Mengenlehre II, SS 2007

Übungsaufgaben I

Exercise 1. Assume that a class $A \neq \emptyset$ is closed with respect to definite terms, i.e., whenever $t(\vec{v})$ is a definite term then $\forall \vec{a} \in A\, t(\vec{a}) \in A$. Then

a) Show that A is transitive.

b) Provide a (long) list of ZF-axioms which hold in A.

c) Show that every definite formula is absolute with respect to A.

Exercise 2. Is there a uniform way to define (in ZF) for a given class A the *definite closure* of A, i.e., the smallest class containing A which is closed with respect to definite terms.

Exercise 3. Assume that $(W_\alpha | \alpha \in \mathrm{Ord})$ is a continuous hierarchy. Prove

$$\forall a \subseteq \bigcup_{\alpha \in \mathrm{Ord}} W_\alpha \, \exists \theta \, a \subseteq W_\theta .$$

Exercise 4. a) Let φ_0, ..., φ_{n-1} be a finite collection of ZF-axioms. Use the reflection theorem to prove that there is θ such that

$$\varphi_0^{V_\theta}, \ldots, \varphi_{n-1}^{V_\theta} .$$

b) Use this to show that the theory ZF cannot be finitely axiomatized if it is consistent.

Übungsaufgaben II

Exercise 5. Prove that every L_α is closed under the operations

a) $x, y \mapsto x \cup y, x \cap y, x \setminus y$

b) $x \mapsto \bigcup x$

c) $x \mapsto \mathrm{TC}(x)$, where $\mathrm{TC}(x)$ is the *transitive hull* of x.

Exercise 6. Prove that

a) $<_n$ is a wellorder of V_n;

b) $<_{n+1}$ end-extends $<_n$;

c) $<_\omega$ is a wellorder of V_ω.

Exercise 7. Prove

a) If r is a strict linear ordering then r^{lex} is a strict linear ordering on ${}^{<\mathrm{Ord}}A$.

b) If (A, r) is a wellorder and $n < \omega$ then r^{lex} is a wellorder of ${}^nA$.

c) If A has more than one element than r^{lex} does not wellorder ${}^\omega A$.

Exercise 8.

a) $<_\alpha$ and $\tilde{<}_\alpha$ are well-defined

b) $\tilde{<}_\alpha$ is a wellordering of $\tilde{L}_\alpha$

c) $<_\alpha$ is a wellordering of L_α

d) $\beta < \alpha$ implies that $\tilde{<}_\beta$ is an initial segment of $\tilde{<}_\alpha$

e) $\beta < \alpha$ implies that $<_\beta$ is an initial segment of $<_\alpha$

Übungsaufgaben III

Exercise 9. a) Show that every countable ordinal can be embedded order-preservingly into the rational numbers $(\mathbb{Q}, <)$. b) Show that $\aleph_1$ cannot be embedded order-preservingly into the real numbers $(\mathbb{R}, <)$.

Exercise 10. H_{ω_1} is the set of all *hereditarily countable* sets. Show that H_{ω_1} is a ZF^--model.

Exercise 11. Let $\varphi = \exists \vec{x}\ \psi(\vec{x})$ be a statement where $\psi(\vec{x})$ is a definite formula. Assume that φ holds in V. Show that φ holds in H_{ω_1}. [Hint: Transitivize a countable substructure of $(V, \in)$.]

Exercise 12. Let S and P be the ordinal *successor* and *predecessor* function, resp.: $S(\xi) = \xi + 1$ and

$$P(\xi) = \begin{cases} \zeta, \text{ if } \xi = \zeta + 1 \\ \xi, \text{ else} \end{cases}$$

Let $\alpha \in \mathrm{Ord}$ and let $X \subseteq \alpha$ be closed with respect to the (partial) functions S and P. Show that there is a unique $\beta \leqslant \alpha$ such that

$$(\beta, <, S \cap \beta^3, P) \cong (X, <, S \cap \alpha^3, P).$$

Übungsaufgaben IV

Exercise 13. Show that every countable strict linear ordering can be order-embedded into $(\mathbb{Q}, <_{\mathbb{Q}})$.

Exercise 14. Carry out the construction of an Aronszajn tree using the standard rational numbers $(\mathbb{Q}, <)$.

Exercise 15. From an Aronszajn tree construct an *Aronszajn line*, i.e., an uncountable strict linear order $(A, <_A)$ such that the following holds:

- $(\omega_1, <)$ cannot be order-embedded into $(A, <_A)$;
- $(\omega_1, >)$ cannot be order-embedded into $(A, <_A)$;
- $(X, <_{\mathbb{R}})$ cannot be order-embedded into $(A, <_A)$ for every uncountable $X \subseteq \mathbb{R}$.

Exercise 16. Assume $V = L$. Show that for any infinite cardinal κ

$$H_\kappa = L_\kappa$$

where $H_\kappa = \{x | \mathrm{card}(\mathrm{TC}(x)) < \kappa\}$.

Übungsaufgaben V

Exercise 17. Let $\theta > \omega_1$ be a regular cardinal. Then

a) If $\mu < \theta$ is regular then $\{\alpha \in \theta | \mathrm{cof}(\alpha) = \mu\}$ is stationary in θ.

b) If $W \subseteq \theta$ is stationary and $f: W \to \eta$ for some $\eta < \theta$ then there is some $\nu < \eta$ such that $\{\alpha \in W | f(\alpha) = \nu\}$ is stationary in θ.

Exercise 18. Show that the principle Global Square implies $\square_\kappa$ for every infinite cardinal κ.

Exercise 19. Show that the principle $\diamondsuit$ is equivalent to: there exists a sequence $(R_\alpha | \alpha < \omega_1)$ such that $\forall R \subseteq \omega_1 \, \exists \alpha < \omega_1 \, (\alpha \neq 0 \wedge R \cap \alpha = R_\alpha)$.

Exercise 20. Consider forcing with the partial order

$$P = \{p | \beta < \omega_1 \, (p: \beta \to \mathcal{P}(\omega_1) \wedge \forall \alpha < \beta \, p(\alpha) \subseteq \alpha)\},$$

partially ordered by reverse inclusion. Show that in a generic extension by $(P, \supseteq)$ the principle $\diamondsuit$ holds. [Hint: use exercise 19]