4. Problem set for "Models of set theory I", Summer 2011

Stefan Geschke, Philipp Schlicht, Anne Fernengel, Allard van Veen

Problem 13. The set $Fn(X, 2)$ of finite partial functions from X to 2 is partially ordered by reverse inclusion, i.e. $p \leq q$ iff $p \supseteq q$. Show:

(1) If $G \subseteq Fn(X, 2)$ is a filter, then $f_G := \bigcup G$ is a function.
(2) For all $x \in X$ the set $D_x := \{p \in Fn(X, 2) : x \in dom(p)\}$ is dense in $Fn(X, 2)$.
(3) If $G \subseteq Fn(X, 2)$ is a $\{D_x : x \in X\}$-generic filter, then $dom(f_G) = X$.
(4) For every function $f : X \to 2$ the set $G_f := \{p \in Fn(X, 2) : p \subseteq f\}$ is a $\{D_x : x \in X\}$-generic filter.

Problem 14. Let $(A, \leq_A)$ and $(B, \leq_B)$ be countably infinite, dense linear orders without endpoints. (Recall that a linear order is dense if strictly between any two elements there is another element of the linear order.) Show that $(A, \leq_A)$ and $(B, \leq_B)$ are isomorphic.

Hint: Use the Rasiowa-Sikorski Theorem. Consider the partial $\mathbb{P}$ of isomorphisms between finite subsets of A and B, ordered by reverse inclusion. Find a countable family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ such that for every $\mathcal{D}$-generic filter $G \subseteq \mathbb{P}$ the function $\bigcup G$ is an isomorphism from A to B. It might help to take another look at the previous problem.

Problem 15. Suppose $\mathbb{P}$ is a partial order and $X \subseteq \mathbb{P}$. Use Zorn's Lemma to show that every antichain A with $A \subseteq X$ is contained in an antichain $B \subseteq X$ which is maximal with the property $B \subseteq X$. Show that every condition $p \in X$ is compatible with some condition $q \in B$.

Problem 16. Suppose $\mathbb{P}$ is a partial order and A is an antichain in $\mathbb{P}$. A set $D \subseteq \mathbb{P}$ is *predense* if the set $\{p \in \mathbb{P} : \exists q \in D(p \leq q)\}$ is dense. Show:

(1) A is a maximal antichain iff A is predense.
(2) Suppose D is dense and $A \subseteq D$ is maximal among all antichains $B \subseteq D$, then A is a maximal antichain.

Please hand in your solutions on 04 May before the lecture.