DIAMOND ON SUCCESSOR CARDINALS

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Abstract. We include a proof to the main result of Shelah's paper 922, e.g,. that for uncountable λ, ($2^\lambda = \lambda^+$ iff $\diamondsuit_{\lambda^+}$). The presentation follows a lecture given by Péter Komjáth at the HUJI seminar on 28/Dec/2007.

Theorem (Shelah)**.** *Suppose λ is a cardinal satisfying $2^\lambda = \lambda^+$.*
Then $\diamondsuit_S$ holds for any stationary $S \subseteq \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) \neq \mathrm{cf}(\lambda)\}$.

Proof. Fix a stationary set S as above. In particular, λ is uncountable. To avoid trivialities, we may also assume that $S \cap \lambda = \emptyset$ and that S contains no successor ordinals. Set $\kappa := \mathrm{cf}(\lambda)$. For each $\delta \in S$, let $\{A_i^\delta \mid i < \kappa\}$ be an increasing chain of elements of $[\delta]^{<\lambda}$ satisfying $\delta = \bigcup_{i<\kappa} A_i^\delta$.

For all $\delta \in S$, since $\mathrm{cf}(\delta) < \lambda$, we may also assume that $\sup(A_0^\delta) = \delta$.

Notation. For $X \subseteq I \times Y$ and $i \in I$, write $(X)_i = \{y \mid (i, y) \in X\}$.

Lemma 1. *Suppose $\{X_\beta \mid \beta < \lambda^+\}$ is an enumeration of $[\kappa \times (\lambda \times \lambda^+)]^{\leq\lambda}$.*
Then there exists some $i < \kappa$ such that for all $Z \subseteq \lambda \times \lambda^+$, the following is stationary:

$$S_{i,Z} := \left\{\delta \in S \mid \sup\{\alpha \in A_i^\delta \mid \exists \beta \in A_i^\delta (Z \cap (\lambda \times \alpha) = (X_\beta)_i)\} = \delta\right\}.$$

Proof. Suppose not. Then for all $i < \kappa$, we may find some $Z_i \subseteq \lambda \times \lambda^+$ and a club $D_i \subseteq \lambda^+$ that avoids S_{i,Z_i}. Define $f : \lambda^+ \to \lambda^+$ by:

$$f(\alpha) := \min\{\beta < \lambda^+ \mid X_\beta = \bigcup_{j<\kappa} \{j\} \times (Z_j \cap (\lambda \times \alpha))\}.$$

Let $D \subseteq \bigcap_{i<\kappa} D_i$ be a club such that $f(\alpha) < \delta$ for all $\alpha < \delta \in D$.

Clearly, for $\delta \in D$:

$$A_0^\delta = \{\alpha \in A_0^\delta \mid \exists \beta < \delta \forall j < \kappa (Z_j \cap (\lambda \times \alpha) = (X_\beta)_j)\}.$$

Fix $\delta \in D \cap S$. For $i < \kappa$, write:

$$B_i^\delta := \{\alpha \in A_0^\delta \mid \exists \beta \in A_i^\delta \forall j < \kappa (Z_j \cap (\lambda \times \alpha) = (X_\beta)_j)\}.$$

By $A_0^\delta = \bigcup_{i<\kappa} B_i^\delta$, $\sup A_0^\delta = \delta$ and $\mathrm{cf}(\delta) \neq \kappa$, there must exist some $i < \kappa$ with $\sup(B_i^\delta) = \delta$. In particular:

$$\sup\{\alpha \in A_i^\delta \mid \exists \beta \in A_i^\delta (Z_i \cap (\lambda \times \alpha) = (X_\beta)_i)\} = \delta,$$

i.e., $\delta \in S_{i,Z_i}$. A contradiction to $\delta \in D_i$. □

Date: January 1, 2008.

Corollary 2. *There exists a sequence $\langle A^\delta \in [\delta]^{<\lambda} \mid \delta \in S\rangle$, and an enumeration $\{X_\beta \mid \beta < \lambda^+\} = [\lambda \times \lambda^+]^{\leq\lambda}$ such that for all $Z \subseteq \lambda \times \lambda^+$, the following set is stationary:*

$$S_Z := \left\{\delta \in S \mid \sup\{\alpha \in A^\delta \mid \exists\beta \in A^\delta(Z \cap (\lambda \times \alpha) = X_\beta)\} = \delta\right\}.$$

Proof. Take i as above, and consider $\langle A^\delta_i \mid \delta \in S\rangle$ and $\{(X_\beta)_i \mid \beta < \lambda^+\}$. $\square$

Let us fix such sequence $\langle A^\delta \mid \delta \in S\rangle$ and enumeration $\{X_\beta \mid \beta < \lambda^+\}$.

We shall now recursively define a sequence of subsets of λ^+, $\langle Y_\tau \mid \tau < \lambda\rangle$, and a $\subseteq$-decreasing sequence of clubs of λ^+, $\langle E_\tau \mid \tau < \lambda\rangle$.

