 68 the model existence theorems imply the main theorem.

Theorem 70. (GÖDEL completeness theorem) *The sequent calculus is complete, i.e.,* $\vDash = \vdash$.

Finally the equality of $\vDash$ and $\vdash$ and the compactness theorem 48 for $\vdash$ imply

Theorem 71. (Compactness theorem) *Let* $\Phi \subseteq L^S$ *and* $\varphi \in \Phi$. *Then*

a)$\Phi \vDash \varphi$ *iff there is a finite subset* $\Phi_0 \subseteq \Phi$ *such that* $\Phi_0 \vDash \varphi$.

b)Φ *is satisfiable iff every finite subset* $\Phi_0 \subseteq \Phi$ *is satisfiable.*

The GÖDEL completeness theorem is the fundamental theorem of mathematical logic. It connects syntax and semantics of formal languages in an optimal way. Before we continue the mathematical study of its consequences we make some general remarks about the wider impact of the theorem:

⟨rigid|−⟩ The completeness theorem gives an *ultimate correctness criterion* for mathematical proofs. A proof is correct if it can (in principle) be reformulated as a formal derivation. Although mathematicians prefer semi-formal or informal arguments, this criterion could be applied in case of doubt.

⟨rigid|−⟩ Checking the correctness of a formal proof in the above sequent calculus is a syntactic task that can be carried out by computer. We shall later consider a prototypical *proof checker* `Naproche` which uses a formal language which is a subset of natural english.

⟨rigid|−⟩ By systematically running through all possible formal proofs, *automatic theorem proving* is in principle possible. In this generality, however, algorithms immediately run into very high algorithmic complexities and become practically infeasable.

⟨rigid|−⟩ Practical automatic theorem proving has become possible in restricted situations, either by looking at particular kinds of axioms and associated intended domains, or by restricting the syntactical complexity of axioms and theorems.

⟨rigid|−⟩ Automatic theorem proving is an important component of *artificial intelligence* (AI) where a system has to obtain logical consequences from conditions formulated in first-order logic. Although there are many difficulties with artificial intelligence this approach is still being followed with some success.

⟨rigid|−⟩ Another special case of automatic theorem proving is given by *logic programming* where programs consist of logical statements of some restricted complexity and a run of a program is a systematic search for a solution of the given statements. The original and still most prominent logic programming language is `Prolog` which is still widely used in linguistics and AI.

⟨rigid|−⟩ There are other areas which can be described formally and where syntax/semantics constellations similar to first-order logic may occur. In the theory of algorithms there is the syntax of programming languages versus the (mathematical) meaning of a program. Since programs crucially involve time alternative logics with time have to be introduced. Now in all such generalizations, the GÖDEL completeness theorem serves as a pattern onto which to model the syntax/semantics relation.

⟨rigid|−⟩ The success of the formal method in mathematics makes mathematics a leading *formal science*. Several other sciences also strive to present and justify results formally, like computer science and parts of philosophy.

⟨rigid| − ⟩ The completeness theorem must not be confused with the famous GÖDEL *incompleteness theorems*: they say that certain axiom systems like PEANO arithmetic are incomplete in the sense that they do not imply some formulas which hold in the standard model of the axiom system.

16 Cardinalities of models

Definition 72. *An S-structure $\mathfrak{A}$ is* finite, infinite, countable, *or* uncountable, *resp., iff the underlying set $|\mathfrak{A}|$ is finite, infinite, countable, or uncountable, resp..*

Theorem 73. *Assume that $\Phi \subseteq L^S$ has arbitrarily large finite models. Then Φ has an infinite model.*

Proof. For $n \in \mathbb{N}$ define the sentence

$$\varphi_{\geqslant n} = \exists v_0, \ldots, v_{n-1} \bigwedge_{i<j<n} \neg v_i \equiv v_j \,,$$

where the big conjunction is defined by

$$\bigwedge_{i<j<n} \psi_{ij} = \psi_{0,1} \wedge \ldots \wedge \psi_{0,n-1} \wedge \psi_{1,2} \wedge \ldots \wedge \psi_{1,n-1} \wedge \ldots \wedge \psi_{n-1,n-1} \,.$$

For any model $\mathfrak{M}$

$$\mathfrak{M} \vDash \varphi_{\geqslant n} \text{ iff } A \text{ has at least } n \text{ elements.}$$

Now set

$$\Phi' = \Phi \cup \{\varphi_{\geqslant n} \,|\, n \in \mathbb{N}\}.$$

(1) Φ' has a model.
Proof. By the compactness theorem 71b it suffices to show that every finite $\Phi_0 \subseteq \Phi$ has a model. Let $\Phi_0 \subseteq \Phi$ be finite. Take $n_0 \in \mathbb{N}$ such that

$$\Phi_0 \subseteq \Phi \cup \{\varphi_{\geqslant n} \,|\, n \leqslant n_0\}.$$

By assumption Φ has a model with at least n_0 elements. Thus $\Phi \cup \{\varphi_{\geqslant n} \,|\, n \leqslant n_0\}$ and Φ_0 have a model. *qed*(1)

Let $\mathfrak{M} \vDash \Phi'$. Then $\mathfrak{M}$ is an infinite model of Φ. □

Theorem 74. (Upward LÖWENHEIM-SKOLEM theorem) *Let $\Phi \subseteq L^S$ have an infinite S-model and let X be an arbitrary set. Then Φ has a model into which X can be embedded injectively.*

Proof. Let $\mathfrak{M}$ be an infinite model of Φ. Choose a sequence $(c_x \,|\, x \in X)$ of pairwise distinct constant symbols which do not occur in S, e.g., setting $c_x = ((x, S), 1, 0)$. Let $S' = S \cup \{c_x \,|\, x \in X\}$ be the extension of S by the new constant symbols. Set

$$\Phi' = \Phi \cup \{\neg c_x \equiv c_y \,|\, x, y \in X, x \neq y\}.$$

(1) Φ' has a model.
Proof. It suffices to show that every finite $\Phi_0 \subseteq \Phi'$ has a model. Let $\Phi_0 \subseteq \Phi'$ be finite. Take a finite set $X_0 \subseteq X$ such that

$$\Phi_0 \subseteq \Phi \cup \{\neg c_x \equiv c_y \,|\, x, y \in X_0, x \neq y\}.$$

