Mathematisches Institut
der Universität Bonn
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Winter 09/10
January 18, 2010
Exercise Sheet 12

Exercises for

Models of Set Theory II

A tree is a subset T of $2^{<\omega}$ such that if $t \in T$ and $s = t \restriction n$ for some $n \in \omega$ then $s \in T$. A tree is perfect if for every $t \in T$ there exists $s \supseteq t$ such that $s^\frown 0 \in T$ and $s^\frown 1 \in T$.

Let $\mathbb{P}$ (Sacks forcing) be the set of all perfect subtrees of $2^{<\omega}$ ordered by inclusion. Assume CH in the ground model M. Work in M.

Let $p \in \mathbb{P}$. A node $s \in p$ is a splitting node if both $s^\frown 0 \in p$ and $s^\frown 1 \in p$. A splitting node is an n-th splitting node if the are exactly n splitting nodes t such that $t \subseteq s$. For each $n \geq 1$, let $p \leq_n q$ iff $p \leq q$ and every n-th splitting node of q is an n-th splitting node of p.

45. A fusion sequence is a sequence of conditions $\langle p_n \mid n \in \omega \rangle$ such that $p_n \leq_n p_{n-1}$ for all $n \geq 1$. Show that if $\langle p_n \mid n \in \omega \rangle$ is a fusion sequence then $\bigcap\{p_n \mid n \in \omega\}$ is a perfect tree.

If s is a node in p, let $p \restriction s$ denote the tree $\{t \in p \mid t \subseteq s \text{ or } t \supseteq s\}$. If A is a set of incompatible nodes of p and for each $s \in A$, $q_s \subseteq p \restriction s$ is a perfect tree, then the amalgamation of $\{q_s \mid s \in A\}$ into p is the perfect tree $\{t \in p \mid$ if $t \supseteq s$ for some $s \in A$ then $t \in q_s\}$.

46. Let $\dot{F}$ be a name and $p \in \mathbb{P}$ be such that $p \Vdash (\dot{F}$ is a function from ω into the ordinals). Use amalgamation and fusion to find a $q \leq p$ such that $q \Vdash rng(\dot{F}) \subseteq \check{A}$ for some $A \in M$ with $card^M(A) = \omega$.

From now on work in V, let G be $\mathbb{P}$-generic over M and

$$a = \bigcup\{s \mid \forall p \in G \; s \in p\}.$$

47. Show that $\mathbb{P}$ preserves cardinals.

48. Show that every $f \in \omega^\omega \cap M[G]$ is dominated by some $g \in \omega^\omega \cap M$.

Every problem will be graded with 8 points.

Please hand in your solutions during the lecture at January 25, 2010.