Local rigidity, Glimm-Effros embeddings, and definable cardinals

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Part I

Introduction

The classical notion of cardinality determines whether X is smaller than Y by asking if there is an injection of X into Y.

Much recent work has focused on the refinement of this notion in which one requires that the injection is suitably definable.

Example 1

There is no classification of subgroups of $\mathbb{Q}$ using elements of $\mathbb{N}$ as invariants, since there are $|\mathbb{R}|$-many isomorphism classes.

Theorem 2 (Folklore?)

There is no classification of subgroups of $\mathbb{Q}$ using elements of $\mathbb{R}$ as invariants, since the definable cardinality of the set of isomorphism classes is strictly greater than the definable cardinality of $\mathbb{R}$.

I. Introduction
Motivating examples

Example 3

There is no classification of ergodic, Lebesgue measure-preserving transformations using elements of $\mathbb{N}$ as invariants, since there are $|\mathbb{R}|$-many isomorphism classes.

Theorem 4 (Hjorth)

There is no classification of ergodic, Lebesgue measure-preserving transformations using isomorphism classes of countable groups as invariants, since the definable cardinality of the set of isomorphism classes is again too large.

Notation

Suppose X and Y are standard Borel spaces.

Notation

Suppose E and F are Borel equivalence relations on X and Y.

Definition

A homomorphism from E to F is a function $\varphi\colon X \to Y$ sending E-equivalent points to F-equivalent points.

A homomorphism is a map factoring over the quotients.

Definition

A reduction of E to F is a homomorphism from E to F sending E-inequivalent points to F-inequivalent points.

A reduction is a map factoring to an injection over the quotients.

Suppose that $\mathscr{B}$ is a class of subsets of standard Borel spaces.

Notation

We write $E \leq_{\mathscr{B}} F$ if there is a $\mathscr{B}$-measurable reduction of E to F.

Notation

When this is the case, we say also that the $\mathscr{B}$-cardinality of X/E is at most that of Y/F, or simply that $|X/E|_{\mathscr{B}} \leq |Y/F|_{\mathscr{B}}$.

I. Introduction

General comments

Borel cardinality is the standard notion of definable cardinality.

It is fine enough to detect many anti-classification results.

It avoids metamathematical difficulties inherent in broader notions.

It often coincides with other natural notions of definable cardinality.

It has deep connections with many other areas of mathematics.

Part II

Background

The sequence $|0|, |1|, \ldots, |\mathbb{N}|$ is an initial segment of the cardinals.

And $|0|_B, |1|_B, \ldots, |\mathbb{N}|_B$ is an initial segment of the Borel cardinals.

The continuum hypothesis (Cantor)

The successor of $|\mathbb{N}|$ is $|\mathbb{R}|$.

Theorem 5 (Cohen, Gödel)

The continuum hypothesis is independent of the standard axioms.

II. Background
An initial segment of the Borel cardinals

Theorem 6 (Silver)

The successor of $|\mathbb{N}|_B$ is $|\mathbb{R}|_B$.

Note that $|X/E|$ is always at most $|\mathbb{R}|$.

But the Vitali argument can be used to show that $|\mathbb{R}|_B < |\mathbb{R}/\mathbb{Q}|_B$.

Beginning in the 1960s with work of Effros and Glimm in operator algebras, the successor of $|\mathbb{R}|_B$ was eventually identified.

Theorem 7 (Harrington-Kechris-Louveau)

The successor of $|\mathbb{R}|_B$ is $|\mathbb{R}/\mathbb{Q}|_B$.

Definition

An equivalence relation is finite if its classes are finite.

Definition

An equivalence relation is countable if its classes are countable.

II. Background
An initial segment of the Borel cardinals

Definition

We say that E is hyperfinite if there is an increasing sequence $(E_n)_{n\in\mathbb{N}}$ of finite Borel equivalence relations whose union is E.

Theorem 8 (Dougherty-Jackson-Kechris)

If E is countable, then E is hyperfinite iff $|X/E|_B \leq |\mathbb{R}/\mathbb{Q}|_B$.

Although Borel reducibility of hyperfinite equivalence relations is understood, many questions concerning hyperfiniteness remain open.

Question

Suppose that $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of hyperfinite equivalence relations. Is $\bigcup_{n\in\mathbb{N}} E_n$ hyperfinite?

Question

Suppose that $\Gamma \curvearrowright X$ is a Borel action of an amenable countable group. Is E_Γ^X hyperfinite?

Hyperfiniteness is a Borel analog of a measure-theoretic notion.

