7. Problem set for "Models of set theory I", Summer 2011

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Please hand in your solutions on Monday, 23 May before the lecture (the lecture on Wednesday, 25 May is cancelled because of Dies Academicus).

Problem 25 (Dense embeddings)**.** Suppose $(\mathbb{P}, \leq)$ and $(\mathbb{Q}, \leq)$ are partial orders. Suppose $f : \mathbb{P} \to \mathbb{Q}$ is a *dense embedding*, i.e. if for all $p, q \in \mathbb{P}$

1. $p \leq q$ implies $f(p) \leq f(q)$,
2. $p \perp q$ implies $f(p) \perp f(q)$, and
3. $f[\mathbb{P}]$ is a dense subset of $\mathbb{Q}$.

Suppose N is a countable transitive model of ZFC with $(\mathbb{P}, \leq), (\mathbb{Q}, \leq), f \in N$. Show:

a) If $H \subseteq \mathbb{Q}$ is a $\mathbb{Q}$-generic filter over N, then $G := f^{-1}[H]$ is a $\mathbb{P}$-generic filter over N.

b) If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over N, then $H := \{p \in \mathbb{P} : \exists q \in f[G] : p \leq q\}$ is a $\mathbb{Q}$-generic filter over N.

c) In a), $H = \{p \in \mathbb{P} : \exists q \in f[G] : p \leq q\}$ and in b), $G = f^{-1}[H]$.

d) In a) and in b), $N[G] = N[H]$.

Solution: a) $G = f^{-1}[H]$ is a filter: All $p, q \in G$ are compatible and if $p \in G$ and $p \leq q$, then $f(p) \leq f(q)$, so $f(q) \in H$ and $q \in f^{-1}[H]$. If $D \subseteq \mathbb{P}$ is dense, $f[D]$ is dense (easy). Let $q \in f[D] \cap H$ and $p \in D$ with $f(p) = q$. Then $p \in D \cap G$.

b) H is a filter since we closed it upwards. If $D \subseteq \mathbb{Q}$ is open dense, $f^{-1}[D]$ is predense. Let $p \in f^{-1}[D] \cap G$ and $q = f(p)$. So $q \in D \cap f[G] \subseteq D \cap H$.

c) In a), both $H \subseteq \{p \in \mathbb{P} : \exists q \in f[G] : p \leq q\}$ and $\{p \in \mathbb{P} : \exists q \in f[G] : p \leq q\}$ are generic filters by a) and b). Since generic filters are maximal by Problem 17, they are equal. In b), both $G \subseteq f^{-1}[H]$ and $f^{-1}[H]$ are generic filters by a) and b), so they are equal.

d) $H \in N[G]$ by c). Since $N[H]$ is the least transitive extension of ZFC of N containing H by Lemma 6.9, $N[H] \subseteq N[G]$. Similarly, c) implies $N[G] \subseteq N[H]$.

Problem 26 (Countable forcings)**.** Let $\mathbb{P} = \{p : n \to \omega : n \in \omega\}$ and $p \leq q :\Leftrightarrow p \supseteq q$ for $p, q \in \mathbb{P}$. Suppose $(\mathbb{Q}, \leq)$ is a countable nonatomic forcing (see 5. problem set) with largest element $1_{\mathbb{Q}}$. Show:

a) For all $q \in \mathbb{Q}$ and p compatible with q, there is an infinite antichain $A \subseteq \mathbb{Q}$ such that $a \leq q$ for every $a \in A$, A is predense below q, and there is some $b \in A$ with $b \leq p$.
b) There is a dense embedding $f : \mathbb{P} \to \mathbb{Q}$ (see Problem 25).

Solution: a) Given q we build a family $(p_s : s \in 2^{<\omega})$ of conditions as in Problem 18 with $p_\emptyset \leq p, q$. Extend (say) the antichain $\{p_{0^n 1} : n \in \omega\}$ to an antichain which is maximal below q (using problem 15).
b) Let $\mathbb{P} = \{p_i : i \geq 1\}$. We can use a) to define a dense embedding f. In step 0 let $f(\emptyset) = 1_{\mathbb{Q}}$. In step 1, find an infinite maximal antichain $A^1 := A_\emptyset = \{a_i : i \in \omega\}$ in $\mathbb{Q}$ with $a_0 \leq p_1$ and let $f(\{(0,i)\}) = a_i$. In step 2, find i such that $a_i || p_2$. Let $A_{(i)} = \{a_{ij} : j \in \omega\}$ be an infinite antichain which is maximal below a_i with $a_{i0} \leq p_2$. Let $A_{(i')} = \{a_{i'j} : j \in \omega\}$ be an infinite antichain which is maximal below $a_{i'}$ for $i' \neq i$. Then $A^2 := \bigcup_{j\in\omega} A_{(j)}$ is a maximal antichain in $\mathbb{Q}$ (easy). In step 3, find (the unique pair) (i,j) such that $a_{ij} || p_3$, etc.
The image of f is dense, since there is $a \in A^n$ with $a \leq p_n$. It is easy to see that f preserves $||$ and $\perp$.

Problem 27 (Collapse)**.** Let κ be an infinite cardinal and $\mathbb{P} := \{p : n \to \kappa : n \in \omega\}$ with $p \leq q :\Leftrightarrow p \supseteq q$. Suppose M is a countable transitive model of ZFC with $(\mathbb{P}, \leq) \in M$ and G is $\mathbb{P}$-generic over M. Show:

a) $f_G := \bigcup G$ is a function from ω (surjective) onto κ.
b) $G = \{p \in \mathbb{P} : p \subseteq f_G\}$.

Solution: a) f_G is a function since G is a filter. $D_\alpha := \{p \in \mathbb{P} : \alpha$ is in the range of $p\}$ is dense for each $\alpha < \kappa$, so f_G is onto κ.
b) If $p \subseteq f_G = \bigcup G$, find $q_n \in G$ with $n \in dom(q_n)$ for each $n \in dom(p)$. Then $\bigcap_{n \in dom(p)} q_n \leq p$, so $p \in G$.

Problem 28 (Names)**.** Suppose $\mathbb{P}$ is a partial order and $p, q \in \mathbb{P}$ with $p \perp q$. Suppose M is a countable transitive model of ZFC with $(\mathbb{P}, \leq) \in M$. Show that in M there is a proper class of names σ with $p \Vdash \sigma = \check{\emptyset}$.

Solution: Let $C = \{\{(\check{x}, q)\} : x \in M\}$.