Since $|\mathfrak{M}|$ is infinite we can choose an injective sequence $(a_x | x \in X_0)$ of elements of $|\mathfrak{M}|$ such that $x \neq y$ implies $a_x \neq a_y$. For $x \in X \setminus X_0$ choose $a_x \in |\mathfrak{M}|$ arbitrarily. Then in the extended model

$$\mathfrak{M}' = \mathfrak{M} \cup \{(c_x, a_x) | x \in X\} \vDash \Phi \cup \{\neg c_x \equiv c_y \,|\, x, y \in X_0, x \neq y\} \supseteq \Phi_0 \,.$$

qed(1)

By (1), choose a model $\mathfrak{M}' \vDash \Phi'$. Then the map

$$i \colon X \to |\mathfrak{M}'|, x \mapsto \mathfrak{M}'(c_x)$$

is injective. The reduction $\mathfrak{M}'' = \mathfrak{M}' \upharpoonright \{\forall\} \cup S$ is as required. □

We define notions which allow to examine the axiomatizability of classes of structures.

Definition 75. *Let S be a language and $\mathcal{K}$ be a class of S-structures.*

a)$\mathcal{K}$ *ist* elementary *or* finitely axiomatizable *if there is an S-sentence φ with* $\mathcal{K} = \mathrm{Mod}^S\varphi$.

b)$\mathfrak{K}$ *is* Δ-elementary *or* axiomatizable*, if there is a set Φ of S-sentences with* $\mathcal{K} = \mathrm{Mod}^S\Phi$.

We state simple properties of the Mod-operator:

Theorem 76. *Let S be a language. Then*

a) For $\Phi \subseteq \Psi \subseteq L_0^S$ *holds* $\mathrm{Mod}^S\Phi \supseteq \mathrm{Mod}^S\Psi$.

b) For $\Phi, \Psi \subseteq L_0^S$ *holds* $\mathrm{Mod}^S(\Phi \cup \Psi) = \mathrm{Mod}^S\Phi \cap \mathrm{Mod}^S\Psi$.

c) For $\Phi \subseteq L_0^S$ *holds* $\mathrm{Mod}^S\Phi = \bigcap_{\varphi \in \Phi} \mathrm{Mod}^S\varphi$.

d) For $\varphi_0, \ldots, \varphi_{n-1} \in L_0^S$ *holds* $\mathrm{Mod}^S\{\varphi_0, \ldots, \varphi_{n-1}\} = \mathrm{Mod}^S(\varphi_0 \wedge \ldots \wedge \varphi_{n-1})$.

e) For $\varphi \in L_0^S$ *holds* $\mathrm{Mod}^S(\neg\varphi) = \mathrm{Mod}^S\emptyset \setminus \mathrm{Mod}^S(\varphi)$.

c) explains the denotation Δ-elementary, since $\mathrm{Mod}^S\Phi$ is the intersection ("**D**urchschnitt") of all single $\mathrm{Mod}^S\varphi$.

Theorem 77. *Let S be a language and $\mathcal{K}, \mathcal{L}$ be classes of S-structures with*

$$\mathcal{L} = \mathrm{Mod}^S\emptyset \setminus \mathcal{K}.$$

Then if $\mathcal{K}$ and $\mathcal{L}$ are axiomatizable, they are finitely axiomatizable.

Proof. Take axiom systems Φ_K and Φ_L such that $\mathfrak{K} = \mathrm{Mod}^S\Phi_K$ and $\mathfrak{L} = \mathrm{Mod}^S\Phi_L$. Assume that $\mathfrak{K}$ is not finitely axiomatizable.
(1) Let $\Phi_0 \subseteq \Phi_K$ be finite. Then $\Phi_0 \cup \Phi_L$ is satisfiable.
Proof: $\mathrm{Mod}^S\Phi_0 \supseteq \mathrm{Mod}^S\Phi_K$. Since $\mathfrak{K}$ is not finitely axiomatizable, $\mathrm{Mod}^S\Phi_0 \neq \mathrm{Mod}^S\Phi_K$. Then $\mathrm{Mod}^S\Phi_0 \cap \mathfrak{L} \neq \emptyset$. Take a model $\mathfrak{A} \in \mathfrak{L}$, $\mathfrak{A} \in \mathrm{Mod}^S\Phi_0$. Then $\mathfrak{A} \vDash \Phi_0 \cup \Phi_L$. *qed*(1)
(2) $\Phi_K \cup \Phi_L$ is satisfiable.
Proof: By the compactness theorem 71 it suffices to show that every finite $\Psi \subseteq \Phi_K \cup \Phi_L$ is satsifiable. By (1), $(\Psi \cap \Phi_K) \cup \Phi_L$ is satisfiable. Thus $\Psi \subseteq (\Psi \cap \Phi_K) \cup \Phi_L$ is satisfiable. *qed*(2)

By (2), $\mathrm{Mod}^S\Phi_K \cap \mathrm{Mod}^S\Phi_L \neq \emptyset$. But the classes $\mathfrak{K}$ and $\mathfrak{L}$ are complements, contradiction. Thus $\mathfrak{K}$ is finitely axiomatizable. ☐

Theorem 78. *Let S be a language.*

a) The class of all finite S-structures is not axiomatizable.

b) The class of all infinite S-structures is axiomatizable but not finitely axiomatizable.

c) Let $\Phi \subseteq L_0^S$ *such that* $\mathrm{Mod}^S\Phi$ *contains infinite structures. Then* $\mathrm{Mod}^S\Phi$ *contains structures of arbitrarily high cardinalities, i.e., for any set X there is a model $\mathfrak{M} \vDash \Phi$ and an injective map from X into M.*

