Higher Set Theory - Classical and Ordinal Computability

Exercise Sheet 2
due on Tuesday, 19 April 2011

Unless noted otherwise, from now on register programs need not be given explicitly and may be sketched in some suitable form of pseudo-code or in prose, making use of loops, subroutines etc..

4. Give a proof of theorem 9 and show that for the class $\mathcal{B}$ of decidable subsets of $\mathbb{A}^*$

   a) $\emptyset \in \mathcal{B}$, $\mathbb{A}^* \in \mathcal{B}$

   b) $\mathcal{B}$ is closed under $\cup$

   c) $\mathcal{B}$ is closed under $\cap$

   d) $\mathcal{B}$ is closed under $\setminus$

(4 points)

5. Let $\mathbb{A}$ be finite. Let $|_{\mathbb{A}} \in \mathbb{A}$ to form numerals $n = (|_{\mathbb{A}})^n$.

   (a) Show that $\mathbb{A}^*$ is computably enumerable.

   (b) Let $F : \mathbb{A}^* \to \mathbb{A}^*$ be a function computable by the program P. Show that the function U_P is computable, where

   $$U_P(w,n) = \begin{cases} \square, \text{ if the computation by } P \text{ on input } w \text{ halts in at most } n \text{ steps} \\ |_{\mathbb{A}}, \text{ else} \end{cases}$$

   (c) Let $F : \mathbb{A}^* \to \mathbb{A}^*$ be bijective and computable. Show that the inverse function F^{-1} is computable. What would happen if one were to apply the resulting algorithm to an only injective function $G : \mathbb{A}^* \to \mathbb{A}^*$?

   (d) A partial function $G : \mathbb{A}^* \rightharpoonup \mathbb{A}^*$ (i.e. a function with $\text{dom}(G) \subseteq \mathbb{A}^*$) is called computable, if there is a program P s.t.

   – $P : w \mapsto G(w)$ for $w \in \text{dom}(G)$
   – $P : w \uparrow$ for $w \notin \text{dom}(G)$

   Show that a set $W \subseteq \mathbb{A}^*$ is computably enumerable iff there is a partial computable function H s.t. $\text{dom}(H) = W$.

(8 points)