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Exercise Sheet 13

Exercises for

# Models of Set Theory II

Let $\langle I_n \mid n \in \omega \rangle$ be a (recursive) enumeration of the open intervals with rational endpoints. Let $\Gamma : \omega \times \omega \to \omega$ be the canonical bijection. We want to code every Borel set by a number $c \in \omega^\omega$:

For $c \in \omega^\omega$ let $u(c), v_i(c) \in \omega^\omega$ be defined by: If $d = u(c)$, then $d(n) = c(n+1)$ for all $n \in \omega$; if $d = v_i(c)$, then $d(n) = c(\Gamma(i,n)+1)$ for all $n \in \omega$.

For $0 < \alpha < \omega_1$, we define sets $\Sigma_\alpha$ and $\Pi_\alpha \subseteq \omega^\omega$ as follows:

$c \in \Sigma_1 \quad$ if $\quad c(0) > 1$

$c \in \Pi_\alpha - \bigcup\{\Sigma_\beta \cup \Pi_\beta \mid \beta < \alpha\} \quad$ if $\quad c(0) = 0$ and $u(c) \in \Sigma_\alpha$

$c \in \Sigma_\alpha - \bigcup\{\Sigma_\beta \cup \Pi_\beta \mid \beta < \alpha\} \quad (\alpha > 1) \quad$ if

$$c(0) = 1 \text{ and } v_i(c) \in \bigcup\{\Sigma_\beta \cup \Pi_\beta \mid \beta < \alpha\} \text{ for all } i \in \omega.$$

A $c \in \omega^\omega$ is called a Borel code if there exists $\alpha < \omega_1$ such that $c \in \Sigma_\alpha$. For every Borel code we can define recursively a Borel set $A_c$ as follows:

if $c \in \Sigma_1$ then $A_c = \bigcup\{I_n \mid c(n) = 1\}$

if $c \in \Pi_\alpha$ and $c(0) = 0$ then $A_c = \mathbb{R} - A_{u(c)}$

if $c \in \Sigma_\alpha$ and $c(0) = 1$ then $A_c = \bigcup\{A_{v_i(c)} \mid i \in \omega\}$.

49. Show that there exists for every Borel set $B$ a Borel code $c \in \omega^\omega$ such that $B = A_c$.

50. Let $M$ be a ground model and $c, d$ be in $M$ Borel codes. Let $M[G]$ be a generic extension of $M$. Show that $A_c \subseteq A_d$ holds in $M$ if and only if it holds in $M[G]$.

51. (continuation of 50) Show that $A_c = \emptyset$ holds in $M$ iff it holds in $M[G]$.

52. (continuation of 50) Show that $A_c = \mathbb{R} - A_d$ holds in $M$ iff it holds in $M[G]$.

Every problem will be graded with 8 points.

Please hand in your solutions during the lecture at February 1, 2010.