Suppose that μ is a Borel probability measure on X.

Definition

We say that E is μ-hyperfinite if there is a μ-conull Borel set $C \subseteq X$ with the property that $E \upharpoonright C$ is hyperfinite.

The measure-theoretic analogs of many open questions concerning hyperfiniteness have well-known solutions.

Theorem 9 (Dye-Krieger)

Suppose that $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of μ-hyperfinite equivalence relations. Then $\bigcup_{n \in \mathbb{N}} E_n$ is μ-hyperfinite.

Theorem 10 (Ornstein-Weiss)

Suppose that $\Gamma \curvearrowright X$ is a Borel action of an amenable countable group. Then E_Γ^X is μ-hyperfinite.

There is a useful middle ground between these notions.

Definition

We say that E is measure hyperfinite if it is μ-hyperfinite for every Borel probability measure μ on X.

Proposition 11 (Kechris)

Under $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$, a countable Borel equivalence relation E is measure hyperfinite iff $|X/E|_{UM} \leq |\mathbb{R}/\mathbb{Q}|_{UM}$.

Corollary 12

Suppose that $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of measure hyperfinite equivalence relations. Then $\bigcup_{n\in\mathbb{N}} E_n$ is measure hyperfinite.

Corollary 13

Suppose that $\Gamma \curvearrowright X$ is a Borel action of an amenable countable group. Then E_Γ^X is measure hyperfinite.

Intuitively, an equivalence relation is measure hyperfinite if it cannot be proven to be non-hyperfinite via measure-theoretic techniques.

Question

Are measure hyperfinite equivalence relations always hyperfinite?

A positive answer to this question would essentially answer all remaining open questions concerning hyperfiniteness.

In order to see how one might obtain such an answer, it is instructive to return to an earlier result, albeit in a somewhat different guise.

Definition

We say that E is smooth if $|X/E|_B \leq |\mathbb{R}|_B$.

Definition

We say that E is μ-smooth if there is a μ-conull Borel set $B \subseteq X$ with the property that $E \upharpoonright B$ is smooth.

Definition

We say that E is measure smooth if it is μ-smooth for every Borel probability measure μ on X.

Corollary 14 (Harrington-Kechris-Louveau)

An equivalence relation is smooth iff it is measure smooth.

Proof

We will show that if E is not smooth, then it is not measure smooth.

If E is not smooth, then $|\mathbb{R}/\mathbb{Q}|_B \leq |X/E|_B$.

So there is a Borel reduction $\varphi\colon \mathbb{R} \to X$ of $E_{\mathbb{Q}}^{\mathbb{R}}$ to E.

Proof of Corollary 14 (continued)

Let m denote the Lebesgue measure on $\mathbb{R}$.

Let μ denote the pushforward of m through φ.

Suppose, towards a contradiction, that E is measure smooth.

Then E is μ-smooth.

Proof of Corollary 14 (continued)

Fix a μ-conull Borel set $C \subseteq X$ for which $E \restriction C$ is smooth.

Then $\varphi^{-1}(C)$ is an m-conull Borel set and $E_{\mathbb{Q}}^{\mathbb{R}} \restriction \varphi^{-1}(C)$ is smooth.

But the Vitali argument shows that this is impossible. ⊠

Corollary 15

Under $\texttt{add}(\texttt{null}) = \mathfrak{c}$, it follows that the notions of Borel, universally measurable, and $\aleph_0$-universally Baire reducibility agree up to $\mathbb{R}$.

These notions coincide because there is a one-element basis for the class of equivalence relations failing to satisfy the stronger notion whose unique element fails to satisfy the weaker notions.

Theorem 16 (Hjorth-Kechris)

Suppose that E is countable. Then $|X/E|_{BP} \leq |\mathbb{R}/\mathbb{Q}|_{BP}$.

Question

Is there a basis for the class of non-hyperfinite equivalence relations consisting of non-measure-hyperfinite equivalence relations?

A positive answer to this question would provide an analog of the Glimm-Effros dichotomy characterizing when $|X/E|_B \not\leq |\mathbb{R}/\mathbb{Q}|_B$.

Question

Is there a Glimm-Effros-style dichotomy characterizing when E is not measure hyperfinite, or equivalently, when $|X/E|_{UM} \not\leq |\mathbb{R}/\mathbb{Q}|_{UM}$?

We have thus far focused on the base of Borel reducibility hierarchy.

One would also like to understand it from a more global viewpoint.

Many results have been established suggesting that the properties of Borel cardinality differ wildly from those of the classical notion.

