Mathematical Logic. An Introduction
Summer 2006

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1 Introduction

Mathematics models real world phenomena like space, time, number, probability, games, etc. It proceeds by rigorous arguments from initial assumptions to conclusions. Its results are "universal", or "logically valid", in that they do not depend on external or implicit conditions which may change with nature or society.

It is remarkable that mathematics is also able to model itself: Mathematical logic defines rigorously what mathematical statements and rigorous arguments are. The mathematical enquiry into the mathematical method leads to deep insights into mathematics, applications to classical field of mathematics, and to new mathematical theories. The study of mathematical language has also influenced the study of formal and natural languages in computer science, linguistics and philosophy.

1.1 A simple proof

We want to indicate that rigorous mathematical proofs can be obtained by a sequence of syntactic manipulations of mathematical statements. Let us consider a fragment of the elementary theory of functions which expresses that the composition of two surjective maps is surjective as well:

> Let f and g be surjective, i.e., for all y there is x such that $y = f(x)$, and for all elements y there is x such that $y = g(x)$.
>
> Theorem. $g \circ f$ is surjective, i.e., for all y there is x such that $y = g(f(x))$.
>
> Proof. Consider any y. Choose z such that $y = g(z)$. Choose x such that $z = f(x)$. Then $y = g(f(x))$. Thus there is x such that $y = g(f(x))$. Thus for all y there is x such that $y = g(f(x))$.
>
> Qed.

These statements and arguments are expressed in an austere and systematic language, which can be normalized even further. Common abbreviations stand for certain figures of language:

Let $\forall y \exists x\, y = f(x)$.
Let $\forall y \exists x\, y = g(x)$.

Theorem. $\forall y \exists x\, y = g(f(x))$.
Proof. Consider y.
$\exists z\, y = g(z)$.
Let $y = g(z)$.
$\exists x\, z = f(x)$.
Let $z = f(x)$.
$y = g(f(x))$.
Thus $\exists x\, y = g(f(x))$.
Thus $\exists x\, y = g(f(x))$.
Thus $\forall y \exists x\, y = g(f(x))$.
Qed.

The lines in this text can be considered as formal sequences of symbols. Certain sequences of symbols are acceptable as mathematical formulas. There are logical rules with allow the transfer from certain formulas to other ones. These rules have a purely formal character and they can be applied irrespectively of the "meaning" of the symbols and formulas.

1.2 Formal proofs

In the example, $\exists x\, y = g(f(x))$ is inferred from $y = g(f(x))$. The rule of *existential quantification*: "put $\exists x$ in front of a formula" can be applied in many circumstances and it is comparable to a kind of left-multiplication of $y = g(f(x))$ by $\exists x$.

$$\exists x\,,\, g(f(x)) \mapsto \exists x\, g(f(x)).$$

It is conceivable that logical rules satisfy certain algebraic laws like associativity. Another interesting operation is *substitution*: From $y = g(z)$ and $z = f(x)$ we inferred $y = g(f(x))$ by a "find-and-replace"-substitution of z by $f(x)$.

Given a sufficient collection of rules, the above sequence of formulas, involving "keywords" like "Let" and "Thus" can be a deduction or derivation in which every line is generated from earlier ones by one of the rules. Mathematical results may be provable simply by the application of formal rules. In analogy with the formal rules of the infinitesimal calculus one calls such a system of rules a *calculus*.

1.3 Syntax and semantics

Obviously we do not just want to describe a formal derivation as a kind of domino but we want to *interpret* the occuring symbols as mathematical objects. Thus we have to let variables $x, y, \ldots$ range over some domain like the real numbers $\mathbb{R}$ and let f and g stand for functions $F, G\colon \mathbb{R} \rightarrow \mathbb{R}$. Observe that the symbol or "name" f is not identical with the function F, and indeed f might also be interpretated as another function F'. To emphasize the distinction between names and objects, we consider symbols, formulas and derivations as *syntax* whereas the interpretations of symbols belong to the realm of *semantics*.

By interpreting $x, y, \ldots$ and f, g in a structure like $(\mathbb{R}, F, G)$ we can define straightforwardly whether a formula like $\exists x\, g(f(x))$ is *satisfied* in the structure. A formula is *logically valid* if it is satisfied under *all* interpretations. The fundamental theorem of mathematical logic and the central result of this course is Gödel's completeness theorem:

Theorem 1. *There is a calculus with finitely many rules such that a formula is derivable in the calculus iff it is logically valid.*

1.4 Set theory

In modern mathematics notion can usually be reduced to set theoretic notions: non-negative integers correspond to cardinalities of finite sets, integers can be obtained via pairs of non-negative integers, rationals via pairs of integers, and real numbers via subsets of the rationals. Geometric notions can be defined from real numbers using analytic geometry. The basic set theoretical axioms can be formulated in the logical language indicated above.

This shows that the mathematical method can be understood abstractly as

mathematics = (first-order) logic + set theory.

