7. Problem set for "Models of set theory I", Summer 2011

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Please hand in your solutions on Monday, 23 May before the lecture (the lecture on Wednesday, 25 May is cancelled because of Dies Academicus).

Problem 25 (Dense embeddings)**.** Suppose $(\mathbb{P}, \leq)$ and $(\mathbb{Q}, \leq)$ are partial orders. Suppose $f : \mathbb{P} \to \mathbb{Q}$ is a *dense embedding*, i.e. if for all $p, q \in \mathbb{P}$

1. $p \leq q$ implies $f(p) \leq f(q)$,
2. $p \perp q$ implies $f(p) \perp f(q)$, and
3. $f[\mathbb{P}]$ is a dense subset of $\mathbb{Q}$.

Suppose N is a countable transitive model of ZFC with $(\mathbb{P}, \leq), (\mathbb{Q}, \leq), f \in N$. Show:

a) If $H \subseteq \mathbb{Q}$ is a $\mathbb{Q}$-generic filter over N, then $G := f^{-1}[H]$ is a $\mathbb{P}$-generic filter over N.

b) If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over N, then $H := \{p \in \mathbb{P} : \exists q \in f[G] : p \leq q\}$ is a $\mathbb{Q}$-generic filter over N.

c) In a), $H = \{p \in \mathbb{P} : \exists q \in f[G] : p \leq q\}$ and in b), $G = f^{-1}[H]$.

d) In a) and in b), $N[G] = N[H]$.

Problem 26 (Countable forcings)**.** Let $\mathbb{P} = \{p : n \to \omega : n \in \omega\}$ and $p \leq q :\Leftrightarrow p \supseteq q$ for $p, q \in \mathbb{P}$. Suppose $(\mathbb{Q}, \leq)$ is a countable nonatomic forcing (see 5. problem set) with largest element $1_{\mathbb{Q}}$. Show:

a) For all $q \in \mathbb{Q}$ and p compatible with q, there is an infinite antichain $A \subseteq \mathbb{Q}$ such that $a \leq q$ for every $a \in A$, A is predense below q, and there is some $b \in A$ with $b \leq p$.

b) There is a dense embedding $f : \mathbb{P} \to \mathbb{Q}$ (see Problem 25).

Problem 27 (Collapse)**.** Let κ be an infinite cardinal and $\mathbb{P} := \{p : n \to \kappa : n \in \omega\}$ with $p \leq q :\Leftrightarrow p \supseteq q$. Suppose M is a countable transitive model of ZFC with $(\mathbb{P}, \leq) \in M$ and G is $\mathbb{P}$-generic over M. Show:

a) $f_G := \bigcup G$ is a function from ω (surjective) onto κ.

b) $G = \{p \in \mathbb{P} : p \subseteq f_G\}$.

Problem 28 (Names)**.** Suppose $\mathbb{P}$ is a partial order and $p, q \in \mathbb{P}$ with $p \perp q$. Suppose M is a countable transitive model of ZFC with $(\mathbb{P}, \leq) \in M$. Show that in M there is a proper class of names σ with $p \Vdash \sigma = \check{\emptyset}$.