THE AUTOMORPHISM TOWER PROBLEM

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Let G be a group with trivial centre. By identifying the elements of G with the corresponding *inner automorphisms* of G, we may view G as a normal subgroup of the group Aut(G) consisting of all automorphisms of G. An easy argument shows that Aut(G) is also a group with trivial centre and we can iterate this process to construct the *automorphism tower* $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ *of* G by setting $G_0 = G$, $G_{\alpha+1} = \mathrm{Aut}(\mathrm{G}_\alpha)$ and $G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$ for every limit ordinal λ. We say that the automorphism tower of a centreless group G *terminates* in α steps if $G_\alpha = G_\beta$ holds for all $\beta \geq \alpha$, i.e. G_α is a *complete* group.

In 1939 Helmut Wielandt showed that the automorphism tower of every finite centreless group terminates in finitely many steps. This result was generalized to infinite groups by Simon Thomas' *automorphism tower theorem*: *if* G *is an infinite centreless group of cardinality* κ*, then the automorphism tower of* G *terminates in less than* $(2^\kappa)^+$*-many steps.* These results imply that for every centreless group G there is a least ordinal $\tau(G)$ with $G_{\tau(G)} = G_{\tau(G)+1}$. Given an infinite cardinal κ, we define

$$\tau_\kappa = \mathrm{lub}\{\tau(G) \mid G \textit{ is a centreless group of cardinality } \kappa\}.$$

The following problem is still open.

Problem (The automorphism tower problem)**.** *Construct a model of set theory* $\mathcal{M}$ *such that it is possible to* compute *the exact value of* τ_κ *in* $\mathcal{M}$ *for some infinite cardinal* κ *in* $\mathcal{M}$*.*

In my talk, I want to present an easy proof of Simon Thomas' *automorphism tower theorem* using methods developed by Itay Kaplan and Saharon Shelah. Following this, I will present some recent results on upper bounds for τ_κ, groups whose automorphism towers are highly malleable by forcing and automorphism towers of countable groups. The proofs of these results combine concepts and results from group theory, set theory, classical descriptive set theory, and abstract recursion theory.