5. Problem set for "Models of set theory I", Summer 2011

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Suppose M is a countable transitive model of ZFC and $(\mathbb{P}, \leq) \in M$ is a partial order. $\mathbb{P}$ is called *nonatomic* if every $p \in \mathbb{P}$ has at least two incompatible extensions.

Problem 17. Show:

(1) If $\mathbb{P}$ is nonatomic, then no filter $G \subseteq \mathbb{P}$ that is $\mathbb{P}$-generic over M is actually an element of M.

(2) Every $\mathbb{P}$-generic filter G over M is a maximal filter.

Hint for (1): Given a filter $G \in M$, find a dense subset $D \in M$ that is disjoint from G.

(1) Show that $\mathbb{P} - G$ is dense. Given $p \in \mathbb{P}$, let $q, r \leq p$ be incompatible extensions of p. They are not both in G, since G is a filter, so one of them is in $\mathbb{P} - G$.
(2) If not, there is a condition $p \notin G$ with $p||q$ for all $q \in G$. $D := \{q \in \mathbb{P} : q \leq p \text{ or } q \perp p\}$ is dense: If $q \not\perp p$ there is $r \leq p, q$, so $r \in D$. $D \cap G = \emptyset$, contradiction.

Problem 18. Show: If $\mathbb{P}$ is nonatomic, then $|\{G \subseteq \mathbb{P} : \text{G is } \mathbb{P}\text{-generic over } M\}| = 2^\omega$.

Hint: Looking at the proof of Theorem 6.4 might be useful. Construct a family $(p_s : s \in 2^{<\omega})$ of conditions with $p_t \leq p_s$ for $s \subseteq t$ such that $G_x := \{p \in \mathbb{P} : \exists n (p_{x \restriction n} \leq p)\}$ is a generic filter for each $x \in 2^\omega$.

Construct $(p_s : s \in 2^{<\omega})$ by recursion on the length of s with $p_t \leq p_s$ for $s \subseteq t$ and $p_{s^\frown 0} \perp p_{s^\frown 1}$ for $s \in 2^{<\omega}$. Suppose $(D_n : n \in \omega)$ is an enumeration of the dense sets in M and make sure that $p_s \in D_n$ for all s of length n.

Problem 19. Given $\mathbb{P}$-names $\sigma, \tau \in M$, find $\mathbb{P}$-names $\mu, \nu, \pi, \rho \in M$ such that for every $\mathbb{P}$-generic filter G over M

(1) $\mu_G = (\sigma_G, \tau_G) = \{\{\sigma_G\}, \{\sigma_G, \tau_G\}\}$,

(2) $\nu_G = \sigma_G \cup \tau_G$,

(3) $\pi_G = \sigma_G \cap \tau_G$, and

(4) $\rho_G = \sigma_G - \tau_G$,

and prove that they have these properties, where you may assume $dom(\sigma) \cup dom(\tau) \subseteq \{\check{x} : x \in y\}$ for some $y \in M$ in (3) and (4).

Hint for (4): Let $\rho = \{(\check{x}, p) : x \in y \text{ and } \exists q \geq p((\check{x}, q) \in \sigma) \text{ and } \forall r || p((\check{x}, r) \notin \tau)\}$, where $r||p$ means that r and p are compatible, i.e. there is some $s \leq r, p$. If $x \in \sigma_G - \tau_G$, there is $q \in G$ with $(\check{x}, q) \in \sigma$ and $(\check{x}, p) \notin \tau$ for all $p \in G$. Show that $D := \{p \in \mathbb{P} : \forall r || p((\check{x}, r) \notin \tau) \text{ or } \exists r \geq p(\check{x}, r) \in \tau\}$ is dense.

(1) $\mu = \{(\{(\sigma, 1), (\tau, 1)\}, 1), (\{\sigma, 1)\}, 1)\}$,
(2) $\nu = \sigma \cup \tau$,
(3) $\pi = \{(\check{x}, p) : x \in y \text{ and } \exists q, r \geq p((\check{x}, q) \in \sigma, (\check{x}, r) \in \tau)\}$, and
(4) $\rho = \{(\check{x}, p) : x \in y \text{ and } \exists q \geq p((\check{x}, q) \in \sigma) \text{ and } \forall r || p((\check{x}, r) \notin \tau)\}$.
The properties are easily checked except for (4): If $x \in \sigma_G - \tau_G$, there is $q \in G$ with $(\check{x}, q) \in \sigma$ and $(\check{x}, p) \notin \tau$ for all $p \in G$. Show that $D := \{p \in \mathbb{P} : \forall r || p((\check{x}, r) \notin \tau) \text{ or } (\exists r \geq p(\check{x}, r) \in \tau)\}$ is dense: If $p \in \mathbb{P}$ and there is $r||p$ with $(\check{x}, r) \in \tau$, let $s \leq r, p$, then $s \in D$. So $E := \{p \in \mathbb{P} : (p \in D \text{ and } p \leq q) \text{ or } p \perp q\}$ is also dense. Let $p \in E \cap G$, so $p \leq q$ and $p \in D$. Since $p \in D$ and $x \notin \tau_G$, $\forall r || p((\check{x}, r) \notin \tau)$. This shows $x \in \rho_G$ witnessed by $(\check{x}, p)$.

Problem 20. Suppose M and $\mathbb{P}$ are as in problem 18 and G is a $\mathbb{P}$-generic filter over M. Assume that $M[G]$ satisfies ZF and show that in $M[G]$ every set can be well-ordered.

Hint: You may use that every name can be well-ordered in M, since M satisfies ZFC. Show that every surjective image of a well-orderable set can be well-ordered.

Working in ZF, suppose the set A can be well-ordered and $f : A \to B$ is a partial function from A onto B. Let $g(b)$ be the least $a \in A$ with $f(a) = b$ for $b \in B$. Then $g : B \to A$ is one-to-one and $b \leq b' :\Leftrightarrow g(b) \leq g(b')$ defines a wellorder on B.
Suppose τ is a name for a set $A \in M[G]$. Let $\pi : \tau \to A$ be the (partial) interpretation map $\pi((\sigma, p)) = \sigma_G$ if $p \in G$. $A = range(\pi)$ by definition. τ can be well-ordered (in M and hence in $M[G]$), so A can be well-ordered.