Proof. a) is immediate by Theorem 73.
b) The class of infinite S-structures is axiomatized by

$$\Phi = \{\varphi_{\geqslant n} \mid n \in \mathbb{N}\}.$$

If that class were *finitely* axiomatizable then the complementary class of finite S-structures would also be (finitely) axiomatizable, contradicting a).
c) Let $\{c_x | x \in X\}$ be a set of "new" constant symbols. Let

$$\Phi_X = \Phi \cup \{\neg c_x \equiv c_y \mid x, y \in X, x \neq y\}.$$

Every finite subset of Φ_X is satisfiable in any infinite model of Φ. By the compactness theorem, Φ_X is consistent and satisfiable. Let $\mathfrak{M}_X \vDash \Phi_X$ and let $\mathfrak{M} = \mathfrak{M}_X \upharpoonright S \vDash \Phi$. Define $f: X \to M$ by

$$f(x) = \mathfrak{M}_X(c_x).$$

Then f is injective as required. $\square$

17 Groups

Definition 79. *The* language of group theory *is the language*

$$S_{\mathrm{Gr}} = \{\circ, e\},$$

where $\circ$ *is a binary function symbol and e is a constant symbol. The* group axioms *are the following set of sentences:*

$$\Phi_{\mathrm{Gr}} = \{\forall v_0 \forall v_1 \forall v_2 \circ \circ v_0\, v_1\, v_2 \equiv \circ\, v_0 \circ v_1\, v_2\,, \forall v_0\, \circ v_0\, e \equiv v_0\,, \forall v_0 \exists v_1 \circ v_0\, v_1 \equiv e\}.$$

A group *is an* S_{Gr}*-structure* $\mathfrak{G}$ *with* $\mathfrak{G} \vDash \Phi_{\mathrm{Gr}}$.

The group axioms may be written in a more familiar way with variables $x, y, z, \ldots$, infix notation and further abbreviations as

⟨rigid| − ⟩ $\forall x, y, z\ (x \circ y) \circ z \equiv x \circ (y \circ z)$ (associativity)

⟨rigid| − ⟩ $\forall x\, x \circ e \equiv x$ (neutral element)

⟨rigid| − ⟩ $\forall x \exists y\ x \circ y \equiv e$ (inverses)

Some elementary facts of group theory have short formal proofs. We show that the neutral element of a group is its own left inverse.

Theorem 80. $\Phi_{\mathrm{Gr}} \vdash \forall v_0\, (v_0 \circ e \equiv e \rightarrow v_0 \equiv e)$.

Proof.

Let $\forall x \forall y \forall z\, ((x * y) * z) = (x * (y * z))$.
Let $\forall x\, (x * e) = x$.
Let $\forall x \exists y\, (x * y) = e$.
Theorem. $\forall u\, ((u * e) = e \rightarrow u = e)$.
Proof. Let $(u * e) = e$. $(u * e) = u$. $u = (u * e)$. $u = e$.
Thus $\forall u\, ((u * e) = e \rightarrow u = e)$. Qed. $\square$

Let us now consider some algebraic details.

Definition 81. *A group* $\mathfrak{G} = (G, \cdot, 1)$ *is a* torsion group *if for all* $g \in G$ *there is* $n \in \mathbb{N} \setminus \{0\}$ *with* $g^n = 1$. *Here,* g^n *is defined recursively by:* $g^0 = 1$, $g^{n+1} = g \cdot g^n$.

Theorem 82. *The class* $\mathcal{T}$ *of all torsion groups is not axiomatizable.*

Proof. Assume $\mathcal{T} = \mathrm{Mod}^{S_{\mathrm{Gr}}}\Phi$, where $\Phi \subseteq L_0^{S_{\mathrm{Gr}}}$. Define

$$\Psi = \Phi \cup \{\neg \underbrace{v_0 \circ \ldots v_0}_{n-\text{times}} \equiv e \,|\, n \in \mathbb{N} \setminus \{0\}\}.$$

Every finite subset of Ψ is satisfiable: Consider a finite $\Psi_0 \subseteq \Psi$. Take $n_0 \in \mathbb{N}$ such that

$$\Psi_0 \subseteq \Phi \cup \{\neg \underbrace{v_0 \circ \ldots v_0}_{n-\text{times}} \equiv e \,|\, 1 \leqslant n \leqslant n_0\}.$$

The right-hand side can be satisfied in every torsion group which has an element of *order* $\geqslant n_0$, e.g., in the additive group of integers *modulo* n_0. Bei the compactness theorem 71, Ψ is satisfiable. Take a model $\mathcal{G} \vDash \Psi$. Then $\mathcal{G}$ is a group in which the element $\mathcal{G}(v_0)$ satisfies all formulas

$$\neg \underbrace{v_0 \circ \ldots v_0}_{n-\text{times}} \equiv e\,.$$

Hence $\mathcal{G}(v_0)$ has infinite order in $\mathcal{G}$ and $\mathcal{G}$ is not a torsion group, although $\mathcal{G} \vDash \Phi$. Contradiction. $\square$

This theorem demonstrates that mathematical logic also examines the limits of its methods: torsion groups *cannot* be axiomatized in the language of group theory. It is however possible to characterize torsion groups in stronger theories, where the formation of powers v_0^n is uniformly available.

There are several ways to logically treat group theory. One could for example include inversion as a function symbol.

Definition 83. *The* extended language of group theory *is the language*

$$S_{\mathrm{Gr}'} = \{\circ, i, e\},$$

where i is a unary function symbol. The extended group axioms *consist of the axioms*

$$\Phi_{\mathrm{Gr}'} = \{\forall v_0 \forall v_1 \forall v_2 \circ \circ v_0\, v_1\, v_2 \equiv \circ\, v_0 \circ v_1\, v_2, \forall v_0 \circ v_0\, e \equiv v_0, \forall v_0 \circ v_0\, i\, v_0 \equiv e\}.$$

An extended group *is an $S_{\mathrm{Gr}'}$-structure $\mathfrak{G}$ with $\mathfrak{G} \vDash \Phi_{\mathrm{Gr}'}$.*

Obviously every extended group can be reduced to a group in the former sense and vice versa. There are, however, model theoretic differences, e.g., concerning substructures.

Theorem 84. *A substructure of a group need not be a group. A substructure of an extended group is an extended group.*

This fact is due to the syntactic structure of the axioms considered.