Notation. Whenever $\langle Y_\tau \mid \tau < \gamma\rangle$ is defined, we shall denote for $\delta \in S$:

$$V^\delta_\gamma = \left\{(\alpha,\beta) \in A^\delta \times A^\delta \mid \forall\tau < \gamma\,(Y_\tau \cap \alpha = (X_\beta)_\tau)\right\}.$$

We start the recursion by letting $E_0 = Y_0 = \lambda^+$. Suppose now $\langle (Y_\tau, E_\tau) \mid \tau < \gamma\rangle$ has been defined for some $\gamma < \lambda$. Clearly, for any set Y_γ, and any $\delta \in S$, we would have $V^\delta_\gamma \supseteq V^\delta_{\gamma+1}$. If there exists a set $Y_\gamma \subseteq \lambda^+$ and a club $E_\gamma \subseteq \bigcap_{\tau<\gamma} E_\tau$ such that for all $\delta \in E_\gamma \cap S$:

$$\sup\{\alpha < \delta \mid \exists\beta < \delta\left((\alpha,\beta) \in V^\delta_\gamma\right)\} = \delta \text{ implies } V^\delta_\gamma \neq V^\delta_{\gamma+1},$$

then continue the recursion with such Y_γ and E_γ. Otherwise, terminate the recursion.

Claim 3. *The recursion must terminate at some $\gamma^* < \lambda$.*

Proof. Suppose not, and let $\langle Y_\tau \mid \tau < \lambda\rangle$, $\langle E_\tau \mid \tau < \lambda\rangle$ be the output sequences. Put $E = \bigcap_{\tau<\lambda} E_\tau$ and $Z = \bigcup_{\tau<\lambda}\{\tau\} \times Y_\tau$.

Fix $\delta \in E \cap S_Z$. Then by definition of S_Z:

$$\sup\{\alpha \in A^\delta \mid \exists\beta \in A^\delta(Z \cap (\lambda \times \alpha) = X_\beta)\} = \delta,$$

In other words:

$$\sup\{\alpha \in A^\delta \mid \exists\beta \in A^\delta\forall\tau < \lambda(Y_\tau \cap \alpha = (X_\beta)_\tau)\} = \delta.$$

It follows that $\sup\{\alpha < \delta \mid \exists\beta < \delta\left((\alpha,\beta) \in V^\delta_\gamma\right)\} = \delta$ for all $\gamma < \lambda$. Since $S_Z \subseteq S$, the recursive construction gives that $\langle V^\delta_\gamma \mid \gamma < \lambda\rangle$ is a strictly $\subseteq$-decreasing sequence of subsets $A^\delta \times A^\delta$, contradicting the fact that $|A^\delta| < \lambda$. $\square$

Thus, let γ^* be the point at which the recursion terminates, and let $\langle Y_\tau \mid \tau < \gamma^*\rangle$,$\langle E_\tau \mid \tau < \gamma^*\rangle$ be the resulted sequences. Set $E = \bigcap_{\tau<\gamma^*} E_\tau$.

For every $\delta \in S \cap E$, put:

$$S_\delta := \bigcup\{(X_\beta)_{\gamma^*} \mid (\alpha,\beta) \in V^\delta_{\gamma^*}\}.$$

Claim 4. $\{S_\delta \mid \delta \in E \cap S\}$ *exemplify* $\diamondsuit_S$.

Proof. Assume towards a contradiction that there exists a set $Y \subseteq \lambda^+$ and a club $C \subseteq E$ such that $S_\delta \neq Y \cap \delta$ for all $\delta \in C \cap S$.

Following the notation of the recursion, write $Y_{\gamma^*} := Y$.

Let $Z = \bigcup_{\tau \leq \gamma^*} \{\tau\} \times Y_\tau$. Then, for $\delta \in C \cap S_Z$, we have:

$$\sup\{\alpha \in A^\delta \mid \exists \beta \in A^\delta \forall \tau \leq \gamma^* (Y_\tau \cap \alpha = (X_\beta)_\tau)\} = \delta.$$

So, $\sup\{\alpha < \delta \mid \exists \beta < \delta \left((\alpha, \beta) \in V^\delta_{\gamma^*}\right)\} = \delta$, and also:

$$Y \cap \delta = \bigcup \{(X_\beta)_{\gamma^*} \mid (\alpha, \beta) \in V^\delta_{\gamma^*+1}\}.$$

It follows that if $V^\delta_{\gamma^*+1} = V^\delta_{\gamma^*}$, then $Y \cap \delta = S_\delta$. However, by the choice of Y and $\delta \in C$, this is not the case, i.e., $V^\delta_{\gamma^*+1} \neq V^\delta_{\gamma^*}$.

But if $\sup\{\alpha < \delta \mid \exists \beta < \delta \left((\alpha, \beta) \in V^{\delta^*}_\gamma\right)\} = \delta$ and $V^\delta_{\gamma^*+1} \neq V^\delta_{\gamma^*}$ for all $\delta \in S \cap C$, this means that the recursion could have been continued using Y and C, while it was terminated at γ^*. A contradiction. $\square$

$\square$

Remark. To see that the above theorem is optimal, we mention the following two results concerning successors of regular and singular cardinals.

Theorem (Shelah). $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ *is consistent with the failure of* $\diamondsuit_S$ *for* $S = \{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \aleph_1\}$.

Theorem (Magidor). *Assume GCH and that κ is a measurable cardinal.*

In the generic extension of prikry forcing, GCH holds, κ^+ is a successor of a singular cardinal of countable cofinality, and $\diamondsuit_S$ fails for some stationary $S \subseteq \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \aleph_0\}$.

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