Abelian groups that (do not) have automatic presentations

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Bonn, 21^{th} of June 2010

Contents

Adding Integers

1	0	1	$\diamond$	$\diamond$	$\diamond$
0	1	1	0	1	$\diamond$
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Question

What structures have a nice encoding that make it simple to compute operations just using local information?

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What structures have a nice encoding that make it simple to compute operations just using local information?

Automatic Structures

Definition (Khoussainov-Nerode)

A countable relational structure $(M; R_1, \ldots, R_k)$ is called **automatic** if there exists a finite alphabet Σ and a regular language $D \subseteq \Sigma^*$ and a bijection $f : D \to M$ such that the relations $f^{-1}(R_1), \cdots, f^{-1}(R_k)$ are regular.

- We can also include languages with function symbols by considering the graphs of the functions
- What does it mean for $f^{-1}(R_i)$ to be regular?

$$f^{-1}(R_i) \subseteq D^s \subseteq (\Sigma^*)^s \hookrightarrow ((\Sigma \cup \{\diamond\})^s)^*$$

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Basic Properties

- The first order theory of an automatic structure is decidable
- The model checking problem is decidable

Few Automatic Structures

If one allows rich algebraic structure in the language, then the only automatic structures are the trivial ones:

- (Khoussainov-Nies-Rubin-Stephan) every automatic Boolean algebra is a finite product of copies of the algebra of all finite and cofinite subsets of $\mathbb{N}$
- (KNRS) every automatic integral domain is finite
- (Oliver-Thomas) a finitely generated group is automatic iff it is has an abelian subgroup of finite index (abelian-by-finite)
- (Nies-Thomas) every finitely generated subgroup of an automatic group is abelian-by-finite

Automatic Abelian Groups

Examples of automatic abelian groups:

- Finite groups
- $\mathbb{Z}$
- $(\mathbb{Z}/p\mathbb{Z})^{(\aleph_0)}$
- $\mathbb{Z}(p^\infty)$
- $\mathbb{Z}[1/m]$
- finite direct sums of those
- finite extensions and automatic amalgamations, e.g.

$$\langle p_1^\infty e_1, p_2^\infty e_2, q^\infty(e_1 + e_2)\rangle \subseteq \mathbb{Q}^2$$

Non-examples

- every group containing $\mathbb{Z}^{(\aleph_0)}$
- $\mathbb{Z}(p^\infty)^{(\aleph_0)}$

Thus it is reasonable to look at abelian groups !

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Automatic abelian groups

Suppose $(G, +)$ is an automatic torsion-free abelian group. Let

- $D \subseteq \Sigma^*$ regular
- $g\colon D \to G$ bijection
- $g^{-1}(+)$ is recognizable by an automaton with r states

$\Longrightarrow \quad D^{\leq n} + D^{\leq n} \subseteq D^{\leq n+r}$

Lemma

Let L_1, L_2 be languages over a finite alphabet and $R \subseteq L_1 \times L_2$ be a regular relation such that the sections $R_x = \{y \in L_2 : (x, y) \in R\}$ are finite. Then for all $(x, y) \in R$, $len(y) \leq len(x) + k$, where k is the number of states of an automaton recognizing R and $len(y)$ is the length of y.

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Automatic abelian groups

Now the graph of the homomorphism $M_m : G \to G$ defined by $M_m(x) = mx$ is also regular. Let

- $h(m)$ be the minimal number of states of an automaton recognizing the graph of M_m
- $l_0 = min\{l \in \mathbb{N} : 0 \in D^{\leq l} \text{ and } |D^{\leq l}| \geq 2\}$
- $A_n = D^{\leq l_0 + nr}$ for $n = 0, 1, \cdots$

$\Longrightarrow$

1. $0 \in A_0$ and $|A_0| \geq 2$ and $G = \bigcup_{n \in \omega} A_n$;
2. $A_n + A_n \subseteq A_{n+1}$ for $n \in \omega$;
3. $|A_{n+1}| \leq C|A_n|$ for $n \in \omega$;
4. $m^{-1}A_n \subseteq A_{n+h(m)}$ for $n \in \omega$ and $m \in \mathbb{N}$.

