**Theorem 1.** *For every term $G$ there is a term $F$ satisfying:*

$$G\colon V\to V \implies F\colon \mathrm{Ord}\to V \wedge \forall\alpha\, F(\alpha)=G(F\restriction\alpha).$$

*Moreover for every term $F'$:*

$$G\colon V\to V \wedge F'\colon \mathrm{Ord}\to V \wedge \forall\alpha\, F'(\alpha)=G(F'\restriction\alpha) \implies F=F'.$$

**Proof.** Set

$$F=\{(\alpha,x)|\exists f\colon\alpha\to V\wedge\forall\xi<\alpha\, f(\xi)=G(f\restriction\xi)\wedge x=G(f)\}.$$

Assume that $G\colon V\to V$.

(1) Let $F'$, $F''$ be terms with

- $F'\colon \mathrm{dom}(F')\to V,\ F''\colon \mathrm{dom}(F'')\to V$,
- $\mathrm{dom}(F')\in\mathrm{Ord}$ or $\mathrm{dom}(F')=\mathrm{Ord}$,
- $\mathrm{dom}(F'')\in\mathrm{Ord}$ or $\mathrm{dom}(F'')=\mathrm{Ord}$,
- $\forall\alpha\in\mathrm{dom}(F')\, F'(\alpha)=G(F'\restriction\alpha)$,
- $\forall\alpha\in\mathrm{dom}(F'')\, F''(\alpha)=G(F''\restriction\alpha)$.

Then

$$\forall\alpha\in\mathrm{dom}(F')\cap\mathrm{dom}(F'')\ F'(\alpha)=F''(\alpha).$$

*Proof*. Assume not and let $\alpha\in\mathrm{Ord}$ be minimal such that $\alpha\in\mathrm{dom}(F')\cap\mathrm{dom}(F'')$ and $F'(\alpha)\neq F''(\alpha)$. Then $\alpha\subseteq\mathrm{dom}(F')$ and, by replacement, $F'\restriction\alpha\in V$. Similarly, $\alpha\subseteq\mathrm{dom}(F'')$ and $F''\restriction\alpha\in V$. By the minimality of $\alpha$, $F'\restriction\alpha=F''\restriction\alpha$. Hence

$$F'(\alpha)=G(F'\restriction\alpha)=G(F''\restriction\alpha)=F''(\alpha).$$

Contradiction. *qed*(1)

Note that (1) proves the "moreover" part of the theorem.

(2) $F$ is a function.

*Proof*. Let $(\alpha,x'),(\alpha,x'')\in F$ and take $f'$, $f''$ satisfying $f'\colon\alpha\to V\wedge\forall\xi<\alpha\, f'(\xi)=G(f'\restriction\xi)\wedge x'=G(f')$ and $f''\colon\alpha\to V\wedge\forall\xi<\alpha\, f''(\xi)=G(f''\restriction\xi)\wedge x''=G(f'')$. By (1), $f'=f''$ and hence

$$x'=G(f')=G(f'')=x''.$$

*qed*(2)

(3) $\mathrm{dom}(F)$ is transitive.

*Proof*. Let $\alpha\in\mathrm{dom}(F)$ and $\beta<\alpha$. Take $f$ satisfying $f\colon\alpha\to V\wedge\forall\xi<\alpha\, f(\xi)=G(f\restriction\xi)\wedge x=G(f)$. Then $f'=f\restriction\beta$ satisfies $f'\colon\beta\to V\wedge\forall\xi<\beta\, f'(\xi)=G(f'\restriction\xi)$. $(\beta,G(f'))\in F$ and so $\beta\in\mathrm{dom}(F)$. *qed*(3)

This implies immediately

(4) $\mathrm{dom}(F)\in\mathrm{Ord}$ or $\mathrm{dom}(F)=\mathrm{Ord}$.

(5) $\forall\alpha\in\mathrm{dom}(F)\ F(\alpha)=G(F\restriction\alpha)$.

*Proof*. Assume not and let $\alpha\in\mathrm{Ord}$ be minimal such that $\alpha\in\mathrm{dom}(F)$ and $F(\alpha)\neq G(F\restriction\alpha)$. Set $f=F\restriction\alpha$. By the minimality of $\alpha$: $\forall\xi<\alpha\, f(\xi)=G(f\restriction\xi)$. By the definition of $F$, $(\alpha,G(f))\in F$ and

$$F(\alpha)=G(f)=G(F\restriction\alpha),$$

contradiction. *qed*(5)

(6) $\mathrm{dom}(F)=\mathrm{Ord}$.

*Proof*. Assume not and let $\alpha=\mathrm{dom}(F)\in\mathrm{Ord}$. Set $f=F\restriction\alpha$. By (5), $\forall\xi<\alpha\, f(\xi)=G(f\restriction\xi)$. By the definition of $F$, $(\alpha,G(f))\in F$ and $\alpha\in\mathrm{dom}(F)=\alpha$, contradiction. □