Mathematisches Institut
der Universität Bonn
P. Koepke, B. Irrgang

Summer Semester 2009
April 29, 2009
Exercise Sheet 2

Exercises for

Models of Set Theory I

A model is a structure (M, E) where $E \subseteq M \times M$. The axiom of

Existence $\exists x \, \forall y \, \neg y \in x$

states that there exists an empty set.

5. (a) Prove that every model of *Existence*, *Extensionality*, *Pairing* and *Union* is infinite.

(b) Define a model for *Extensionality*, *Pairing*, *Union* and the negation of *Existence*.

6. Let $(M, \epsilon) \models ZF$. Let $F : M \to M$ be bijective such that there exists a formula φ and $x_1, \ldots, x_n \in M$ with $F = \{(x, y) \in M \times M \mid (M, \epsilon) \models \varphi[x, y, x_1, \ldots, x_n]\}$. For all $x, y \in M$ define $x \; \epsilon' \; y$ by $x \; \epsilon \; F(y)$.

Prove that $(M, \epsilon') \models$ *Extensionality*, *Pairing*, *Union*, *Powerset*.

7. (Continuation of problem 6) Show that:

(a) $(M, \epsilon') \models$ *Replacement*, *Infinity*

(b) F can be chosen in such a way that $(M, \epsilon') \models \neg$*Foundation*.

This shows together with exercise 4 that *Foundation* is independent of ZF.

8. Prove: For every formula $\varphi(x_1, \ldots, x_n)$, there exists a closed unbounded class $C_\varphi \subseteq Ord$ such that for each $\alpha \in C_\varphi$ and all $x_1, \ldots, x_n \in V_\alpha$

$$\varphi^{V_\alpha}(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n).$$

Every problem will be graded with 8 points.

Please hand in your solutions during the lecture at May 6, 2009.