Higher Set Theory - Classical and Ordinal Computability

Exercise Sheet 9
due on Tuesday, 7 June 2011

Consider the signature $\{<, G, f\}$ (G is a ternary relation, f a unary function) to define structures on ordinals $(\alpha, <, G \cap \alpha^3, f)$ where $f : \alpha \to \alpha$ is some unary function on α and G is the characteristic relation of the Gödel pairing function $G : \alpha \times \alpha \to \mathrm{Ord}$.

20. Starting with variables $v_n = G(0, n)$ for $n \in \omega$ and constant symbols $c_\xi = G(1, \xi)$, use G to code terms and formulas into ordinals (you may want to keep ex. 22 in mind, but straightforward definitions should suit ex. 22 just fine).

(4 points)

21. Let f be ordinal computable. Show that the satisfaction relation

$$\models_\alpha (\beta, \gamma) \leftrightarrow (\alpha, <, G \cap \alpha^3, f) \models \varphi_\beta[a_\gamma]$$

is ordinal computable, where φ_β is interpreted as a formula (say, with n free variables) and a_γ – via Gödel pairing – as n-tuple on α (the assignment for φ_β). If β does not code a formula, $\models_\alpha (\beta, \gamma)$ shall be false.

(4 points)

22. Define a truth predicate $T : \mathrm{Ord} \to \{0, 1\}$ by

$$T(\alpha) = 1 \leftrightarrow \models_\alpha (G_0(\alpha), G_1(\alpha))$$

where G_0 and G_1 are the inverses of the Gödel pairing function. Show that this definition is OTM-recursive, i.e., that $T(\alpha)$ can be OTM-computably determined from $T \restriction \alpha$. Modify your definition from ex. 20 if needed.

(4 points)