Here we will focus on the fact that they are not linearly ordered.

Theorem 17 (Louveau-Velickovic, Woodin)

There is an uncountable family of incomparable Borel cardinals.

The proof actually yields incomparable Baire property cardinals.

Unfortunately, this means that their techniques cannot be used to obtain incomparable countable Borel equivalence relations.

This leads to the question of whether analogous results can be established via measure-theoretic techniques.

An answer was eventually obtained using Zimmer superrigidity.

Theorem 18 (Adams-Kechris)

There is a family of continuum-many incomparable Borel cardinals associated with countable Borel equivalence relations.

Their argument produced a family of continuum-many incomparable Lebesgue-measurable cardinals.

II. Background
Non-linearity

Hjorth-Kechris eventually found a simpler proof of this result eliminating the need for Zimmer superrigidity.

However, both arguments relied critically on rigidity properties of actions of product groups.

This places severe restrictions on the extent to which their arguments can be pushed towards the base of the Borel reducibility hierarchy.

Definition

A Borel equivalence relation is treeable if its equivalence classes coincide with the connected components of an acyclic Borel graph.

Arguments of Adams, Hjorth, and Kechris rule out the treeability of orbit equivalence relations of product group actions.

For many years, the question of whether there are incomparable countable treeable Borel equivalence relations was open.

Shortly before his recent passing, Hjorth discovered the answer.

Theorem 19 (Hjorth)

There is a family of continuum-many incomparable Borel cardinals associated with countable treeable Borel equivalence relations.

The argument again produced a family of continuum-many incomparable Lebesgue-measurable cardinals.

While not literally at the base of the reducibility hierarchy, the countable treeable Borel equivalence relations are in some sense the simplest ones that appear naturally.

As a result, it is not immediately clear what it would mean to push the incomparability results further towards the base of the Borel reducibility hierarchy.

Question

Suppose that $\mathscr{B}$ is a basis for the family of countable treeable Borel equivalence relations E for which $|\mathbb{R}/\mathbb{Q}|_B < |X/E|_B$. What can be said about the structure of Borel reducibility on $\mathscr{B}$?

In the rest of this lecture, we discuss some recent progress towards answering this question.

Part III

Local rigidity

Definition

Let $\mathrm{Hom}(E, \mu, F)$ be the set of μ-measurable homomorphisms φ.

Definition

Let $\mathrm{Hom}_{\mathbb{E}_0}(E, \mu, F)$ be the set of μ-hyperfinite-to-one such φ.

Definition

Given $\varphi, \psi \colon X \to Y$, let $D(\varphi, \psi)$ denote $\{x \in X \mid \varphi(x) \neq \psi(x)\}$.

Definition

Given a group Δ, a function $\rho \colon E \to \Delta$, and an action $\Delta \curvearrowright Y$, we say that a function $\varphi \colon X \to Y$ is ρ-invariant if

$$\forall (x, y) \in E \ \varphi(x) = \rho(x, y) \cdot \varphi(y).$$

Definition

We say that an action $\Delta \curvearrowright Y$ generating F is locally rigid if whenever X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho\colon E \to \Delta$ is Borel, and $\varphi, \psi \in \mathrm{Hom}_{\mathbb{E}_0}(E, \mu, F)$ are ρ-invariant, the relation $E \restriction D(\varphi, \psi)$ is μ-hyperfinite.

Definition

Let $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ denote the group of transformations $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the form $T(x) = Ax + b$, where $A \in \mathrm{SL}_2(\mathbb{Z})$ and $b \in \mathbb{Z}^2$.

Definition

Let $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ denote the corresponding action.

III. Local rigidity
An example

Definition

Let $\mathbb{T}$ be the space of infinite rays through $\mathbb{R}^2$ rooted at the origin.

Definition

Let $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ denote the action induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$.

Definition

Let $\mathbb{T}^2$ denote $\mathbb{R}^2/\mathbb{Z}^2$.

Definition

Let $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ denote the action induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$.

Theorem 20

The action $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ is locally rigid.

Proof

Let F denote the orbit equivalence relation of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \curvearrowright \mathbb{R}^2$.

Suppose X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to \Delta$ is Borel, μ is a Borel probability measure on X, $\varphi \in \mathrm{Hom}_{\mathbb{E}_0}(E, \mu, F)$, and $\psi \in \mathrm{Hom}(E, \mu, F)$.

