Higher Set Theory - Classical and Ordinal Computability

Exercise Sheet 7
due on Tuesday, 24 May 2011

15. Recall the axiom system PM of Peano-style set theory. It can be formulated in first order logic with the signature $S = \{\emptyset, \in, \{\cdot\}, \cdot \cup \cdot\}$ in the following way:

(Empty) $\forall x \, \neg x \in \emptyset$

(Union) $\forall x \forall y \forall z \, (z \in x \cup y \leftrightarrow (z \in x \vee z \in y))$

(Singleton) $\forall x \forall z \, (y \in \{x\} \leftrightarrow y = x)$

(Ext) $\forall x \forall y \, (x = y \leftrightarrow \forall z \, (z \in x \leftrightarrow z \in y))$

(Ind$_\phi$) $(\phi(\emptyset) \wedge ((\phi(x) \wedge \phi(y)) \rightarrow \phi(x \cup \{y\}))) \rightarrow \forall z \phi(z)$

where (Ind$_\phi$) is an axiom for every unary S-predicate ϕ.

(a) Show that in PM every set is Dedekind-finite, i.e.

$$\forall x \forall y \, (x \subsetneq y \rightarrow x \text{ is not bijective to } y)$$

(b) Prove that in PM every injective $f : x \rightarrow x$ is in fact surjective.

(6 points)

16. Consider the bijective coding of heriditarily finite sets and natural numbers introduced in the lecture. Abuse notation and let $\ulcorner x \urcorner$ denote the "Gödel number" of a set x and let $\ulcorner n \urcorner$ denote the "Gödel set" of a number n.

(a) Give a number theoretical relation ϕ s.t.

$$u \in v \leftrightarrow \phi(\ulcorner u \urcorner, \ulcorner v \urcorner)$$

(b) Find a number theoretical function f s.t.

$$\ulcorner f(n) \urcorner = \mathcal{P}(\ulcorner n \urcorner)$$

(6 points)