18 Fields

Fields are *arithmetical structures*, i.e., a field allows addition and multiplication. We describe filed in the *language of arithmetic*

$$S_{\mathrm{Ar}} = \{+, \cdot, 0, 1\}$$

with the usual conventions for infix notation and bracket notation. The axiom system Φ_{Fd} of *field theory* consists of the following axioms:

⟨rigid| – ⟩ $\forall x \forall y \forall z\, (x+y)+z \equiv x+(y+z)$

⟨rigid| – ⟩ $\forall x \forall y \forall z\, (x \cdot y) \cdot z \equiv x \cdot (y \cdot z)$

⟨rigid| – ⟩ $\forall x \forall y x + y \equiv y + x$

⟨rigid| – ⟩ $\forall x \forall y x \cdot y \equiv y \cdot x$

⟨rigid| – ⟩ $\forall x\, x + 0 \equiv x$

⟨rigid| – ⟩ $\forall x\, x \cdot 1 \equiv x$

⟨rigid| – ⟩ $\forall x \exists y x + y \equiv 0$

⟨rigid| – ⟩ $\forall x (\neg x \equiv 0 \rightarrow \exists y x \cdot y \equiv 1)$

⟨rigid| – ⟩ $\neg 0 \equiv 1$

⟨rigid| – ⟩ $\forall x \forall y \forall z\, x \cdot (y+z) \equiv (x \cdot y) + (x \cdot z)$

A *field* is an S_{Ar}-model satisfying Φ_{Fd}. The axiom system Φ_{Fd} is not *complete*. There are, e.g., finite and infinite fields and thus there is a natural number n such that the axioms do not decide the sentence $\varphi_{=n}$ which expresses that there are exactly n elements.

Substantial parts of mathematics can be carried out within field theory. Vectors of a finite-dimensional vector space over a field $\mathbb{K}$ can be represented as finite tuples from $\mathbb{K}$. The laws of vector and matrix calculus are sentences about appropriately indexed field elements. Thus the theory of finite-dimensional vector spaces can be carried out within field theory. Technically we say that the theory of n-dimensional vector spaces can be *interpreted* within the theory of fields. That $(z_0, \ldots, z_{n-1})$ is the *vector sum* of $(x_0, \ldots, x_{n-1})$ and $(y_0, \ldots, y_{n-1})$ can be expressed by the S_{Ar}-formula

$$z_0 \equiv x_0 + x_1 \wedge \ldots \wedge z_{n-1} \equiv x_{n-1} + y_{n-1}.$$

The *linear independence* of $(x_0, \ldots, x_{n-1})$ and $(y_0, \ldots, y_{n-1})$ is formalizable by

$$\forall\lambda\forall\mu((\bigwedge_{i=0}^{n-1} \lambda \cdot x_i + \mu \cdot y_i \equiv 0) \to (\lambda \equiv 0 \wedge \mu \equiv 0)).$$

Analytic geometry provides means to translate geometric statements into field theory.

18.1 The characteristic of a field

We study some logical aspects of an important field invariant, namely its *characteristic.*

Definition 85. *A field* $\mathbb{K} = (\mathbb{K}, +, \cdot, 0, 1)$ *has* characteristic p*, if* p *is the minimal integer* >0 *such that*

$$\underbrace{1 + \ldots + 1}_{p-times} = 0.$$

If such a p *exists then* p *is a prime number. Otherwise the* characteristic *of* $\mathbb{K}$ *is defined to be* 0*.*

Fields of characteristic p can be axiomatized by

$$\Phi_{\mathrm{Fd},p} = \Phi_{\mathrm{Fd}} \cup \{\underbrace{1 + \ldots + 1}_{p-\text{times}} \equiv 0\},$$

and fields of characteristic 0 by

$$\Phi_{\mathrm{Fd},0} = \Phi_{\mathrm{Fd}} \cup \{\underbrace{1 + \ldots + 1}_{n-\text{times}} \not\equiv 0 | n \in \mathbb{N} \setminus \{0\}\}.$$

The axiom system $\Phi_{\mathrm{Fd},0}$ is infinite.

Theorem 86. *The class of fields of characteristic* 0 *cannot be finitely axiomatized.*

Proof. Assume for a contradiction that the sentence φ_0 axiomatizes the class under consideration. Then

$$\Phi_{\mathrm{Fd},0} \vDash \varphi_0 \text{ and } \{\varphi_0\} \vDash \Phi_{\mathrm{Fd},0}.$$

By the compactness theorem there is a finite $\Phi_0 \subseteq \Phi_{\mathrm{Fd},0}$ such that

$$\Phi_0 \vDash \varphi_0 \text{ and } \{\varphi_0\} \vDash \Phi_0.$$

Without loss of generality, Φ_0 is of the form

$$\Phi_0 = \Phi_{\mathrm{Fd}} \cup \{\underbrace{1 + \ldots + 1}_{n-\text{times}} \not\equiv 0 | n = 1, \ldots, n_0\}.$$

This set is equivalent to $\Phi_{\mathrm{Kp},0}$ and also axiomatizes the class of fields of characteristic 0. Take a prime number $p > n_0$. Then the field $\mathbb{K}_p$ of integers *modulo* p has characteristic p and $\mathbb{K}_p \vDash \Phi_0$. But then Φ_0 does *not* axiomatize the class of fields of characteristic 0. Contradiction. ☐

18.2 Algebraically closed fields

Definition 87. *A field* $\mathbb{K}$ *is* algebraically closed *if every polynomial of degree* $\geqslant 1$ *has a zero in* $\mathbb{K}$*.*

A polynomial

$$x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

is determined by the sequence $a_{n-1}, \ldots, a_0$ of coefficients. The following axiomatizes algebraically closed fields:

$$\Phi_{\mathrm{acf}} = \Phi_{\mathrm{Fd}} \cup \{\forall a_{n-1} \ldots \forall a_0 \exists x\, x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \equiv 0 | n \in \mathbb{N} \setminus \{0\}\}.$$

Here x^i denotes the term $\underbrace{x \cdot x \cdots x}_{i-\text{times}}$.

19 Dense linear orders

The structure $\mathbb{Q} = (\mathbb{Q}, <)$ is an example of a *dense linear order*.

Definition 88. *Let* $S_{\text{so}} = \{<\}$ *be the language of* strict orders. *The system* Φ_{slo} *axiomatizing* strict linear orders *consists of the sentences*

⟨rigid| − ⟩ $\forall x \neg x < x$

⟨rigid| − ⟩ $\forall x \forall y \forall z\, (x < y \wedge y < z \rightarrow x < z)$

⟨rigid| − ⟩ $\forall x \forall y\, (x < y \vee x = y \vee y < x)$

The system Φ_{dlo} *axiomatizing* dense linear orders (without endpoints) *consists of* Φ_{slo} *and*

⟨rigid| − ⟩ $\forall x \exists y\, x < y$

⟨rigid| − ⟩ $\forall x \exists y\, y < x$

⟨rigid| − ⟩ $\forall x \forall y\, (x < y \rightarrow \exists z\, (x < z \wedge z < y))$

The following theorem was shown by GEORG CANTOR.

