8. Problem set for "Models of set theory I", Summer 2011

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Problem 29 (Strategic closure)**.** Suppose M is a countable transitive model of ZFC and $(\mathbb{P}, \leq)$ is a partial order. Consider the following two player game that lasts ω many rounds: Let $p_0 = q_0 = 1_{\mathbb{P}}$. In the n-th round the first player choses a condition $p_{n+1} \leq q_n$ and the second player replies by chosing a condition $q_{n+1} \leq p_{n+1}$. The second player wins the game iff there is a common extension $p \in \mathbb{P}$ of the p_n. Suppose that in M, the second player has a winning strategy for this game (i.e. $\mathbb{P}$ is ω-*strategically closed*). Show that for every G that is $\mathbb{P}$-generic over M, $M[G]$ does not contain any new function from ω into the ordinals.

Solution: Suppose α is least such that there is $f : \omega \to \alpha$ in $M[G] - M$ towards a contradiction. Let $\dot{f}$ be a $\mathbb{P}$-name for f, i.e. $\dot{f}_G = f$. Let σ be a name for $(\alpha^\omega)^M$, i.e. $\sigma_G = (\alpha^\omega)^M$. Some condition q forces $\dot{f} \notin \sigma$ by Corollary 6.30. We will find a play $p_0 \geq q_0 \geq p_1 \geq q_1 \geq ...$ in the game in which the second player follows her winning strategy. Let $\beta_0 \in Ord$ and $p_0 \leq q$ with $p_0 \Vdash \dot{f}(\check{0}) = \check{\beta}_0$ (to see that this exists, take some generic G_0 with $p_0 \in G_0$, then $\dot{f}^{G_0}(0) = \beta_0$ for some β_0, and this is forced by some p_0 by Corollary 6.30). Let the second player answer with $q_0 \leq p_0$ following her winning strategy. Let $p_1 \leq q_0$ decide $\dot{f}(1)$ etc. Let $p \leq p_n$ for all n. Then $p \Vdash \dot{f}(\check{n}) = \check{\beta}_n$ for all n. Hence $p \Vdash \dot{f} \in \sigma$. Contradiction.

Do Exercise 7.6 in the exercise classes.

Problem 30 (ω_1-closed forcings)**.** Suppose M is a transitive model of ZFC and $(\mathbb{P}, \leq)$ is a partial order in M. Suppose $\mathbb{P}$ is ω_1-*closed* in M, i.e. for every decreasing ω-sequence $p_0 \geq p_1 \geq ... \geq p_n \geq ...$ in M of conditions in $\mathbb{P}$, there is a condition p with $p \leq p_n$ for all $n \in \omega$. Suppose every ω-sequence of elements of M is an element of M. Prove similar to the Rasiowa-Sikorsky Theorem 6.4 that for every family $\mathcal{D} \in M$ of size ω_1, there is a $\mathcal{D}$-generic filter over M.

Solution: Let $(D_\alpha : \alpha < \omega_1)$ enumerate $\mathcal{D}$. Construct a sequence $(p_\alpha : 1 < \alpha < \omega_1)$ with $p_\alpha < p_\beta$ for $\alpha > \beta$ and $p_\alpha \in D_\alpha$ as follows. Let $p_1 \in D_0$ and $p_{\alpha+1} \leq p_\alpha$ in D_α for $\alpha < \omega_1$. For limits $\alpha < \omega$, the sequence $(p_\beta : \beta < \alpha)$

constructed up to α is an element of M by assumption. So we can find $p_\alpha \leq p_\beta$ for all $\beta < \alpha$ (pick a cofinal sequence $(p_{\beta_n} : n < \omega)$ and let $p \leq p_{\beta_n}$ for all n). Let $G := \{p \in \mathbb{P} : \exists \alpha < \omega_1 (p_\alpha \leq p)\}$. Then any two elements of G are compatible, G is upwards closed, and $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

Problem 31 (Cohen subsets of ω_1)**.** Suppose M is a countable transitive model of ZFC. Let $\mathbb{Q} = \{p : A \to 2 : A \subseteq \omega_1 \text{ countable}\}^M$ and $p \leq q$ iff $p \supseteq q$. Suppose G is a $\mathbb{Q}$-generic filter over M. Show that CH holds in $M[G]$.

Solution: It is easy to show this by mapping the generic function to a function $f : \omega_1 \times \omega \to 2$ and observing that generically all ground model reals appear as some $f(\alpha, .)$, and no new reals are added. Instead we can identify elements of the forcing $\mathbb{P}$ in Definition 7.1 with functions $\omega_1 \times \omega \to 2$ and this is a dense subforcing of the previous forcing (when we work with partial functions $p : \omega_1 \times \omega \to 2$, but the forcings are isomorphic); the functions are those such that when $(\alpha, n) \in dom(p)$, then $(\alpha, m) \in dom(p)$ for all $m \in \omega$. Hence $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, i.e. every $\mathbb{P}$-generic extension is a $\mathbb{Q}$-generic extension and conversely.

Problem 32 (Cohen forcing is not ω^ω-bounding)**.** Let $\mathbb{P} = \{p : n \to \omega : n \in \omega\}$ and $p \leq q :\Leftrightarrow p \supseteq q$ for $p, q \in \mathbb{P}$. Note that $\mathbb{P}$ is *equivalent* to Cohen forcing $Fn(\omega, 2)$ by Problem 26, i.e. every Cohen generic extension is a $\mathbb{P}$-generic extension and vice versa. Suppose M is a countable transitive model of ZFC. Let $f \leq^* g$ for $f, g : \omega \to \omega$ iff $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. Show that in every $\mathbb{P}$-generic extension of M, there is a function $f : \omega \to \omega$ such that $f \not\leq^* g$ for all $g \in M$.

Solution: Let G be $\mathbb{P}$-generic over M and $f = \bigcup G$. Suppose (towards a contradiction) that there is a function $g : \omega \to \omega$ in M with $f \leq^* g$. Let $\dot{f}$ be a name for f. Let $p \in \mathbb{P}$ with $p \Vdash \dot{f} \leq^* \check{g}$. Let $q \leq p$ and $n \in \omega$ with $q \Vdash \dot{f}(i) \leq \check{g}(i)$ for all $i \geq \check{n}$. Suppose $q : m \to \omega$. Let $r := q \cup \{(m, g(m) + 1)\}$. Then $r \Vdash \exists i \geq \check{n}(\dot{f}(i) > \check{g}(i))$. Contradiction.

Please hand in your solutions on Wednesday, 01 June before the lecture.