Definite Notions

We shall study transitive $\in$-structures $(M, \in)$ where M is a non-empty transitive set or class. We are mainly interested in situations where $(M, \in)$ is a model of ZF^-, i.e., φ^M holds for every axiom in ZF^-. We want to show that many properties are *absolute* between $(M, \in)$ and the set-theoretical universe V.

Definition 1. *Let $\psi(\vec{v})$ be an $\in$-formula and let $t(\vec{v})$ be a term, both in the free variables $\vec{v}$. Then*

a) ψ is definite *iff for every transitive* ZF^-*-model $(M, \in)$*

$$\forall \vec{x} \in M\, (\psi^M(\vec{x}) \leftrightarrow \psi(\vec{x})).$$

b) t is definite *iff for every transitive* ZF^-*-model $(M, \in)$*

$$\forall \vec{x} \in M\, t^M(\vec{x}) \in M \text{ and } \forall \vec{x} \in M\, t^M(\vec{x}) = t(\vec{x}).$$

Recall that if t is of the form $t = \{u \,|\, \varphi\}$ then $t^M = \{u \in M \,|\, \varphi^M\}$; for $t = x$ a variable term, set $x^M = x$. We shall prove that a majority of set-theoretical notions are definite. We shall shall work inductively: some basic notions are definite and many set-theoretical operations lead from definite notions to definite notions.

Lemma 2. *Let $\varphi(x, \vec{y})$ be a formula and $t(\vec{z})$ be a term and M be a class. Assume that $\forall \vec{z} \in M\, t(\vec{z}) \in M$. Then*

$$\forall \vec{y}, \vec{z} \in M\, (\varphi(t(\vec{z}), \vec{y}))^M \leftrightarrow \varphi^M(t^M(\vec{z}), \vec{y})).$$

Proof. If $t = t(\vec{z})$ is of the form $t = z$ then there is nothing to show. Assume otherwise that t is of the form $t = \{u|\psi(u, \vec{z})\}$. We work by induction on the complexity of φ. Assume that $\varphi \equiv x = y$ and $y, \vec{z} \in M$. Then

$$\begin{aligned}
(t(\vec{z}) = y)^M &\leftrightarrow (\{u|\psi(u, \vec{z})\} = y)^M \\
&\leftrightarrow (\forall u\, (\psi(u, \vec{z}) \leftrightarrow u \in y))^M \\
&\leftrightarrow \forall u \in M\, (\psi^M(u, \vec{z}) \leftrightarrow u \in y) \\
&\leftrightarrow \{u \in M | \psi^M(u, \vec{z})\} = y \\
&\leftrightarrow t^M(\vec{z}) = y \\
&\leftrightarrow \varphi^M(t^M(\vec{z}), y)
\end{aligned}$$

Assume that $\varphi \equiv y \in x$ and $y, \vec{z} \in M$. Then

$$\begin{aligned}
(y \in t(\vec{z}))^M &\leftrightarrow \psi^M(\frac{y}{u}, \vec{z}) \\
&\leftrightarrow y \in \{u \in M | \psi^M(u, \vec{z})\} \\
&\leftrightarrow y \in t^M(\vec{z}) \\
&\leftrightarrow \varphi^M(t^M(\vec{z}), y)
\end{aligned}$$

Assume that $\varphi \equiv x \in y$ and $y, \vec{z} \in M$. Then

$$\begin{aligned}
(t(\vec{z}) \in y)^M &\leftrightarrow (\exists u\, (u = t(\vec{z}) \wedge u \in y)^M \\
&\leftrightarrow \exists u \in M\, ((u = t(\vec{z}))^M \wedge u \in y) \\
&\leftrightarrow \exists u \in M\, (u = t^M(\vec{z}) \wedge u \in y), \text{ by the first case,} \\
&\leftrightarrow \exists u\, (u = t^M(\vec{z}) \wedge u \in y), \text{ since } M \text{ is closed w.r.t. } t, \\
&\leftrightarrow t^M(\vec{z}) \in y \\
&\leftrightarrow \varphi^M(t^M(\vec{z}), y)
\end{aligned}$$

The induction steps are obvious. □

Theorem 3.

a) The formulas $x = y$ and $x \in y$ are definite.

b) If the formulas φ and ψ are definite then so are $\neg\varphi$ and $\varphi \wedge \psi$.

c) Let the formula $\varphi(x, \vec{y})$ and the term $t(\vec{z})$ be definite. Then so are $\varphi(t(\vec{z}), \vec{y})$ and $\forall x \in t(\vec{z})\, \varphi(x, \vec{y})$.