Theorem 89. *Let* $X = (X, <^X)$ *and* $Y = (Y, <^Y)$ *be countable dense linear orders. Then* X *and* Y *are isomorphic.*

Proof. Let $X = \{x_i | i \in \omega\}$ and $Y = \{y_j | j \in \omega\}$. Define a sequence $(f_n | n \in \omega)$ of maps $f_n: X_n \rightarrow Y_n$ such that
(1) $X_n \subseteq X$ and $Y_n \subseteq Y$ have cardinality n;
(2) $f_n: (X_n, <^X \cap X_n^2) \rightarrow (Y_n, <^Y \cap Y_n^2)$ is an isomorphism.

Set $f_0 = X_0 = Y_0 = \emptyset$.

Assume that f_{2n} is constructed according to (1) and (2). Let

$$X_{2n} = \{u_0, \ldots, u_{2n-1}\} \text{ with } u_0 <^X u_1 <^X \ldots <^X u_{2n-1}$$

and

$$Y_{2n} = \{v_0, \ldots, v_{2n-1}\} \text{ with } v_0 <^Y v_1 <^Y \ldots <^Y v_{2n-1}.$$

Take $i \in \omega$ minimal such that $x_i \notin X_{2n}$.
Case 1: $x_i <^X u_0$. Then take $j \in \omega$ minimal such that $y_j <^Y v_0$.
Case 2: $u_0 <^X x_i <^X u_{2n-1}$. Take $k < 2n - 1$ such that $u_k <^X x_i <^X u_{k+1}$. Take $j \in \omega$ minimal such that $v_k <^Y y_j <^Y v_{k+1}$.
Case 3: $u_{2n-1} <^X x_i$. Take $j \in \omega$ minimal such that $v_{2n-1} <^Y y_j$.

In all three cases set

$$X_{2n+1} = X_{2n} \cup \{x_i\}, Y_{2n+1} = Y_{2n} \cup \{y_j\}, f_{2n+1} = f_{2n} \cup \{(x_i, y_j)\}.$$

Then f_{2n+1} is constructed according to (1) and (2).

Now let

$$X_{2n+1} = \{u_0, \ldots, u_{2n}\} \text{ with } u_0 <^X u_1 <^X \ldots <^X u_{2n}$$

and

$$Y_{2n+1} = \{v_0, \ldots, v_{2n}\} \text{ with } v_0 <^Y v_1 <^Y \ldots <^Y v_{2n}.$$

Take $j \in \omega$ minimal such that $y_j \notin Y_{2n+1}$.
Case 1': $y_j <^Y v_0$. Then take $i \in \omega$ minimal such that $x_i <^X u_0$.
Case 2': $v_0 <^Y y_j <^Y v_{2n}$. Take $k < 2n$ such that $v_k <^Y y_j <^Y v_{k+1}$. Take $i \in \omega$ minimal such that $u_k <^X x_i <^X u_{k+1}$.
Case 3': $v_{2n} <^Y y_j$. Take $i \in \omega$ minimal such that $u_{2n} <^X x_i$.

In all three cases set

$$X_{2n+2} = X_{2n+1} \cup \{x_i\}, Y_{2n+2} = Y_{2n+1} \cup \{y_j\}, f_{2n+2} = f_{2n+1} \cup \{(x_i, y_j)\}.$$

Then f_{2n+2} is constructed according to (1) and (2).

Obviously, $f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots$. Let $f = \bigcup_{n \in \omega} f_n$. Then

$$f \colon (X, <^X) \cong (Y, <^Y). \qquad \square$$

We draw some logical consequences from this isomorphism result.

Definition 90. *Let S be a language. An S-*theory *is a consistent set $\Phi \subseteq L_0^S$ of sentences. A set $\Phi \subseteq L_0^S$ is* complete *if for every $\varphi \in L_0^S$*

$$\Phi \vdash \varphi \textit{ gdw. } \Phi \nvdash \neg\varphi.$$

A complete theory $\Phi \subseteq L_0^S$ "decides" all "questions" which can be posed in the language S. The theories Φ_{Gr} and Φ_{Fd} are not complete. Obviously:

Proposition 91. *Let $\mathfrak{A}$ be an S-structure. Let*

$$\mathrm{Th}(\mathfrak{A}) = \{\varphi \in L_0^S \,|\, \mathfrak{A} \vDash \varphi\}$$

be the theory *of $\mathfrak{A}$. Then $\mathrm{Th}(\mathfrak{A})$ is complete.*

Definition 92. *Let S be a language and $\Phi \subseteq L_0^S$. Then Φ is* ω-categorical, *if all countably infinite structures $\mathfrak{A} \vDash \Phi$ and $\mathfrak{B} \vDash \Phi$ are isomorphic.*

Theorem 93. *Let S be a countable language and let $\Phi \subseteq L_0^S$ be a consistent ω-categorical set of sentences which has no finite models. Then Φ is complete.*

Proof. Let $\varphi \in L_0^S$. Assume $\Phi \vdash \varphi$. Then $\Phi \nvdash \neg\varphi$ since Φ is consistent.

Conversely assume $\Phi \nvdash \neg\varphi$. Assume for a contradiction that $\Phi \nvdash \varphi$. Then $\Phi \cup \{\varphi\}$ und $\Phi \cup \{\neg\varphi\}$ are consistent. By the LÖWENHEIM-SKOLEM theorem 69 there are countable models $\mathfrak{A}_0 \vDash \Phi \cup \{\varphi\}$ and $\mathfrak{A}_1 \vDash \Phi \cup \{\neg\varphi\}$. Since Φ has not finite models, $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are both countably infinite. By ω-categoricity, $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are isomorphic. But $\mathfrak{A}_0 \vDash \varphi$ and $\mathfrak{A}_1 \vDash \neg\varphi$. Contradiction. $\square$

As an immediate corollary of the previous theorems we obtain:

Theorem 94. *The theory Φ_{dlo} is complete.*

By a main theorem of algebra an algebraically closed field is determined by its characteristic and its *transcendence degree* up to isomorphism. Given an appropriate theory of uncountable cardinalities this implies that two algebraically closed fields of characteristic 0 and of the same *uncountable* cardinality are isomorphic. By arguments similar to the countable case one can show:

Theorem 95. *The theory of algebraically closed fields of characteristic 0 is complete.*

20 Peano arithmetic

The language of arithmetic can also be interpreted in the structure $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ of integers. We formulate a theory which attempts to describe this structure.

