11. Problem set for "Models of set theory I", Summer 2011

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Problem 41 (Amoeba forcing)**.** Let μ denote Lebesgue measure on $\mathbb{R}$ and let $\epsilon > 0$. Let $\mathbb{P}_\epsilon$ consist of all open sets $A \subseteq \mathbb{R}$ with $\mu(A) < \epsilon$, and for $A, B \in \mathbb{P}_\epsilon$ let $A \leq B$ iff $A \supseteq B$. Show that $(\mathbb{P}_\epsilon, \leq)$ has the c.c.c. You may use the fact that for every set $A \in \mathbb{P}_\epsilon$ and every $\delta > 0$, there is a finite union C_A of open intervals with rational end points such that $\mu(A \triangle C_A) < \delta$.

Problem 42 (Lebesgue null sets)**.** Let μ denote Lebesgue measure on $\mathbb{R}$. Use the previous problem to show that MA_κ implies that the union of κ many Lebesgue null sets has measure 0. To do this, show that for every null set $A \subseteq \mathbb{R}$ the set $D_A := \{B \in \mathbb{P}_\epsilon : A \subseteq B\}$ is dense in $\mathbb{P}_\epsilon$.

Problem 43 (Martin's Axiom and absoluteness)**.** Note that the proof of Lemma 9.13 shows that MA_{ω_1} is equivalent to MA_{ω_1} for partial orders of size $\leq \omega_1$. Let $tc(x)$ denote the transitive closure of $\{x\}$, i.e. $tc(x) = Ext^\infty_\in(\{x\})$ as in section 2.2. Let $H_\kappa := \{x : |tc(x)| < \kappa\}$ denote the set of sets herditarily of size less than κ for cardinals κ. If $M \subseteq N$, let $M \prec_{\Sigma_1} N$ mean that for every Σ_1 formula φ and parameters $a_0, ..., a_n \in M$, $M \models \varphi(a_0, ..., a_n) \Leftrightarrow N \models \varphi(a_0, ..., a_n)$. Let M be a countable transitive model of ZFC. Show that if $H^M_{\omega_2} \prec_{\Sigma_1} H^{M[H]}_{\omega_2}$ holds for every forcing $\mathbb{P} \in M$ such that $M \models$ ($\mathbb{P}$ is a c.c.c. forcing) and for every $\mathbb{P}$-generic filter H over M, then MA_{ω_1} holds in M.

Problem 44 (Pseudo-intersections)**.** Suppose $F \subseteq \mathcal{P}(\omega)$ has the property that $\bigcap T$ is infinite for every finite $T \subseteq F$. Let $\mathbb{P}$ consist of all pairs (r, T) such that $r \in 2^{<\omega}$ and $T \subseteq F$ is finite. Let $(r, T) \leq (s, U)$ iff $s \subseteq r$, $U \subseteq T$, and for all $n \in dom(s) - dom(t)$, $s(n) = 1$ implies $n \in \bigcap U$. Use $(\mathbb{P}, \leq)$ to show that MA_{ω_1} implies that for every set F as above with $|F| \leq \omega_1$, there is an infinite set $x \subseteq \omega$ (a *pseudo-intersection* of F) such that $x - y$ is finite for all $y \in F$.

Please hand in your solutions on Wednesday, 29 June before the lecture.