1.5 Course overview

We shall cover the following topics:

1. Words
2. Calculi
3. Induction and recursion on calculi
4. Terms and formulas
5. Structures
6. The satisfaction relation
7. Logical implication and propositional connectives
8. Substitution and quantification rules
9. A sequent calculus

10. Examples of formal proofs

2 Words

The languages of mathematical logic can be defined within the framework of finite sequences.

Definition 2. *Let* $\mathbb{A}$ *be a non-empty set.* w *is a* finite sequence *over* $\mathbb{A}$ *if there is* $n\in\mathbb{N}$ *such that* w *is a function satisfying* $w\colon\{0, 1, ..., n-1\}\rightarrow\mathbb{A}$. *For* w *a finite sequence over* $\mathbb{A}$ *with* $\mathrm{dom}(w)=\{0, 1, ..., n-1\}$ *call* n *the* length *of* w. *For* w *a finite sequence of length* n *write* $w(0)...w(n-1)$ *instead of* w. *We also write* $w_0...w_{n-1}$ *for* $w(0)...w(n-1)$.

We also say that $\mathbb{A}$ *is an* alphabet *and call finite sequences over* $\mathbb{A}$ words *over* $\mathbb{A}$*; a* symbol *is an element of the alphabet* $\mathbb{A}$. *Let* $\mathbb{A}^*$ *be the set of all words over* $\mathbb{A}$. *The* empty sequence *or the* empty word *is the empty set* $\emptyset$.

Note that in our convention, w_0 may denote the symbol w_0 as well as the length-1 word w_0 . This ambiguity will usually pose no problem in concrete situations.

Definition 3. *For words* $w=w_0...w_{m-1}$ *and* $w'=w_0'...w_{n-1}'$ *let* $w\,\hat{}\,w'=w_0...w_{m-1}w_0'...w_{n-1}'$ *be the* concatenation *of* w *and* w'. *One can also define the word* $w\,\hat{}\,w'\colon\{0,1,...,m+n-1\}\rightarrow V$ *by*

$$w\,\hat{}\,w'(i)=\begin{cases} w(i), \text{ if } i<m\\ w'(i-m), \text{ if } i\geqslant m\end{cases}$$

We also write ww' *instead of* $w\,\hat{}\,w'$.

Note that V stands for the class of all mathematical objects, the *universe*. We prove that $\hat{}$ is an associative operation on words:

Theorem 4. *For words* w, w', w'' *over* $\mathbb{A}$ *holds*

a) $(w\,\hat{}\,w')\,\hat{}\,w''=w\,\hat{}\,(w'\,\hat{}\,w'')$

b) $\emptyset\,\hat{}\,w=w\,\hat{}\,\emptyset=w$

c) $w\,\hat{}\,w'=w\,\hat{}\,w''\rightarrow w'=w''$

d) $w'\,\hat{}\,w=w''\,\hat{}\,w\rightarrow w'=w''$

This means that the set $\mathbb{A}$ of words together with $\hat{}$ form a *monoid* which also satisfies the cancellation rules c) and d).

Proof. a) Let $n,n',n''\in\mathbb{N}$ such that $w=w_0...w_{n-1}$, $w'=w_0'...w_{n'-1}'$, $w''=w_0''...w_{n''-1}''$. Then

$$\begin{aligned}(w\,\hat{}\,w')\,\hat{}\,w'' &= (w_0...w_{n-1}w_0'...w_{n'-1}')\,\hat{}\,w_0''...w_{n''-1}''\\ &= w_0...w_{n-1}w_0'...w_{n'-1}'w_0''...w_{n''-1}''\\ &= w_0...w_{n-1}\,\hat{}\,(w_0'...w_{n'-1}'w_0''...w_{n''-1}'')\\ &= w_0...w_{n-1}\,\hat{}\,(w_0'...w_{n'-1}'\,\hat{}\,w_0''...w_{n''-1}'')\\ &= w\,\hat{}\,(w'\,\hat{}\,w'').\end{aligned}$$

The trouble with this proof is the intuitive but somewhat vague use of the ellipses "...". In mathematical logic we are particularly attentative to such vagueness. It can be avoided as follows. In set theory, the natural number n is defined as the set $\{0,1,...,n-1\}$:

$$n=\{0,1,...,n-1\}.$$

This means that

$$\begin{aligned}0 &= \{0,...,-1\}=\emptyset\\ 1 &= \{0,...,0\}=\{0\}\\ 2 &= \{0,1\}\\ 3 &= \{0,1,2\}\\ &\vdots\end{aligned}$$