Definition 96. *The axiom system $\mathrm{PA} \subseteq L^{S_{\mathrm{AR}}}$ of* PEANO arithmetic *consists of the following sentences*

$\langle$rigid$|-\rangle$ $\forall x\, x + 1 \neq 0$

$\langle$rigid$|-\rangle$ $\forall x \forall y\, x + 1 = y + 1 \rightarrow x = y$

$\langle$rigid$|-\rangle$ $\forall x\, x + 0 = x$

$\langle$rigid$|-\rangle$ $\forall x \forall y\, x + (y + 1) = (x + y) + 1$

$\langle$rigid$|-\rangle$ $\forall x\, x \cdot 0 = 0$

$\langle \text{rigid} | - \rangle$ $\forall x \forall y \, x \cdot (y+1) = x \cdot y + x$

$\langle \text{rigid} | - \rangle$ *Schema of induction: for every formula* $\varphi(x_0, \ldots, x_{n-1}, x_n) \in L^{S_{\mathrm{AR}}}$*:*

$$\forall x_0 \ldots \forall x_{n-1} (\varphi(x_0, \ldots, x_{n-1}, 0) \wedge \forall x_n (\varphi \rightarrow \varphi(x_0, \ldots, x_{n-1}, x_n + 1)) \rightarrow \forall x_n \varphi)$$

Then $\mathbb{N} \vDash \mathrm{PA}$. The first incompleteness theorem of Gödel shows that PA is *not* complete, i.e., there are arithmetic sentences which are not decided by PA although in the standard model they have to be either true or false, and they really are true if one is working in a meta-theory which is able to construct the model $\mathbb{N}$.

21 Nonstandard analysis

Analysis was developed using *infinitesimal* numbers. Although infinitesimals in most cases lead to correct results, they are nevertheless paradoxical object (arbitrarily small but not equal to 0) which gave rise to severe foundational controversies.

The following is a caricature of the use of infinitesimals: To determine the derivation of $f = x^2$ in a take an infinitesimal ε and form the difference quotient

$$\frac{(a+\varepsilon)^2 - a^2}{\varepsilon} = \frac{a^2 + 2a\varepsilon + \varepsilon^2 - a^2}{\varepsilon} = 2a + \varepsilon \,.$$

Setting $\varepsilon = 0$, after all, we obtain

$$f'(a) = 2a \,.$$

It is difficult to account for this recipe in terms of a single structure. It seems that there is a structure of *standard* numbers like $0, 2, a, \ldots$ in which we want to know the result of the argument. For the argument, however, one seems to enrich the domain by *nonstandard* numbers like ε, $a + \varepsilon, \ldots$. The nonstandard numbers are then projected back into the standard numbers.

This idea was put on firm foundations by Abraham Robinson, the inventor of *nonstandard analysis*. We give a small impression of this field, emphasizing logical aspects. We extend the structure $\mathbb{R}$ of standard reals to a structure $\mathbb{R}^*$ which also contains “infinitesimals”. There is a partial map $\mathrm{st}\colon \mathbb{R} \rightharpoonup \mathbb{R}^*$ which maps an infinitesimal ε to 0.

So let

$$\mathbb{R} = (\mathbb{R}, <, +, \cdot, (r | r \in \mathbb{R}), f, g)$$

be the standard strictly ordered field of reals enriched by constants r for every $r \in \mathbb{R}$ and by unary functions f and g. Let S be an appropriate symbol set for this structure. For simplicity we identify the symbols with their interpretation in $\mathbb{R}$. Let

$$T = \mathrm{Th}(\mathbb{R}) = \{\varphi \in L_0^S | \mathbb{R} \vDash \varphi\}$$

be the theory of $\mathbb{R}$. Let ε be a new constant symbol (for an infinitesimal) and $S^* = S \cup \{\varepsilon\}$. The set

$$T^* = T \cup \{0 < \varepsilon \wedge \varepsilon < r | r \in \mathbb{R} \wedge 0 < r\}$$

of S^*-sentences expresses that ε lies between 0 and all positive standard reals, i.e., that ε is an infinitesimal. Every finite subset $T' \subseteq T^*$ can be satisfied by the structure

$$\mathbb{R}' = (\mathbb{R}, <, +, \cdot, (r | r \in \mathbb{R}), f, g, e)$$

where ε is interpreted by a positive real number e which is smaller than the finitely many positive reals r such that r occurs in the finite set T'. Hence T^* is consistent and satisfiable, and we let

$$(\mathbb{R}^*, <^*, +^*, \cdot^*, (r^* | r \in \mathbb{R}), f^*, g^*, \varepsilon^*) \vDash T^*$$

where ε^* interprets ε . Restrict that structure to the language S to obtain

$$\mathbb{R}^* = (\mathbb{R}^*, <^*, +^*, \cdot^*, (r^* | r \in \mathbb{R}), f^*, g^*) \vDash T \,.$$

Embed $\mathbb{R}$ into $\mathbb{R}^*$ by

$$r \mapsto r^* .$$

Since the theory T contains all first-order information about all elements of $\mathbb{R}$ we get that the embedding is elementary. Via the embedding, we can identify r and r^* for $r \in \mathbb{R}$. Moreover, the relations and functions of $\mathbb{R}^*$ are extension of the corresponding functions in $\mathbb{R}$. We may thus denote the components of $\mathbb{R}^*$ just like the components of $\mathbb{R}$:

$$\mathbb{R}^* = (\mathbb{R}^*, <, +, \cdot, (r | r \in \mathbb{R}), f, g).$$

After the identification we get

Proposition 97. *$\mathbb{R}$ is a proper elementary substructure of $\mathbb{R}^*$: $\mathbb{R} \prec \mathbb{R}^*$.*