Proof of Theorem 20 (continued)

Define $\pi \colon D(\varphi, \psi) \to \mathbb{T}$ by $\pi(x) = \mathrm{proj}_{\mathbb{T}}(\varphi(x) - \psi(x))$.

Define $\sigma \colon E \restriction D(\varphi, \psi) \to \mathrm{SL}_2(\mathbb{Z})$ by $\sigma(x, y) = \mathrm{proj}_{\mathrm{SL}_2(\mathbb{Z})} \circ \rho(x, y)$.

Then π is σ-invariant.

In particular, it is a homomorphism from $E \restriction D(\varphi, \psi)$ to $E^{\mathbb{T}}_{\mathrm{SL}_2(\mathbb{Z})}$.

Proof of Theorem 20 (continued)

Jackson-Kechris-Louveau have shown that $E^{\mathbb{T}}_{\mathrm{SL}_2(\mathbb{Z})}$ is hyperfinite.

And the existence of a μ-hyperfinite-to-one Borel homomorphism to a hyperfinite equivalence relation implies μ-hyperfiniteness.

So we must check that if $\theta \in \mathbb{T}$, then $E \restriction \pi^{-1}(\theta)$ is μ-hyperfinite.

Proof of Theorem 20 (continued)

Set $\Gamma = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(\theta)$.

Then Γ is cyclic, so $E_\Gamma^{\mathbb{T}^2}$ is hyperfinite by a result of Slaman-Steel.

Proof of Theorem 20 (continued)

Note that $\mathrm{proj}_{\mathbb{T}^2} \circ \varphi$ is σ-invariant.

And $\sigma[E \restriction \pi^{-1}(\theta)] \subseteq \Gamma$.

So φ is a homomorphism from $E \restriction \pi^{-1}(\theta)$ to $E_{\Gamma}^{\mathbb{T}^2}$.

Thus $E \restriction \pi^{-1}(\theta)$ is μ-hyperfinite. ⊠

Definition

Define d_μ on $\mathrm{Hom}(E, \mu, F)$ by $d_\mu(\varphi, \psi) = \mu(D(\varphi, \psi))$.

Definition

We say that F has separable homomorphisms if whenever X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is a Borel probability measure on X, at least one of the following holds:

- The equivalence relation E is μ-somewhere hyperfinite.
- The space $\mathrm{Hom}_{\mathbb{E}_0}(E, \mu, F)$ is separable.

Proposition 21

Suppose that Y and Y' are standard Borel spaces, F and F' are countable Borel equivalence relations on X and Y, F has separable homomorphisms, and there is a countable-to-one Borel homomorphism from F' to F. Then F' has separable homomorphisms.

Proof

The separabililty of $\mathrm{Hom}_{\mathbb{E}_0}(E, \mu, F)$ is equivalent to the existence of a Borel set $R \subseteq X \times Y$ with countable vertical sections such that $\mu(\{x \in X \mid (x, \varphi(x)) \in R\}) = 1$ for all $\varphi \in \mathrm{Hom}_{\mathbb{E}_0}(E, \mu, F)$.

But any such set can be pulled back to one for F'. ⊠

Theorem 22

Suppose that Y is a standard Borel space and F is a countable Borel equivalence relation on Y generated by a locally rigid Borel action $\Delta \curvearrowright Y$. Then F has separable homomorphisms.

Proof

Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is a Borel probability measure on X for which E is μ-nowhere hyperfinite.

Proof of Theorem 22 (continued)

Write $F = \bigcup_{n \in \mathbb{N}} R_n$, where $(R_n)_{n \in \mathbb{N}}$ is an increasing sequence of Borel sets whose vertical sections are of bounded finite cardinality.

Fix countable families $\mathscr{F}_n$ of functions $\rho \colon R_n \to \Delta$ dense in the space of μ-measurable functions $\sigma \colon R_n \to \Delta$ equipped with d_μ.

For all $n \in \mathbb{N}$, $\rho \in \mathscr{F}_n$, and rational $\epsilon > 0$ for which it is possible, fix σ in the ϵ-ball of ρ and a σ-invariant function $\varphi \in \mathrm{Hom}_{\mathbb{E}_0}(E, \mu, F)$.

Then the set of such φ is dense in $\mathrm{Hom}_{\mathbb{E}_0}(E, \mu, F)$. ⊠

Part IV

Applications

Theorem 23

Suppose that X is a standard Borel space, E is a countable treeable Borel equivalence relation on X with separable homomorphisms, and μ is an E-invariant Borel probability measure on X. Then exactly one of the following holds:

1. The equivalence relation E is μ-hyperfinite.
2. There are continuum-many pairwise incomparable μ-cardinals associated with Borel subequivalence relations of E.

