4. Problem set for "Models of set theory I", Summer 2011

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Problem 13. The set $Fn(X, 2)$ of finite partial functions from X to 2 is partially ordered by reverse inclusion, i.e. $p \leq q$ iff $p \supseteq q$. Show:

(1) If $G \subseteq Fn(X, 2)$ is a filter, then $f_G := \bigcup G$ is a function.
(2) For all $x \in X$ the set $D_x := \{p \in Fn(X, 2) : x \in dom(p)\}$ is dense in $Fn(X, 2)$.
(3) If $G \subseteq Fn(X, 2)$ is a $\{D_x : x \in X\}$-generic filter, then $dom(f_G) = X$.
(4) For every function $f : X \to 2$ the set $G_f := \{p \in Fn(X, 2) : p \subseteq f\}$ is a $\{D_x : x \in X\}$-generic filter.

(1) Suppose $(x, i), (x, j) \in \bigcup G$ with $i \neq j$. Let $p, q \in G$ with $(x, i) \in p$ and $(y, j) \in q$. Then there is a function r (in G) with $p \cup q \subseteq r$, contradiction.
(2) Given $p \in Fn(X, 2)$ with $x \notin dom(p)$, let $q = p \cup \{(x, 0)\}$, then $q \in D_x$.
(3) By (1) and (2).
(4) G_f is a filter since it is downwards closed and any $p, q \in G_f$ are compatible. Clearly $G \cap D_x \neq \emptyset$.

Problem 14. Let $(A, \leq_A)$ and $(B, \leq_B)$ be countably infinite, dense linear orders without endpoints. (Recall that a linear order is dense if strictly between any two elements there is another element of the linear order.) Show that $(A, \leq_A)$ and $(B, \leq_B)$ are isomorphic.

Hint: Use the Rasiowa-Sikorski Theorem. Consider the partial $\mathbb{P}$ of isomorphisms between finite subsets of A and B, ordered by reverse inclusion. Find a countable family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ such that for every $\mathcal{D}$-generic filter $G \subseteq \mathbb{P}$ the function $\bigcup G$ is an isomorphism from A to B. It might help to take another look at the previous problem.

Let $\mathcal{D}$ be the set of $D_a := \{p : A \to B : p$ is a finite partial isomorphism with $a \in dom(p)\}$ and $D^b := \{p : A \to B : p$ is a finite partial isomorphism with $b \in range(p)\}$.
To see that D_a is dense, suppose $p : A \to B$ is a partial isomorphism with $a \notin dom(p)$. Let $dom(p) = \{a_i : i < n\}$ with $a_0 < a_1 < ... < a_{n-1}$ and $k \leq n$ with $a_0 < ... < a_{k-1} < a < a_k < ... < a_{n-1}$. We can choose some $b \in B$ with $p(a_{k-1}) < b < p(a_k)$ by density of B (if $k = 0$ choose $b < p(a_0)$ and

if $k = n-1$ choose $b > p(a_{k-1})$, possible since B has no end points). Let $q = p \cup \{(a, b)\} \in D_a$.
Density of D^b is similar.

Problem 15. Suppose $\mathbb{P}$ is a partial order and $X \subseteq \mathbb{P}$. Use Zorn's Lemma to show that every antichain A with $A \subseteq X$ is contained in an antichain $B \subseteq X$ which is maximal with the property $B \subseteq X$. Show that every condition $p \in X$ is compatible with some condition $q \in B$.

Let S be the set of antichains B in $\mathbb{P}$ with $A \subseteq B \subseteq X$. Every chain in S has an upper bound (its union), so there is a maximal element B by Zorn's Lemma. If some $p \in X$ is incompatible with all $q \in B$, then B is not maximal.

Problem 16. Suppose $\mathbb{P}$ is a partial order and A is an antichain in $\mathbb{P}$. A set $D \subseteq \mathbb{P}$ is *predense* if the set $\{p \in \mathbb{P} : \exists q \in D(p \leq q)\}$ is dense. Show:

(1) A is a maximal antichain iff A is predense.
(2) Suppose D is dense and $A \subseteq D$ is maximal among all antichains $B \subseteq D$, then A is a maximal antichain.

(1) Suppose A is a maximal antichain. If A is not predense, it can be extended to a strictly larger antichain, contradiction.
Suppose A is predense. If A is not a maximal antichain, there is $p \in \mathbb{P}$ with $p \perp q$ for all $q \in A$. Let $D := \{p \in \mathbb{P} : \exists q \in A(p \leq q)\}$ (this is dense). Let $r \leq p$ with $r \in D$ and $q \in A$ with $r \leq q$. Then $p||q$, contradiction.

(2) If A is a maximal antichain (and hence predense by (1)), D is predense.
Suppose D is predense and A is not a maximal antichain, i.e. there is $p \in \mathbb{P}$ which is incompatible with every $q \in A$. Let $r \in D$ with $r \leq p$ (since D is dense). There is $q \in A$ with $q||r$ since A is maximal with $A \subseteq D$. So $p||q$, contradiction.