# 6. Problem set for “Models of set theory I”, Summer 2011

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**Problem 21** (Regular open sets)**.** Suppose $(\mathbb{P}, \leq)$ is a partial order.

a) Let $\mathcal{F} \subseteq ro(\mathbb{P})$ be a family of regular open sets. Show that $\bigcap \mathcal{F}$ is regular open.
b) Let $A \subseteq \mathbb{P}$. Show that $\neg A$ is a regular open subset of $\mathbb{P}$.
c) Suppose $\mathbb{P} = Fn(\omega, 2)$. Find regular open sets $A, B \subseteq \mathbb{P}$ such that $A \cup B$ is not regular open.
d) Suppose $M$ is a countable transitive model of ZFC and $(\mathbb{P}, <) \in M$. Show that for every formula $\varphi(\tau_1, ..., \tau_n)$ in the forcing language, the truth value $[\![\varphi(\tau_1, ..., \tau_n)]\!]$ is a regular open subset of $\mathbb{P}$. Lemma 6.19 might be useful.

Solution: a) If $p \in reg(\bigcap \mathcal{F})$ (i.e. $\bigcap \mathcal{F}$ is predense below $p$), then for every $A \in \mathcal{F}$, $A$ is predense below $p$. Since each $A \in \mathcal{F}$ is regular open, $p \in A$. So $reg(\bigcap \mathcal{F}) \subseteq \bigcap \mathcal{F}$.
b) Let $p \in reg(\neg A)$ (i.e. $\neg A$ is predense below $p$, recall $\neg A = \{p : \forall q \in A(p \perp q)\}$). To show $p \in \neg A$, suppose $p||q$ for some $q \in A$. Let $r \leq p, q$. Let $s \in \neg A$ with $s||r$. Then $s||q$. Contradiction.
c) Let $A = \{p : p(0) = 0\}$ and $B = \{p : p(0) = 1\}$. Then $A \cup B$ is predense below $\emptyset$, but $\emptyset \notin A \cup B$.
d) Suppose $[\![\varphi(\tau_1, ..., \tau_n)]\!]$ is predense below $p$. We want to show $p \in [\![\varphi(\tau_1, ..., \tau_n)]\!]$. If $G$ is generic over $M$ and $p \in G$, then $G \cap [\![\varphi(\tau_1, ..., \tau_n)]\!] \neq \emptyset$ by Lemma 6.19 c). Hence $M[G] \vDash \varphi(\tau_1, ..., \tau_n)$, and (by definition) $p \in [\![\varphi(\tau_1, ..., \tau_n)]\!]$.

**Problem 22** (Separative partial orders)**.** A partial order $(\mathbb{P}, <)$ is called *weakly separative* if for all $p, q \in \mathbb{P}$ we have

$$p = q \Leftrightarrow \forall r \in \mathbb{P}(r \perp p \Leftrightarrow r \perp q).$$

$\mathbb{P}$ is called *separative* if for all $p, q \in \mathbb{P}$ with $p \not\leq q$, there is $r \leq p$ with $r \perp q$. Suppose $\mathbb{B}$ is a Boolean algebra with smallest element 0. A set $A \subseteq \mathbb{B}$ is called dense in $\mathbb{B}$ if $A - \{0\}$ is dense in $\mathbb{B} - \{0\}$. Let $e : \mathbb{P} \to ro(\mathbb{P})$ be the map defined in section 6.3. Show:

a) Every separative partial order is weakly separative.

b) Give an example of a finite partial order that is weakly separative, but not separative. It is sufficient to find a partial order without a largest element.
c) The range of $e$ is dense in $ro(\mathbb{P})$.
d) $e$ is one-to-one iff $\mathbb{P}$ is weakly separative.

Solution: a) Suppose $\forall r \in \mathbb{P}(r \perp p \Leftrightarrow r \perp q)$ but $p \neq q$, say $p \not\leq q$. Let $r \leq p$ with $r \perp q$. Contradiction.
b) $a, b, c$ minimal, $a, b \leq d$, $a, b, c \leq e$.
c) Given $A \in ro(\mathbb{P})$, let $p \in A$. Then $e(p) = reg(\{p\}) \subseteq reg(A) = A$, so $e(p) \leq A$ in $ro(\mathbb{P})$.
d) Suppose $e$ is one-to-one. If $p \neq q$, then $e(p) \neq e(q)$, so there is $r \in \mathbb{P}$ with $\{p\}$ predense below $r$ and $\{q\}$ not predense below $r$. Let $s \leq r$ with $s \perp q$. Then $s||p$ but $s \perp q$. The other direction is easy.