d) The formulas $\mathrm{Trans}(x)$, $\mathrm{Ord}(x)$, $\mathrm{Succ}(x)$, and $\mathrm{Lim}(x)$ are definite.

e) Let the terms $t(x, \vec{y})$ and $r(\vec{z})$ be definite. Then so is $t(r(\vec{z}), \vec{y})$.

f) The terms x, $\emptyset$, $\{x, y\}$, $\bigcup x$ and ω are definite.

g) Let the formula $\varphi(x, \vec{y})$ be definite. Then so is the term $\{x \in z \,|\, \varphi(x, \vec{y})\}$.

h) Let the term $t(x, \vec{y})$ be definite. Then so is the term $\{t(x, \vec{y}) | x \in z\}$.

Proof. Let M be a transitive ZF^--model.
a) is obvious since $(x = y)^M = (x = y)$ and $(x \in y)^M = (x \in y)$.
b) Assume that φ and ψ are definite and that $(M, \in)$ is a transitive ZF^--model. Then $\forall \vec{x} \in M\, (\varphi^M(\vec{x}) \leftrightarrow \varphi(\vec{x}))$ and $\forall \vec{x} \in M\, (\psi^M(\vec{x}) \leftrightarrow \psi(\vec{x}))$. Thus

$$\forall \vec{x} \in M\, ((\varphi \wedge \psi)^M(\vec{x}) \leftrightarrow (\varphi^M(\vec{x}) \wedge \psi^M(\vec{x})) \leftrightarrow (\varphi(\vec{x}) \wedge \psi(\vec{x})) \leftrightarrow (\varphi \wedge \psi)(\vec{x})).$$

A similar argument works for $\neg\varphi$.
c) Let $(M, \in)$ be a transitive ZF^--model. Let $\vec{y}, \vec{z} \in M$. $t(\vec{z}) \in M$ since t is definite. Then

$$\begin{aligned}
(\varphi(t(\vec{z}), \vec{y}))^M &\leftrightarrow \varphi^M(t^M(\vec{z}), \vec{y}), \text{ by the Lemma,}\\
&\leftrightarrow \varphi^M(t(\vec{z}), \vec{y}), \text{ since } t \text{ is definite,}\\
&\leftrightarrow \varphi(t(\vec{z}), \vec{y}), \text{ since } \varphi \text{ is definite.}
\end{aligned}$$

Also

$$\begin{aligned}
(\forall x \in t(\vec{z})\, \varphi(x, \vec{y}))^M &\leftrightarrow (\forall x\, (x \in t(\vec{z}) \to \varphi(x, \vec{y})))^M\\
&\leftrightarrow \forall x \in M\, ((x \in t(\vec{z}))^M \to \varphi^M(x, \vec{y}))\\
&\leftrightarrow \forall x \in M\, (x \in t^M(\vec{z}) \to \varphi^M(x, \vec{y}))\\
&\leftrightarrow \forall x \in M\, (x \in t(\vec{z}) \to \varphi(x, \vec{y})), \text{ since } t \text{ and } \varphi \text{ are definite,}\\
&\leftrightarrow \forall x\, (x \in t(\vec{z}) \to \varphi(x, \vec{y})), \text{ since } t(\vec{z}) \subseteq M,\\
&\leftrightarrow \forall x \in t(\vec{z})\, \varphi(x, \vec{y})).
\end{aligned}$$

d) follows immediately from c).

e) is obvious.

f) A variable term x is trivially definite, since $x^M = x$.

Consider the term $\emptyset = \{u | u \neq u\}$. Since M is non-empty and transitive, $\emptyset \in M$. Also

$$\emptyset^M = \{u \in M | u \neq u\} = \emptyset.$$

Consider the term $\{x,y\}$. For $x,y\in M$:

$$\{x,y\}^M=\{u\in M\,|\,u=x\vee u=y\}=\{u|u=x\vee u=y\}=\{x,y\}.$$

The pairing axiom in M states that

$$(\forall x,y\exists z\; z=\{x,y\})^M.$$

This implies

$$\forall x,y\in M\exists z\in M\,z=\{x,y\}^M=\{x,y\}$$

and

$$\forall x,y\in M\;\{x,y\}\in M.$$

Consider the term $\bigcup x$. For $x\in M$:

$$(\bigcup x)^M=\{u\in M\,|\,(\exists v\in x\;u\in v)^M\}=\{u\in M\,|\,\exists v\in x\cap M\,u\in v\}=\{u\,|\,\exists v\in x\;u\in v\}=\bigcup x.$$