If $w = w_0 \ldots w_{n-1}$ is a word of length n then $n = \mathrm{dom}(w)$. So let, again, w, w', w'' be words, and let $n = \mathrm{dom}(w)$, $n' = \mathrm{dom}(w')$, $n'' = \mathrm{dom}(w'')$. Then

$$\begin{aligned}\mathrm{dom}(w\hat{\ }w') &= n + n' \\ \mathrm{dom}((w\hat{\ }w')\hat{\ }w'') &= n + n' + n'' \\ \mathrm{dom}(w'\hat{\ }w'') &= n' + n'' \\ \mathrm{dom}(w\hat{\ }(w'\hat{\ }w'')) &= n + n' + n''\end{aligned}$$

To show that $(w\hat{\ }w')\hat{\ }w'' = w\hat{\ }(w'\hat{\ }w'')$ we have to show that for all $i < n + n' + n''$ holds

$$((w\hat{\ }w')\hat{\ }w'')(i) = (w\hat{\ }(w'\hat{\ }w''))(i).$$

Let $i < n + n' + n''$.
Case 1: $i < n$. Then

$$\begin{aligned}((w\hat{\ }w')\hat{\ }w'')(i) &= (w\hat{\ }w')(i) \\ &= w(i) \\ &= (w\hat{\ }(w'\hat{\ }w''))(i).\end{aligned}$$

Case 2: $n \leqslant i < n + n'$. Then

$$\begin{aligned}((w\hat{\ }w')\hat{\ }w'')(i) &= (w\hat{\ }w')(i) \\ &= w'(i - n) \\ &= (w'\hat{\ }w'')(i - n) \\ &= (w\hat{\ }(w'\hat{\ }w''))(i).\end{aligned}$$

Case 3: $n + n' \leqslant i < n + n' + n''$. Then

$$\begin{aligned}((w\hat{\ }w')\hat{\ }w'')(i) &= w''(i - (n + n')) \\ &= w'\hat{\ }w''(i - (n + n') + n') = w'\hat{\ }w''(i - n) \\ &= (w\hat{\ }(w'\hat{\ }w''))(i - n + n) \\ &= (w\hat{\ }(w'\hat{\ }w''))(i).\end{aligned}$$

Thus $((w\hat{\ }w')\hat{\ }w'')(i) = (w\hat{\ }(w'\hat{\ }w''))(i)$ holds in all cases. $\square$

3 Calculi

Let us fix a non-empty alphabet $\mathbb{A}$. We want to express abstractly how words like $\exists x\, y = g(f(x))$ can be obtained from words like $y = g(f(x))$.

Definition 5. *A relation $R \subseteq (\mathbb{A}^*)^n \times \mathbb{A}^*$ is called a* rule *(over $\mathbb{A}$). A* calculus *(over $\mathbb{A}$) is a set $\mathcal{C}$ of rules (over $\mathbb{A}$).*

A rule is often indicated as a *production rule* of the form

$$\frac{\text{arguments}}{\text{production}} \quad \text{or} \quad \frac{\text{preconditions}}{\text{conclusion}}.$$

For the above existential quantification we may for example write

$$\frac{\varphi}{\exists x \varphi}$$

where the production is the concatenation of $\exists x$ and φ.

Definition 6. *Let $\mathcal{C}$ be a calculus over $\mathbb{A}$. Let $R \subseteq (\mathbb{A}^*)^n \times \mathbb{A}^*$ be a rule of $\mathcal{C}$. For $X \subseteq \mathbb{A}^*$ set*

$$R[X] = \{w \in \mathbb{A}^* \mid \textit{there are words } u_0, \ldots, u_{n-1} \in X \textit{ such that } R(u_0, \ldots, u_{n-1}, w) \textit{ holds}\}.$$

Then the product *of $\mathcal{C}$ is the smallest subset of $\mathbb{A}^*$ closed under the rules of $\mathcal{C}$:*

$$\mathrm{Prod}(\mathcal{C}) = \bigcap \{X \subseteq \mathbb{A}^* \mid \textit{for all rules } R \in \mathcal{C} \textit{ holds } R[X] \subseteq X\}.$$

The product of a calculus can also be described "from below" by:

Definition 7. *Let $\mathcal{C}$ be a calculus over $\mathbb{A}$. A sequence $w^{(0)}, \ldots, w^{(k-1)} \in \mathbb{A}^*$ is called a* derivation *in $\mathcal{C}$ if for every $l < k$ there exists a rule $R \in \mathcal{C}$, $R \subseteq (\mathbb{A}^*)^n \times \mathbb{A}^*$ and $l_0, \ldots, l_{n-1} < l$ such that*

$$R(w^{(l_0)}, \ldots, w^{(l_{n-1})}, w^{(l)}).$$

This means that every word of the derivation can be derived from earlier words of the derivation by application of one of the rules of the calculus. We shall later define a calculus such that the sequence of sentences

Let $\forall y \exists x\, y = f(x)$.
Let $\forall y \exists x\, y = g(x)$.
Consider y.
$\exists z\, y = g(z)$.
Let $y = g(z)$.
$\exists x\, z = f(x)$.
Let $z = f(x)$.
$y = g(f(x))$.
Thus $\exists x\, y = g(f(x))$.
Thus $\exists x\, y = g(f(x))$.
Thus $\forall y \exists x\, y = g(f(x))$.
Qed.

is a derivation in that calculus.