Proof. Since $0 < \varepsilon < r$ for every positive $r \in \mathbb{R}$ we have $\varepsilon \notin \mathbb{R}$ and $\mathbb{R} \neq \mathbb{R}^*$. □

We now connect the structure $\mathbb{R}^*$ back to $\mathbb{R}$:

Definition 98.

a) $u \in \mathbb{R}^*$ *is* finite *if there are* $a, b \in \mathbb{R}$ *such that* $a < u < b$.

b) $u \in \mathbb{R}^*$ *is* infinite *if* u *is not finite.*

c) For finite $u \in \mathbb{R}^*$ *define the* standard part

$$\mathrm{st}(u) = \sup_{\mathbb{R}} \{r \in \mathbb{R} | r < u\}$$

as a supremum in the standard numbers. Note that $\mathrm{st}: \mathbb{R}^* \rightharpoonup \mathbb{R}$ *is a partial function defined on the finite elements of* $\mathbb{R}^*$.

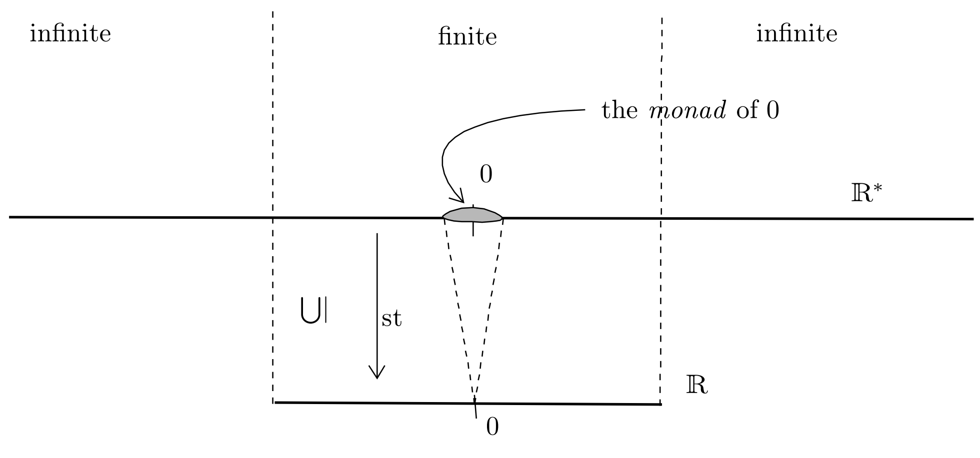
d) $u \in \mathbb{R}^*$ *is* infinitesimal *if* $\mathrm{st}(u) = 0$.

e) $u, v \in \mathbb{R}^*$ *are* infinitesimally near, $u \sim v$, *if* $u - v$ *is infinitesimal.*

Note that by the inequalities $0 < \varepsilon \wedge \varepsilon < r \in T^*$

$$\mathrm{st}(\varepsilon^*) = \sup_{\mathbb{R}} \{r \in \mathbb{R} | r < \varepsilon^*\} = \sup_{\mathbb{R}} \{r \in \mathbb{R} | r \leqslant 0\} = 0.$$

So $\mathbb{R}^*$ possesses an infinitesimal element $\neq 0$. The two models may be represented graphically by



Proposition 99.

a) If $s \in \mathbb{R}$ *then* $\mathrm{st}(s) = s$.

b) $u \in \mathbb{R}^*$ *is infinitesimal iff* $\forall s \in \mathbb{R} (s > 0 \to |u| < s)$.

c) If $u \sim 0$ *and* $|v| < |u|$ *then* $v \sim 0$.

d) Let $u \sim u'$ and $v \sim v'$. Then $u + v \sim u' + v'$.

e) Let $u \sim u'$, $v \sim v'$, and u, v be finite. Then $u \cdot v \sim u' \cdot v'$.

Proof. a) $\mathrm{st}(s) = \sup_{\mathbb{R}} \{r \in \mathbb{R} | r < s\} = s$.
b) Let $\mathrm{st}(u) = 0$. Let $s \in \mathbb{R}$, $s > 0$. Assume for a contradiction that $-s \geqslant u$. Then

$$\mathrm{st}(u) \leqslant \mathrm{st}(-s) = -s < 0,$$

contradiction. Assume for a contradiction that $u \geqslant s$. Then

$$\mathrm{st}(u) \geqslant \mathrm{st}(s) = s > 0,$$

contradiction. Thus $-s < u < s$, i.e., $|u| < s$.
c) follows immediately from b).
d) Let $s \in \mathbb{R}$, $s > 0$. By assumption, $|u - u'| < \frac{s}{2}$ and $|v - v'| < \frac{s}{2}$. Then

$$|(u + v) - (u' + v')| = |(u - u') + (v - v')| \leqslant |u - u'| + |v - v'| < \frac{s}{2} + \frac{s}{2} = s.$$

By b), $u + v \sim u' + v'$.
e) Choose $a \in \mathbb{R}$ such that $|u|, |v|, |u'|, |v'| < a$. Let $s \in \mathbb{R}$, $s > 0$. By assumption, $|u - u'| < \frac{s}{2a}$ and $|v - v'| < \frac{s}{2a}$. Then

$$\begin{aligned} |u \cdot v - u' \cdot v'| &= |u \cdot v - u \cdot v' + u \cdot v' - u' \cdot v'| \\ &\leqslant |u \cdot v - u \cdot v'| + |u \cdot v' - u' \cdot v'| \\ &= |u| \cdot |v - v'| + |u - u'| \cdot |v'| \\ &\leqslant a \cdot \frac{s}{2a} + a \cdot \frac{s}{2a} = s. \end{aligned}$$

By b), $u \cdot v \sim u' \cdot v'$. □

To demonstrate the potential of the standard-nonstandard setup we give a nonstandard characterization of when the function $g: \mathbb{R} \to \mathbb{R}$ is the derivative f' of the function f.

Theorem 100. *$g = f'$ iff the following criterion holds:*

$$\forall x \in \mathbb{R} \forall \xi \in \mathbb{R}^* \setminus \{0\} (\xi \sim 0 \to g(x) \sim \frac{f(x + \xi) - f(\xi)}{\xi}).$$

Proof. To deal with difference quotients we use the common absolute value notation

$$\left| a - \frac{b - c}{d} \right| < e.$$

This abbreviates the formula

$$(d > 0 \to -de < da - b + c \wedge da - b + c < de) \wedge (d < 0 \to de < da - b + c \wedge da - b + c < -de)$$

where we assume $d \neq 0$.

