# 3. Problem set for "Models of set theory I", Summer 2011

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In the exercise courses, you could prove Exercise 4.7: The Mostowski collapse $\mu : (C, \in) \to (D, \in)$ is the identity on transitive classes $T \subseteq C$. Induction: $\mu(a) = \{\mu(b) : b \in C, b \in a\} = a$ for $a \in T$.

**Problem 9.** Suppose $M$ is a transitive class such that for every set $x \subseteq M$ there is a set $y \in M$ with $x \subseteq y$ and $(M, \in)$ satisfies the Separation Axiom for $\Delta_0$ formulas. Prove that

1. $L$ satisfies the assumptions on $M$,
2. if $x \in M$, then $\bigcup x \in M$, and
3. if $x \in M$, then there is some $y \in M$ with $(M, \in) \models$ '$y$ is the power set of $x$'.

1. Let $\alpha = \sup\{\rho(z) + 1 : z \in x\}$ where $\rho$ is the $L$-rank, so $x \subseteq L_\alpha$. For $\Delta_0$ separation, suppose $a \in L_\alpha$, so $a \subseteq L_{\alpha+1}$. Since $\Delta_0$ formulas are absolute between transitive classes, the subset of $a$ defined by some $\Delta_0$ formula $\varphi$ is an element of $L_{\alpha+2}$.
2. Since $M$ is transitive, $\bigcup x \subseteq M$. Find a superset $y$ and apply $\Delta_0$ separation.
3. Let $y = \mathcal{P}(x) \cap M$. Find a superset $z \in M$ of $y$ and apply $\Delta_0$ separation to show $y \in M$.

**Problem 10.** Suppose $M = (X, R_i, f_i : i \in \omega)$ is a structure on a set $X \supseteq \aleph_1$ in a countable language. A set $C \subseteq \aleph_1$ is closed in $\aleph_1$ if $\alpha \in C$ for all $\alpha < \aleph_1$ with $C \cap \alpha$ unbounded in $\alpha$. Prove that there is a closed unbounded (*club*) subset $C \subseteq \aleph_1$ such that for every $\alpha \in C$ there is a substructure $N = (\bar{X}, \bar{R}_i, \bar{f}_i : i \in \omega) \prec M$ with $X \cap \aleph_1 = \alpha$.

Hint: Use Skolem hulls (closure under Skolem functions, see the proof of Theorem 4.4) to construct a sequence $(X_\alpha : \alpha < \aleph_1)$ such that each $(X_\alpha, R_i \upharpoonright X_\alpha, f_i \upharpoonright X_\alpha : i \in \omega)$ is a countable elementary substructure of $M$ and

i. $X_\alpha \subseteq X_\beta$ for all $\alpha < \beta < \aleph_1$,
ii. $X_\gamma = \bigcup_{\beta<\gamma} X_\beta$ for all limit ordinals $\gamma < \aleph_1$,
iii. $X_\alpha \cap \aleph_1 \in \aleph_1$ for each $\alpha < \aleph_1$,

and the set of $X_\alpha \cap \aleph_1$ forms a club subset of $\aleph_1$.

In the successor step, let $X_\alpha^0 = X_\alpha$, $X_\alpha^{n+1} = sk(X_\alpha^n \cup \sup(X_\alpha^n \cap \aleph_1) \cup \{\sup(X_\alpha^n \cap \aleph_1)\})$, and $X_{\alpha+1} = \bigcup_{n\in\omega} X_\alpha^n$. Take unions at limits. All $X_\alpha$ will be countable. Check the required properties (trivial).

**Problem 11.** Let $\sqsubset$ be a well-ordering of a set $X$.

1. For all $a_1, ..., a_n, b_1, ..., b_n \in X$ let $\bar{a} = (a_1, ..., a_n) \sqsubset^n \bar{b} = (b_1, ..., b_n)$ iff $\bar{a} \neq \bar{b}$ and $a_i \sqsubset b_i$ for the minimal $i$ with $a_i \neq b_i$. Prove that $\sqsubset^n$ is a well-ordering of $X^n$.
2. For all $\bar{a}, \bar{b} \in X^{<\omega} = \bigcup_{n\in\omega} X^n$ let $\bar{a} \sqsubset^{<\omega} \bar{b}$ if either $\bar{a}$ is a shorter finite sequence than $\bar{b}$ or for some $n \in \omega$, $\bar{a}, \bar{b} \in X^n$ and $\bar{a} \sqsubset^n \bar{b}$. Prove that $\sqsubset^{<\omega}$ is a well-ordering of $X^{<\omega}$.

1. Suppose $Y \subseteq X^n$. Find $(a_1, ..., a_{n-1}) \in Y$ $\sqsubset^{n-1}$-minimal such that there is $a_n \in X$ with $(a_1, ..., a_n) \in Y$, then choose $a_n$ $\sqsubset$-minimal with $(a_1, ..., a_n) \in Y$. Check that this is $\sqsubset^n$-minimal.
2. Find a tuple in $Y$ of minimal length $m$ and a $\sqsubset^m$-minimal $(a_1, ..., a_m) \in Y$. Check that this is $\sqsubset^{<\omega}$-minimal.

**Problem 12.** Suppose $V = L$ and $\alpha < \aleph_1$. Give an explicit example of a subset $a$ of $\omega$ such that $a \in L_{\aleph_1} - L_\alpha$.

Hint: See Exercise 5.15 in the lecture notes. You may use that the Reflection Principle holds for $(L_\alpha : \alpha < \aleph_1)$ (this is true since $\aleph_1$ has uncountable confinality). Note that $L_{\aleph_1}$ satisfies ZF without the Power set Axiom by Lemma 5.12. You may use that there is a finite fragment ZF* of ZF without the Power set Axiom which implies the following version of Theorem 4.5 (Mostowski collapse): If $E$ is a binary relation on a set $X$ and $(X, E)$ is extensional and well-founded, then there are a unique transitive set $Y$ and a unique isomorphism $\mu : (X, E) \to (Y, \in)$.

There is $\beta < \aleph_1$ with $\beta \geq \alpha$ and $\beta \geq \omega + 2$ so that $L_\beta$ satisfies ZF* by the Reflection Principle for $(L_\delta : \delta < \aleph_1)$ (choose the least such $\beta$ if you like). $L_\beta$ is countable in $L$ by Lemma 5.6. The image of $\in$ under a bijection $f : \omega \to L_\beta$ in $L$ (the $\lhd$-least such $f$ if you like) is a subset of $\omega \times \omega$, its image under a (simple) bijection $g : \omega \times \omega \to \omega$ in $L_{\omega+2}$ is a set $A \subseteq \omega$. Suppose $A \in L_\beta$. In $L_\beta$, we can form the Mostowski collapse $\mu : (\omega, g^{-1}[A]) \to (X, \in)$. Since $\mu$ and $f$ are isomorphisms in $L$, $\mu = f$ by the uniqueness of the Mostowski collapse (in $L$) and hence $X = L_\beta$. Then $X = L_\beta$ is a set in $L_\beta$, so $L_\beta \in L_\beta$. Contradiction. Hence $A \notin L_\beta$.

Please hand in your solutions on 27 April before the lecture.