Everything in the product of a calculus can be obtained by a derivation:

Theorem 8. *Let $\mathcal{C}$ be a calculus over $\mathbb{A}^*$. Then*

$$\mathrm{Prod}(\mathcal{C}) = \{w \,|\, \textit{there is a derivation } w^{(0)}, \ldots, w^{(k-1)} = w \textit{ in } \mathcal{C}\}.$$

Proof. The equality of sets can be proved by two inclusions.
($\subseteq$) The set

$$X = \{w \,|\, \text{there is a derivation } w^{(0)}, \ldots, w^{(k-1)} = w \text{ in } \mathcal{C}\}$$

satisfies the closure property $R[X] \subseteq X$ for all rules $R \in \mathcal{C}$. Since $\mathrm{Prod}(\mathcal{C})$ is the intersection of all such sets, $\mathrm{Prod}(\mathcal{C}) \subseteq X$.
($\supseteq$) Consider $w \in X$. Consider a derivation $w^{(0)}, \ldots, w^{(k-1)} = w$ in $\mathcal{C}$. We show by induction on $l < k$ that $w^{(l)} \in \mathrm{Prod}(\mathcal{C})$. Let $l < k$ and assume that for all $i < l$ holds $w^{(i)} \in \mathrm{Prod}(\mathcal{C})$. Take a rule $R \in \mathcal{C}$, $R \subseteq (\mathbb{A}^*)^n \times \mathbb{A}^*$ and $l_0, \ldots, l_{n-1} < l$ such that $R(w^{(l_0)}, \ldots, w^{(l_{n-1})}, w^{(l)})$. Since $\mathrm{Prod}(\mathcal{C})$ is closed under application of R we get $w^{(l)} \in \mathrm{Prod}(\mathcal{C})$. Thus $w = w^{(k-1)} \in \mathrm{Prod}(\mathcal{C})$. □

Exercise 1. (Natural numbers 1) Consider the alphabet $\mathbb{A} = \{|\}$. The set $\mathbb{A}^* = \{\emptyset, |, ||, |||, \ldots\}$ of words may be identified with the set $\mathbb{N}$ of natural numbers. Formulate a calculus $\mathcal{C}$ such that $\mathrm{Prod}(\mathcal{C}) = \mathbb{A}^*$.

4 Induction and recursion on calculi

5 Terms and formulas

6 Structures

We *interpret* formulas like $\forall y \exists x\, y = g(f(x))$ in adequate *structures*. The realm of structures and interpretations is usually called *semantics*. Fix a symbol set

$$S = ((R_n)_{n \geq 1}, (F_n)_{n \geq 1}, C).$$

Definition 9. *An S-*structure *is a pair* $\mathfrak{A} = (A, a)$ *mit:*

a) $A \neq \emptyset$*; A is the* underlying set *of* $\mathfrak{A}$*;*

b) $a: S = (\bigcup_{n \geqslant 1} R_n) \cup (\bigcup_{n \geqslant 1} F_n) \cup C \to V$ *satisfying:*

1. *for* $r \in R_n$*:* $a(r)$ *is an n-ary* relation *on A, i.e.,* $a(r) \subseteq A^n$*;*
2. *for* $f \in F_n$*:* $a(f)$ *is an n-ary* function *on A, i.e.,* $a(f): A^n \to A$*;*
3. *for* $c \in C$*:* $a(c)$ *is a* constant *in A, i.e.,* $a(c) \in A$*.*

We use various simplifying notations like $r^{\mathfrak{A}}$, $f^{\mathfrak{A}}$, or $c^{\mathfrak{A}}$ instead of $a(r)$, $a(f)$, or $a(c)$ resp. In concrete cases, we simply list the values of a, i.e., the components of the structure $\mathfrak{A}$.

Example 10. Formalize the structure $\mathbb{R} = (\mathbb{R}, a)$ of the *ordered real numbers* as follows. Take the language of ordered fields

$$S_{\mathrm{oF}} = \{<, +, \cdot, 0, 1\}.$$

Define the interpretation function $a: S_{\mathrm{oF}} \to V$ by

$$\begin{aligned} a(<) = <^{\mathbb{R}} &= \{(u, v) \in \mathbb{R}^2 \,|\, u < v\} \\ a(+) = +^{\mathbb{R}} &= \{(u, v, w) \in \mathbb{R}^3 \,|\, u + v = w\} \\ a(\cdot) = \cdot^{\mathbb{R}} &= \{(u, v, w) \in \mathbb{R}^3 \,|\, u \cdot v = w\} \\ a(0) = 0^{\mathbb{R}} &= 0 \in \mathbb{R} \\ a(1) = 1^{\mathbb{R}} &= a \in \mathbb{R} \end{aligned}$$

Instead of $(\mathbb{R}, a)$ one also writes $(\mathbb{R}, <^{\mathbb{R}}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, 0^{\mathbb{R}}, 1^{\mathbb{R}})$ or $(\mathbb{R}, <, +, \cdot, 0, 1)$.