Assume $g = f'$. Let $x \in \mathbb{R}$, $\delta \in \mathbb{R}^* \setminus \{0\}$, and $\delta \sim 0$. To check whether $g(x) \sim \frac{f(x + \xi) - f(\xi)}{\xi}$ let $\eta \in \mathbb{R}$, $\eta > 0$. Since $g(x) = f'(x)$ there exists $\delta \in \mathbb{R}$, $\delta > 0$ such that

$$\mathbb{R} \vDash \forall \delta' \neq 0 (|\delta'| < \delta \to \left| g(x) - \frac{f(x + \delta') - f(\delta')}{\delta'} \right| < \eta).$$

This S-sentence is an element of the theory T, and therefore it also holds in $\mathbb{R}^*$:

$$\mathbb{R}^* \vDash \forall \delta' \neq 0 (|\delta'| < \delta \to \left| g(x) - \frac{f(x + \delta') - f(\delta')}{\delta'} \right| < \eta).$$

The process of going from $\mathbb{R}$ to $\mathbb{R}^*$ like this or vice versa is called *transfer*; it is one of the most important techniques of nonstandard analysis. We can set $\delta' = \xi$ and get

$$\left| g(x) - \frac{f(x + \xi) - f(\xi)}{\xi} \right| < \eta.$$

Since this holds for every positive $\eta \in \mathbb{R}$ we have

$$g(x) \sim \frac{f(x+\xi) - f(\xi)}{\xi}$$

as required.

Conversely assume that $g \neq f'$. Take $x \in \mathbb{R}$ such that $g(x) \neq f'(x)$. Then there is $\eta \in \mathbb{R}$, $\eta > 0$ such that

$$\mathbb{R} \vDash \forall \delta > 0 \exists \delta', \delta' \neq 0 |\delta'| < \delta \left| g(x) - \frac{f(x+\delta') - f(\delta')}{\delta'} \right| \geqslant \eta.$$

We transfer this property to $\mathbb{R}^*$:

$$\mathbb{R}^* \vDash \forall \delta > 0 \exists \delta', \delta' \neq 0 |\delta'| < \delta \left| g(x) - \frac{f(x+\delta') - f(\delta')}{\delta'} \right| \geqslant \eta.$$

Take some positive *infinitesimal* $\delta \in \mathbb{R}^*$, $\delta > 0$ and apply the last property: there exists $\xi \in \mathbb{R}^* \setminus \{0\}$, $|\xi| < \delta$ such that

$$\left| g(x) - \frac{f(x+\xi) - f(\xi)}{\xi} \right| \geqslant \eta.$$

Since $|\xi| < \delta$ we have that $\xi \sim 0$. Hence

$$g(x) \nsim \frac{f(x+\xi) - f(\xi)}{\xi}.$$

This shows that the criterion is false in case $g \neq f'$. □

The nonstandard criterion for the derivation can be applied in proving the usual laws of the differential calculus. As an example we show the product rule.

Theorem 101. *Let $f, g: \mathbb{R} \to \mathbb{R}$ be differentiable functions. Then the product $f \cdot g$ is differentiable and*

$$(f \cdot g)' = f' \cdot g + f \cdot g'.$$

Proof. The criterion of the previous theorem is satisfied by f', f and g', g respectively. We now show the criterion for $f' \cdot g + f \cdot g'$ and $f \cdot g$. Let $x \in \mathbb{R}$ and $\xi \in \mathbb{R}^* \setminus \{0\}$, $\xi \sim 0$. Calculate in $\mathbb{R}^*$:

$$\begin{aligned}
\frac{(f \cdot g)(x+\xi) - (f \cdot g)(x)}{\xi} &= \frac{f(x+\xi) \cdot g(x+\xi) - f(x) \cdot g(x)}{\xi} \\
&= \frac{f(x+\xi) \cdot g(x+\xi) - f(x) \cdot g(x+\xi) + f(x) \cdot g(x+\xi) - f(x) \cdot g(x)}{\xi} \\
&= \frac{f(x+\xi) - f(x)}{\xi} \cdot g(x+\xi) + f(x) \cdot \frac{g(x+\xi) - g(x)}{\xi}.
\end{aligned}$$

By assumption, $\frac{f(x+\xi) - f(x)}{\xi} \sim f'(x)$ and $\frac{g(x+\xi) - g(x)}{\xi} \sim g'(x)$. The latter near-equality also implies $g(x+\xi) \sim g(x)$. Since $\sim$ commutes with arithmetic operations,

$$\begin{aligned}
\frac{(f \cdot g)(x+\xi) - (f \cdot g)(x)}{\xi} &= \frac{f(x+\xi) - f(x)}{\xi} \cdot g(x+\xi) + f(x) \cdot \frac{g(x+\xi) - g(x)}{\xi} \\
&\sim f'(x) \cdot g(x) + f(x) \cdot g'(x)
\end{aligned}$$

as required. □

The treatment of differentiation has demonstrated that the nonstandard theory allows different argumentations from the standard theory. The relation $\sim$ of nearness allows to to dispense with some explicit calculations of inequalities. Of course the basic laws of the $\sim$-relation were proved using explicit estimates. The use of infinitesimals also seems to eliminate some quantifiers: some familiar properties of the form $\forall \varepsilon \exists \delta \ldots$ can be replaced by properties of the form $\forall \xi \sim 0 \ldots$.

On the other side, one has to be caefully distinguish whether one is working in the standard model or the nonstandard extension. Particular combinations of standard and nonstandard variables are often crucial. A function $f: \mathbb{R} \to \mathbb{R}$ is continuous iff

$$\forall x \in \mathbb{R} \forall x' \in \mathbb{R}^* (x \sim x' \to f(x) \sim f(x')).$$

The similar looking property

$$\forall x \in \mathbb{R}^* \forall x' \in \mathbb{R}^* (x \sim x' \to f(x) \sim f(x'))$$

where both variables range over $\mathbb{R}^*$ is much more restrictive and describes some class of "strongly continuous" functions.

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