The union axiom in M states that

$$(\forall x\exists z\;z=\bigcup x)^M.$$

This implies

$$\forall x\in M\exists z\in M\;z=(\bigcup x)^M=\bigcup x$$

and

$$\forall x\in M\;\bigcup x\in M.$$

Consider the term $\omega=\bigcap\{x|x \text{ is inductive}\}$. Since M satisfies the axiom of infinity,

$$\exists x\in M\,(x=\omega)^M.$$

Take $x_0\in M$ such that $(x_0=\omega)^M$. Then $(\mathrm{Lim}(x_0))^M$, $(\forall y\in x_0\;\neg\mathrm{Lim}(y))^M$. By definiteness, $\mathrm{Lim}(x_0)$, $\forall y\in x_0\;\neg\mathrm{Lim}(y)$, i.e., x_0 is equal to the smallest limit ordinal ω. Hence $\omega\in M$. The formula "x is inductive" has the form

$$\emptyset\in x\wedge\forall y\in x\bigcup\{y,\{y\}\}\in x$$

and is definite by previous considerations. Now

$$\begin{aligned}
\omega^M &= (\bigcap\{x|x\text{ is inductive}\})^M\\
&= (\{y|\forall x(x\text{ is inductive}\rightarrow y\in x)\})^M\\
&= \{y\in M|\forall x\in M\,(x\text{ is inductive}\rightarrow y\in x)\},\text{ since "}x\text{ is inductive" is definite,}\\
&= \bigcap\{x\in M|x\text{ is inductive}\}\\
&= \bigcap\{x\cap\omega|x\in M\text{ is inductive}\},\text{ since }\omega\in M,\\
&= \bigcap\{\omega\},\text{ since }\omega\text{ is the smallest inductive set,}\\
&= \omega.
\end{aligned}$$

g) Let $\vec{y},z\in M$. By the separation schema in M,

$$(\exists w\;w=\{x\in z\,|\,\varphi(x,\vec{y})\})^M,$$

i.e., $\{x \in z \mid \varphi(x, \vec{y}\,)\}^M \in M$. Moreover by the definiteness of φ

$$\{x \in z \mid \varphi(x, \vec{y}\,)\}^M = \{x \in M \mid x \in y \wedge \varphi^M(x, \vec{y}\,)\} = \{x \mid x \in y \wedge \varphi(x, \vec{y}\,)\} = \{x \in z \mid \varphi(x, \vec{y}\,)\}.$$

h) Since t is definite, $\forall x, \vec{y} \in M\, t^M(x, \vec{y}\,) \in M$. This implies

$$\forall x, \vec{y} \in M \exists w \in M\, w = t^M(x, \vec{y}\,)$$

and $(\forall x, \vec{y}\, \exists w\; w = t(x, \vec{y}\,))^M$. Let $\vec{y}\,, z \in M$. By replacement in M,

$$(\exists a\; a = \{t(x, \vec{y}\,) | x \in z\})^M.$$

Hence $\{t(x, \vec{y}\,) | x \in z\}^M \in M$. Moreover

$$\begin{aligned}
\{t(x, \vec{y}\,) | x \in z\}^M &= \{w | \exists x \in z\; w = t(x, \vec{y}\,)\}^M \\
&= \{w \in M \mid \exists x \in z\; w = t^M(x, \vec{y}\,)\} \\
&= \{w \mid \exists x \in z\; w = t^M(x, \vec{y}\,)\}, \text{ since } M \text{ is closed w.r.t. } t^M, \\
&= \{w \mid \exists x \in z\; w = t(x, \vec{y}\,)\}, \text{ since } t \text{ is definite,} \\
&= \{t(x, \vec{y}\,) | x \in z\}.
\end{aligned}$$

$\square$

We may view this theorem as a “definite” form of the ZF^--axioms: common notions and terms of set theory and mathematics are definite, and natural operations lead to further definite terms. Since the recursion principle is so important, we shall need a definite recursion schema:

Theorem 4. *Let $G(w, \vec{y}\,)$ be a definite term, and let $F(\alpha, \vec{y}\,)$ be the canonical term defined by* Ord*-recursion with G:*

$$\forall \alpha\, F(\alpha, \vec{y}\,) = G(F \upharpoonright \alpha, \vec{y}\,).$$

Then the term $F(\alpha)$ is definite.

Proof. Let M be a transitive ZF^--model. $\square$