Observe that the symbols could in principle be interpreted in completely different, counterintuitive ways:

$$\begin{aligned} a'(<) = <^{\mathfrak{A}} &= \{(u, v) \in \mathbb{R}^2 \,|\, u > v\} \\ a'(+) = +^{\mathfrak{A}} &= \{(u, v, w) \in \mathbb{R}^3 \,|\, u \cdot v = w\} \\ a'(\cdot) = \cdot^{\mathfrak{A}} &= \{(u, v, w) \in \mathbb{R}^3 \,|\, u + v = w\} \\ a'(0) = 0^{\mathfrak{A}} &= 1 \in \mathbb{R} \\ a'(1) = 1^{\mathfrak{A}} &= 0 \in \mathbb{R} \end{aligned}$$

An S-structure interprets the symbols in S. To interpret a formula in a structure one also has to interpret the (occuring) variabels.

Definition 11. *Let* $\mathfrak{A} = (A, a)$ *be an S-structure. An* assignment *in* $\mathfrak{A}$ *is a function*

$$\beta: \{v_n \,|\, n \in \mathbb{N}\} \to A.$$

The pair $\mathfrak{I} = (\mathfrak{A}, \beta)$ *is called an S-*interpretation*.*

The value $\beta(v_n)$ is the interpretation of the variable v_n in $\mathfrak{A}$. It will sometimes be important to alter the interpretation of a specific variable.

Definition 12. *Let* $\mathfrak{A} = (A, a)$ *be an S-structure and let* $\beta: \{v_n \,|\, n \in \mathbb{N}\} \to A$ *be an* assignment *in* $\mathfrak{A}$*. For* $n \in \mathbb{N}$ *and* $a \in A$ *let*

$$\beta\frac{a}{v_n} = (\beta \setminus \{(v_n, \beta(v_n))\}) \cup \{(v_n, a)\}.$$

7 The satisfaction relation

Given an S-interpretation for a fixed language S we may interpret terms and formulas.

Definition 13. *Let* $\mathfrak{I}=(\mathfrak{A},\beta)$*,* $\mathfrak{A}=(A,a)$ *be an* S*-interpretation. Define the* interpretation $\mathfrak{I}(t)$ *of a term* t *by recursion on the term calculus:*

a) $\mathfrak{I}(v_n)=\beta(v_n)$*, for* $n\in\mathbb{N}$*;*

b) $\mathfrak{I}(c)=c^{\mathfrak{A}}$*, for* $c\in C$*;*

c) $\mathfrak{I}(ft_0....t_{n-1})=f^{\mathfrak{A}}(\mathfrak{I}(t_0),...,\mathfrak{I}(t_{n-1}))$*, for* $f\in F_n$ *and terms* $t_0,...,t_{n-1}$*.*

This explains the standard interpretation of a term like $v_3^2+v_{200}^3$ in the reals. The following satisfaction relation is the fundamental logical relation which links syntax and semantics.

Definition 14. *Let* $\mathfrak{I}=(\mathfrak{A},\beta)$*,* $\mathfrak{A}=(A,a)$ *be an* S*-interpretation. Define the* satisfaction relation

$$\mathfrak{I}\vDash\varphi$$

for formulas $\varphi\in L^S$ *by recursion on the formula calculus:*

a) $\mathfrak{I}\vDash t_0\equiv t_1$ *iff* $\mathfrak{I}(t_0)=\mathfrak{I}(t_1)$*;*

b) $\mathfrak{I}\vDash Rt_0...t_{n-1}$ *iff* $R^{\mathfrak{A}}(\mathfrak{I}(t_0),...,\mathfrak{I}(t_{n-1}))$*;*

c) $\mathfrak{I}\vDash\neg\varphi$ *iff* not $\mathfrak{I}\vDash\varphi$*;*

d) $\mathfrak{I}\vDash(\varphi\wedge\psi)$ *iff* $\mathfrak{I}\vDash\varphi$ and $\mathfrak{I}\vDash\psi$*;*

e) $\mathfrak{I}\vDash(\varphi\vee\psi)$ *iff* $\mathfrak{I}\vDash\varphi$ or $\mathfrak{I}\vDash\psi$*;*

f) $\mathfrak{I}\vDash(\varphi\to\psi)$ *iff* $\mathfrak{I}\vDash\varphi$ implies $\mathfrak{I}\vDash\psi$*;*

g) $\mathfrak{I}\vDash(\varphi\leftrightarrow\psi)$ *iff* $\mathfrak{I}\vDash\varphi$ is equivalent to $\mathfrak{I}\vDash\psi$*;*

h) $\mathfrak{I}\vDash\forall v_n\varphi$ *iff* for all $a\in A$ *holds* $(\mathfrak{A},\beta\frac{a}{v_n})\vDash\varphi$*;*

i) $\mathfrak{I}\vDash\exists v_n\varphi$ *iff* there exists $a\in A$ *such that* $(\mathfrak{A},\beta\frac{a}{v_n})\vDash\varphi$*.*