**Problem 23** (Boolean algebras)**.** Suppose $\mathbb{B}$ is a complete Boolean algebra and $S \subseteq \mathbb{B} - \{0\}$. Let $p||q$ mean that $p, q$ are *compatible* in the partial order $\mathbb{B} - \{0\}$, i.e. there is $r \leq p, q$ in $\mathbb{B} - \{0\}$. Show for all $p, q \in \mathbb{B} - \{0\}$:

a) $p \perp q$ iff $p \wedge q = 0$, and $p \leq q$ iff $p \wedge q = p$ iff $p \wedge \neg q = 0$.
b) $p \leq q$ implies $r \wedge p \leq r \wedge q$, and $(\mathbb{B}, \leq)$ is separative (see Problem 22).
c) $p \wedge \bigvee S = \bigvee_{s \in S}(p \wedge s)$ (to show this, write $s = (p \wedge s) \vee (\neg p \wedge s)$ for each $s \in S$).
d) $S$ is predense below $p$ iff $\forall q \leq p(q|| \bigvee S)$ iff $p \leq \bigvee S$.

Solution: a) $p \wedge q = 0$ iff there is no $0 \neq r \leq p, q$. $p \leq q$ iff $p \wedge q = p$ is easy. If $p \wedge \neg q = 0$, then $p = (p \wedge \neg q) \vee (p \wedge q) = p \wedge q$. If $p \wedge q = p$ and $p \wedge \neg q \neq 0$, $p \wedge \neg q$ is a common extension of $q$ and $\neg q$, contradiction.
b) If $p \wedge q = p$, then $r \wedge p = r \wedge (p \wedge q) \leq r \wedge q$. $\mathbb{B}$ is separative: If $p \not\leq q$, let $r := p \wedge \neg q \neq 0$ by a). Then $r \leq p$ and $r \perp q$.
c) For $\geq$ note that $p \wedge \bigvee S$ is an upper bound of all $p \wedge s$. For $\leq$ let $q = \bigvee_{s \in S}(p \wedge s)$ and write $s = (p \wedge s) \vee (\neg p \wedge s)$ for all $s \in S$. Then $q \vee \neg p$ is an upper bound of all $s \in S$, so $\bigvee S \leq q \vee \neg p$. Then $p \wedge \bigvee S \leq q$.
d) That $S$ is predense below $p$ means that for all $q \leq p$ there is $s \in S$ with $s||q$ (i.e. $s \wedge q \neq 0$). This is equivalent to $\forall q \leq p(q \wedge \bigvee S \neq 0)$ by c). If $p \leq \bigvee S$, this is clearly true. If $p \not\leq \bigvee S$, let $r \leq p$ with $r \perp \bigvee S$. Then $\bigvee_{s \in S}(r \wedge s) = 0$ by c), so $\forall s \in S(r \wedge s = 0)$. So $S$ if not predense below $p$.

**Problem 24** (Generic filters)**.** Suppose $N$ is a countable transitive model of ZFC and $(\mathbb{P}, \leq) \in N$ is a partial order.

a) Suppose $H$ is an arbitrary subset of $\mathbb{P}$. If $\sigma, \tau$ are $\mathbb{P}$-names, let $\sigma <_H \tau$ iff there is $p \in H$ with $(\sigma, p) \in \tau$. Show that $<_H$ is well-founded on $N^{\mathbb{P}}$. Find the function $F$ that is used in the Recursion Theorem 2.4 to define $\tau_H$ for $\tau \in N^{\mathbb{P}}$ (as in Definition 6.8).

b) Suppose $H \subseteq \mathbb{P}$ and $H \in N$. Show by induction on $<_H$ that $N[H] \subseteq N$.

c) Let $\alpha := Ord^N$. Find a set $a \subseteq \omega$ such that $\alpha \in M$ for every model of ZFC with $a \in M$. You may use that there is a bijection $g : \omega \times \omega \to \omega$ which is an element of every transitive model of ZFC.

d) Let $\mathbb{P} = Fn(\omega, 2)$ and $\alpha := Ord^N$. Let $\chi_a : \omega \to 2$ be the characteristic function of the set $a$ in c). Use Problem 13 to find a filter $H \subseteq \mathbb{P}$ such that $\alpha \in M$ for every transitive model $M$ of ZFC with $H \in M$. Conclude that $H \notin N[G]$ for every generic extension $N[G]$ of $N$.

Solution: a) $<_H$ is well-founded since $\sigma <_H \tau$ implies $rank(\sigma) < rank(\tau)$. The function in the Recursion Theorem is $F(a, h) = range(h)$.

b) If $H \in N$, we can apply the Recursion Theorem in $N$ and define an interpretation that we could call $\tau_{H,N}$ for $\tau \in N^{\mathbb{P}}$, so $\tau_{H,N} = \{\sigma_{H,N} : \exists p \in H((\sigma, p) \in \tau)\}$. Then $\tau_H = \tau_{H,N}$ by induction on $<_H$.

c) Let $f : \omega \to \alpha$ be a bijection and let $E$ be the preimage of $\in$. Let $g : \omega \times \omega \to \omega$ be a bijection (for example we can choose $g \in L_{\omega+2}$) and let $a$ be the image of $E$. Every transitive model of ZFC which contains $a$ contains $g^{-1}[a]$ and hence $\alpha$ by applying the Mostowski collapse.

d) Let $H = G_{\chi_a}$ in Problem 13 d). Then every transitive model of ZFC with $H \in M$ contains $\chi_a = \bigcup G_{\chi_a}$ and hence $\alpha$ by c).

Due Wednesday 18 May.