Proof

We can assume that E is μ-nowhere hyperfintie.

Fix a sequence $(F_r)_{r \in \mathbb{R}}$ of Borel subequivalence relations of E with the property that $B \cap [x]_{F_r} \subset B \cap [x]_{F_s}$ for μ-almost every $x \in X$ whenever $B \subseteq X$ is μ-positive and $r < s$.

Ensure, moreover, that $F = \bigcap_{r \in \mathbb{R}} F_r$ is μ-nowhere hyperfinite.

Proof of Theorem 23 (continued)

Then for each $s \in \mathbb{R}$, the space $\mathrm{Hom}_{\mathbb{E}_0}(F, \mu, F_s)$ is separable.

So there are only countably many $r \in \mathbb{R}$ for which there is a μ-measurable reduction of F_r to F_s.

Thus the set of (r, s) for which there is such a reduction is meager.

Fix a perfect set on which there are no such pairs. ⊠

Remark 24

For each $n \in \{2, 3, \ldots, \aleph_0\}$, one can ensure that the equivalence relations are induced by Borel free actions $\mathbb{F}_n \curvearrowright X$.

Remark 25

One can similarly obtain families lying entirely within the countable-to-one Borel homomorphism class of E.

Remark 26

One can also mimic the arguments of Adams-Kechris to obtain analogs of their complexity results.

Definition

A cohomomorphism from a σ-ideal $\mathcal{I}$ on X to a σ-ideal $\mathcal{J}$ on Y is a function $\pi\colon X \to Y$ sending $\mathcal{I}$-positive sets to $\mathcal{J}$-positive sets.

Theorem 27

Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y, F has separable homomorphisms, μ is a Borel probability measure on X, and $\mathcal{I}$ is a σ-ideal on Y. Then one of the following holds:

- The equivalence relation E is somewhere μ-hyperfinite.
- There is an $\mathcal{I}$-conull Borel set $C \subseteq Y$ for which every μ-hyperfinite-to-one μ-measurable homomorphism from E to $F \upharpoonright C$ is a cohomomorphism from the μ-null ideal to $\mathcal{I} \upharpoonright C$.

Proof

By the separability of $\mathrm{Hom}_{\mathbb{E}_0}(E, \mu, F)$, there is a countable maximal pairwise disjoint family of Borel $\mathcal{I}$-null sets N for which there is a μ-hyperfinite-to-one μ-measurable homomorphism from E to $F \restriction N$.

Let C be the complement of the union of these sets. ⊠

Definition

We say that μ captures the complexity of E if for no μ-conull Borel set $C \subseteq X$ and μ-null Borel set $N \subseteq X$ is there μ-hyperfinite-to-one μ-measurable homomorphism from $E \upharpoonright C$ to $E \upharpoonright N$.

Proposition 28

Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is an E-ergodic Borel measure on X capturing the complexity of E. Then there is no $(\mu \times 2)$-measurable reduction from $E \times \Delta(2)$ to E.

Corollary 29

Suppose X is a standard Borel space, E is a countable Borel equivalence relation on X with separable homomorphisms, and μ is a Borel probability measure on X. Then exactly one of the following holds:

- The equivalence relation E is μ-hyperfinite.
- There is a μ-conull Borel set $C \subseteq Y$ such that $\mu \upharpoonright C$ captures the complexity of $E \upharpoonright C$.

Theorem 30

Suppose that $\mathscr{B}$ is a basis for the class of non-measure-hyperfinite treeable countable Borel equivalence relations on standard Borel spaces. Then $|\mathscr{B}| \geq \texttt{add}(\texttt{null})$.

Proof

Suppose, towards a contradiction, that $|\mathscr{B}| < \texttt{add}(\texttt{null})$.

Proof of Theorem 30 (continued)

Fix a set $\mathscr{E}$ of $\mathtt{add}(\mathtt{null})$-many non-$\mu$-hyperfinite treeable countable Borel equivalence relations which are pairwise incomparable under μ-measurable reducibility and with respect to which μ is ergodic.

By replacing each relation in $\mathscr{E}$ with its restriction to a μ-conull Borel set, we can assume that every Borel reduction of a relation in $\mathscr{B}$ to a relation in $\mathscr{E}$ has μ-positive image.

Then each relation in $\mathscr{B}$ is Borel reducible to at most one $E \in \mathscr{E}$.

So $\mathscr{B}$ is too small to be a basis for $\mathscr{E}$. ⊠