We say $\mathfrak{I}$ satisfies φ *or* $\mathfrak{I}$ *is a* model *of* φ*. For a set* $\Phi\subseteq L^S$ *of* S*-formulas define*

$$\mathfrak{I}\vDash\Phi \quad \textit{iff for all } \varphi\in\Phi \textit{ holds}: \mathfrak{I}\vDash\varphi.$$

We also write $\mathfrak{A}\vDash\varphi[\beta]$ *and* $\mathfrak{A}\vDash\Phi[\beta]$ *instead of* $\mathfrak{I}\vDash\varphi$ *and* $\mathfrak{I}\vDash\Phi$ *resp.*

Definition 15. *Let* S *be a language and* $\Phi\subseteq L^S$*.* Φ *is* universally valid *if for every* S*-Interpretation* $\mathfrak{I}$ *holds* $\mathfrak{I}\vDash\Phi$*.* Φ *is* satisfiable *if there is an* S*-Interpretation* $\mathfrak{I}$ *such that* $\mathfrak{I}\vDash\Phi$*.*

With the notion of $\vDash$ we can now formally define what it means that a group is commutative or that a function is differentiable, using adequate structures and formulas.

It is intuitively obvious that the interpretation of a term should only depend on the occuring variables, and that the satisfaction for a formula should only depend on its free, non-bound variables.

Definition 16. *For* $t\in T^S$ *define* $\mathrm{var}(t)\subseteq\{v_n|n\in\mathbb{N}\}$ *by recursion on the term calculus:*

- $\mathrm{var}(x)=\{x\}$;
- $\mathrm{var}(c)=\emptyset$;
- $\mathrm{var}(ft_0...t_{n-1})=\bigcup_{i<n}\mathrm{var}(t_i)$.

Definition 17. *Für* $\varphi\in L^S$ *define the set of* free variables $\mathrm{free}(\varphi)\subseteq\{v_n|n\in\mathbb{N}\}$ *by recursion on the formula calculus:*

- $\mathrm{free}(t_0\equiv t_1)=\mathrm{var}(t_0)\cup\mathrm{var}(t_1)$;
- $\mathrm{free}(Rt_0...t_{n-1})=\mathrm{var}(t_0)\cup...\cup\mathrm{var}(t_{n-1})$;
- $\mathrm{free}(\neg\varphi)=\mathrm{free}(\varphi)$;
- $\mathrm{free}((\varphi\wedge\psi))=\mathrm{free}((\varphi\vee\psi))=\mathrm{free}((\varphi\to\psi))=\mathrm{free}((\varphi\leftrightarrow\psi))=\mathrm{free}(\varphi)\cup\mathrm{free}(\psi)$.

– $\text{free}(\forall x\varphi) = \text{free}(\exists x\varphi) = \text{free}(\varphi) \setminus \{x\}$.

Example 18.

$$\begin{aligned}\text{free}((Ryx \to \forall y \neg y = z)) &= \text{free}(Ryx) \cup \text{free}(\forall y \neg y = z)\\ &= \text{free}(Ryx) \cup (\text{free}(\neg y = z) \setminus \{y\})\\ &= \text{free}(Ryx) \cup (\text{free}(y = z) \setminus \{y\})\\ &= \{y, x\} \cup (\{y, z\} \setminus \{y\})\\ &= \{y, x\} \cup \{z\}\\ &= \{x, y, z\}.\end{aligned}$$

Definition 19. *a) For $n \in \mathbb{N}$ let $L_n^S = \{\varphi \in L^S \mid \text{free}(\varphi) \subseteq \{v_0, \ldots, v_{n-1}\}\}$.*

b) $\varphi \in L^S$ is an S-sentence *if* $\text{free}(\varphi) = \emptyset$*; L_0^S is the set of S-sentences.*

Theorem 20. *Let t be an S-term and let $\mathfrak{I} = (\mathfrak{A}, \beta)$, $\mathfrak{A} = (A, a)$ and $\mathfrak{I}' = (\mathfrak{A}, \beta')$ be S-interpretations with $\beta \upharpoonright \text{var}(t) = \beta' \upharpoonright \text{var}(t)$. Then $\mathfrak{I}(t) = \mathfrak{I}'(t)$.*

Theorem 21. *Let φ be an S-formula, and let $\mathfrak{I} = (\mathfrak{A}, \beta)$, $\mathfrak{A} = (A, a)$ and $\mathfrak{I}' = (\mathfrak{A}, \beta')$ be S-interpretations with $\beta \upharpoonright \text{free}(\varphi) = \beta' \upharpoonright \text{free}(\varphi)$. Then*

$$\mathfrak{I} \vDash \varphi \quad \textit{iff} \quad \mathfrak{I}' \vDash \varphi.$$