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Automatic abelian groups

Why?....

$$m^{-1}D^{\leq n} \subseteq D^{\leq n+h(m)}$$

and

Lemma

Let $L \subseteq \Sigma^*$ be a regular language. Then there exists a constant C such that $|L^{\leq n+1}| \leq C|L^{\leq n}|$ for all n.

Tsankov's Theorem

Theorem (Todor Tsankov)

There does not exist a sequence $\langle A_n : n \in \omega \rangle$ of finite subsets of $\mathbb{Q}$ that satisfies the conditions

1. $0 \in A_0$ *and* $|A_0| \geq 2$;
2. $A_n + A_n \subseteq A_{n+1}$ *for* $n \in \omega$;
3. $|A_{n+1}| \leq C|A_n|$ *for* $n \in \omega$;
4. $m^{-1} A_n \subseteq A_{n+h(m)}$ *for* $n \in \omega$ *and* $m \in \mathbb{N}$.

Hence the additive group of rational numbers does not have an automatic presentation.

The Local Theorem

Theorem (Local Version)

Given a constant C_1 *and a function* $h : \Pi \to \mathbb{N}$ *there are integers* $d, K \in \mathbb{Z}^+$ *and a constant* $C = C(C_1) \geq C_1$ *such that the following hold for any sequence* $p_d < \cdots < p_0$ *of primes with*

$$p_d > C(4dK)^d \text{ and } p_{i-1} > p_i C^{h(p_i)d} d^{Cd^4} \text{ for } i = d, d-1, \cdots, 1.$$

There is no sequence $A_0, \cdots, A_{h(p_0)+\cdots+h(p_d)} \subseteq \mathbb{Q}$ *of finite subsets of* $\mathbb{Q}$ *such that*

1. $0 \in A_0$, $|A_0| \geq 2$
2. $|A_{n+1}| \leq C_1 |A_n|$ *for* $n < h(p_0) + \cdots + h(p_d)$
3. $A_n + A_n \subseteq A_{n+1}$ *for* $n < h(p_0) + \cdots + h(p_d)$
4. $p_i^{-1} A_n \subseteq A_{n+h(p_i)}$ *for* $n + h(p_i) \leq h(p_0) + \cdots + h(p_d)$ *and* $i = 0, \ldots, d$.

The Finite Theorem

Theorem (Finite Version)

Given a constant C_1 *and a function* $h : \Pi \to \mathbb{N}$ *there are integers* $d, K \in \mathbb{Z}^+$ *and a constant* $C = C(C_1) \geq C_1$ *such that the following hold for any sequence* $p_d < \cdots < p_0$ *of primes with*

$$p_d > C(4dK)^d \text{ and } p_{i-1} > p_i C^{h(p_i)d} d^{Cd^4} \text{ for } i = d, d-1, \cdots, 1.$$

There is p^* *such that for any prime* $p \geq p^*$*, there is no sequence* $A_0, \cdots, A_{h(p_0)+\cdots+h(p_d)} \subseteq \mathbb{Z}/p\mathbb{Z}$ *of finite subsets of* $\mathbb{Z}/p\mathbb{Z}$ *such that*

1. $0 \in A_0$, $2 \leq |A_0| \leq C_1$
2. $|A_{n+1}| \leq C_1 |A_n|$ *for* $n < h(p_0) + \cdots + h(p_d)$
3. $A_n + A_n \subseteq A_{n+1}$ *for* $n < h(p_0) + \cdots + h(p_d)$
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The Main Theorem

Proposition

For every automatic torsion-free abelian group G there is a free finite rank subgroup H such that G/H is p-divisible by almost all primes p.

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Every automatic torsion-free abelian group is the extenson of a finite rank free group by a direct sum of finitely many $\mathbb{Z}(p^\infty)$.

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