# Higher Set Theory - Classical and Ordinal Computability

Exercise Sheet 6
due on Tuesday, 17 May 2011

13. A set $A \subseteq \omega^n$ is called *recursively enumerable (r.e.)* iff there is a recursive relation $R \subseteq \omega^n \times \omega$ s.t.

$$\forall \vec{x} \in \omega^n (\vec{x} \in A \leftrightarrow \exists y \in \omega R(\vec{x}, y))$$

For a set $A \subseteq \omega^n \times \omega$ and a partial function $\phi : \omega^n \rightharpoonup \omega$ we say that $\phi$ *uniformizes* $A$ iff

$$\forall \vec{x} \in \omega^n (\exists y \in \omega A(\vec{x}, y) \leftrightarrow A(\vec{x}, \phi(\vec{x})))$$

(a) Show that $A$ is r.e. iff it is register enumerable.

(b) Prove that $A \subseteq \omega$ is recursive iff both $A$ and $\omega \setminus A$ are r.e..

(c) Show that a set $A \subseteq \omega$ is recursive iff either $A$ is finite or there is an recursive, injective and strictly increasing total function $f : \omega \to \omega$ s.t. $f[\omega] = A$.

(d) Prove that every r.e. set $A \subset \omega^n \times \omega$ is uniformized by a partial recursive function $\phi : \omega^n \rightharpoonup \omega$.

(e) Show that the function $f(\vec{x}) \simeq \mu y A(\vec{x}, y)$ for a r.e. relation $A$ is not necessarily partial recursive.

(8 points)

14. We have seen that every register enumerable (and hence ever r.e.) set is the domain of some partial computable (partial recursive) function. We define:

$$W^n(e, \vec{x}) = \exists g T^n(e, \vec{x}, g) \qquad \text{for } n \in \omega$$
$$W^n_e(\vec{x}) = \exists g T^n(e, \vec{x}, g) \qquad \text{for } n, e \in \omega$$

(a) Show that the r.e. sets are closed under $\cup$ and $\cap$.

(b) Prove that the above closure can be made effective: There are recursive functions $f_\cup$ and $f_\cap$ s.t.

$$W^n_d \cup W^n_e = W^n_{f_\cup(d,e)}$$
$$W^n_d \cap W^n_e = W^n_{f_\cap(d,e)}$$

(4 points)