Proof. By induction on the formula calculus.
$\varphi = t_0 \equiv t_1$: Then $\text{var}(t_0) \cup \text{var}(t_1) = \text{frei}(\varphi)$ and

$$\begin{aligned}\mathfrak{I} \vDash \varphi \ &\text{iff } \ \mathfrak{I}(t_0) = \mathfrak{I}(t_1)\\ &\text{iff } \ \mathfrak{I}'(t_0) = \mathfrak{I}'(t_1) \ \text{ by the previous Theorem,}\\ &\text{iff } \ \mathfrak{I}' \vDash \varphi.\end{aligned}$$

$\varphi = \psi \wedge \chi$ and assume the claim to be true for ψ and χ. Then

$$\begin{aligned}\mathfrak{I} \vDash \varphi \ &\text{iff } \ \mathfrak{I} \vDash \psi \text{ und } \mathfrak{I} \vDash \chi\\ &\text{iff } \ \mathfrak{I}' \vDash \psi \text{ und } \mathfrak{I}' \vDash \chi \text{ by the inductive assumption,}\\ &\text{iff } \ \mathfrak{I}' \vDash \varphi.\end{aligned}$$

$\varphi = \exists v_n \psi$ and assume the claim to be true for ψ. Then $\text{free}(\psi) \subseteq \text{free}(\varphi) \cup \{v_n\}$. For all $a \in A$: $(\beta\frac{a}{v_n}) \upharpoonright \text{free}(\psi) = (\beta'\frac{a}{v_n}) \upharpoonright \text{free}(\psi)$ and so

$$\begin{aligned}\mathfrak{I} \vDash \varphi \ &\text{iff } \ \text{there exists } a \in A \text{ with } (\mathfrak{A}, \beta\frac{a}{v_n}) \vDash \psi\\ &\text{iff } \ \text{there exists } a \in A \text{ with } (\mathfrak{A}, \beta'\frac{a}{v_n}) \vDash \psi \text{ by the inductive assumption,}\\ &\text{iff } \ \mathfrak{I}' \vDash \varphi.\end{aligned}$$

□

This allows further simplifications in the notations for $\vDash$:

Definition 22. *Let $\mathfrak{A}$ be an S-structure and let $(a_0, \ldots, a_{n-1})$ be a sequence of elements of A. Let t be an S-term with $\text{var}(t) \subseteq \{v_0, \ldots, v_{n-1}\}$. Then define*

$$t^{\mathfrak{A}}[a_0, \ldots, a_{n-1}] = \mathfrak{I}(t),$$

*where $\mathfrak{I} = (\mathfrak{A}, \beta)$ is an interpretation with $\beta(0) = a_0, \ldots, \beta(n-1) = a_{n-1}$.
Let φ be an S-formula with $\text{free}(t) \subseteq \{v_0, \ldots, v_{n-1}\}$. Then define*

$$\mathfrak{A} \vDash \varphi[a_0, \ldots, a_{n-1}] \quad gdw. \quad \mathfrak{I} \vDash \varphi,$$

*where $\mathfrak{I} = (\mathfrak{A}, \beta)$ is an interpretation with $\beta(0) = a_0, \ldots, \beta(n-1) = a_{n-1}$.
In case $n = 0$ also write $t^{\mathfrak{A}}$ instead of $t^{\mathfrak{A}}[a_0, \ldots, a_{n-1}]$ and $\mathfrak{A} \vDash \varphi$ instead of $\mathfrak{A} \vDash \varphi[a_0, \ldots, a_{n-1}]$. In this case we also say: $\mathfrak{A}$ is a* model *of φ, $\mathfrak{A}$* satisfies *φ or φ* is true *in $\mathfrak{A}$.*

For $\Phi \subseteq L_0^S$ *a set of sentences also write*

$$\mathfrak{A} \vDash \Phi \quad \textit{iff for all} \quad \varphi \in \Phi \ \textit{holds}{:}\, \mathfrak{A} \vDash \varphi.$$

Example 23. *Groups.* $S_{Gr}{:}=\{\circ, e\}$ with a binary function symbol $\circ$ and a constant symbol e is the *language of groups theory*. The group axioms are

a) $\forall v_0 \forall v_1 \forall v_2 \circ v_0 \circ v_1 v_2 \equiv \circ \circ v_0 v_1 v_2$;

b) $\forall v_0 \circ v_0 e \equiv v_0$;

c) $\forall v_0 \exists v_1 \circ v_0 v_1 \equiv e$.

This define the axiom set

$$\Phi_{\mathrm{Gr}} = \{\forall v_0 \forall v_1 \forall v_2 \circ v_0 \circ v_1 v_2 \equiv \circ \circ v_0 v_1 v_2, \forall v_0 \circ v_0 e \equiv v_0, \forall v_0 \exists v_1 \circ v_0 v_1 \equiv e\}.$$

An S-structure $\mathfrak{G} = (G, *, k)$ satisfies Φ_{Gr} iff it is a group in the ordinary sense.

Definition 24. *Let* S *be a language and* $\Phi \subseteq L_0^S$ *be a set of* S*-sentences. Then let*

$$\mathrm{Mod}^S \Phi = \{\mathfrak{A} \,|\, \mathfrak{A} \textit{ is an } S\textit{-structure and } \mathfrak{A} \vDash \Phi\}$$

be the model class *of* Φ*.*

Thus $\mathrm{Mod}^{S_{\mathrm{Gr}}} \Phi_{\mathrm{Gr}}$ is the model class of all groups. Model classes are studied in generality within *model theory* which is a branch of mathematical logic. For specific Φ the model class $\mathrm{Mod}^S \Phi$ is examined in various fields of mathematics: group theory, ring theory, graph theory, etc. Some typical questions questions are: Is $\mathrm{Mod}^S \Phi \neq \emptyset$, i.e., is Φ satisfiable? Can we extend $\mathrm{Mod}^S \Phi$ by adequate morphisms between models?

8 Logical implication and propositional connectives

Definition 25. $\Phi \vDash \varphi$

Theorem 26. *Properties of* $\Phi \vDash \varphi$ *with respect to propositional connectives.*

9 Substitution and quantification rules

Definition 27. *For a term* $s \in T^S$*, pairwise distinct variables* $x_0, \ldots, x_{r-1}$ *and terms* $t_0, \ldots, t_{r-1} \in T^S$ *define the* (simultaneous) substitution

$$s \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$$

of $t_0, \ldots, t_{r-1}$ *for* $x_0, \ldots, x_{r-1}$ *by recursion:*

a) $x \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = \begin{cases} x, \textit{if } x \neq x_0, \ldots, x \neq x_{r-1} \\ t_i, \ \textit{if } x = x_i \end{cases}$ *for all variables* x*;*

b) $c \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = c$ *for all constant symbols* c*;*

c) $(f s_0 \ldots s_{n-1}) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = f s_0 \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} \ldots s_{n-1} \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$ *for all* n*-ary function symbols* f*.*

Note that the simultaneous substitution

$$s \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$$

is in general different from a succesive substitution

$$s \frac{t_0}{x_0} \frac{t_1}{x_1} \ldots \frac{t_{r-1}}{x_{r-1}}$$

which depends on the order of substitution. E.g., $x \frac{y x}{x y} = y$, $x \frac{y}{x} \frac{x}{y} = y \frac{x}{y} = x$ and $x \frac{x}{y} \frac{y}{x} = x \frac{y}{x} = y$.

Definition 28. *For a formula $\varphi \in L^S$, pairwise distinct variables $x_0, \ldots, x_{r-1}$ and terms $t_0, \ldots, t_{r-1} \in T^S$ define the* (simultaneous) substitution

$$\varphi \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$$

of $t_0, \ldots, t_{r-1}$ for $x_0, \ldots, x_{r-1}$ by recursion:

a) $(s_0 \equiv s_1) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = s_0 \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} \equiv s_1 \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$ *for all terms $s_0, s_1 \in T^S$;*

b) $(R s_0 \ldots s_{n-1}) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = R s_0 \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} \ldots s_{n-1} \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$ *for all n-ary relation symbols R and terms $s_0, \ldots, s_{n-1} \in T^S$;*

c) $(\neg\varphi) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = \neg(\varphi \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}})$;

d) $(\varphi \vee \psi) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = (\varphi \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} \vee \psi \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}})$;

e) for $(\exists x \varphi) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$ *distinguish two cases:*

– *if $x \in \{x_0, \ldots, x_{r-1}\}$, assume that $x = x_0$. Choose $i \in \mathbb{N}$ minimal such that $u = v_i$ does not occur in $\exists x \varphi$, $t_0, \ldots, t_{r-1}$ and $x_0, \ldots, x_{r-1}$. Then set*

$$(\exists x \varphi) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = \exists u (\varphi \frac{t_1 \ldots t_{r-1}\, u}{x_1 \ldots x_{r-1}\, x}).$$

– *if $x \notin \{x_0, \ldots, x_{r-1}\}$, choose $i \in \mathbb{N}$ minimal such that $u = v_i$ does not occur in $\exists x \varphi$, $t_0, \ldots, t_{r-1}$ and $x_0, \ldots, x_{r-1}$ and set*

$$(\exists x \varphi) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = \exists u (\varphi \frac{t_0 \ldots t_{r-1}\, u}{x_0 \ldots x_{r-1}\, x}).$$

f) for $(\forall x \varphi) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}$ *distinguish two cases:*

– *if $x \in \{x_0, \ldots, x_{r-1}\}$, assume that $x = x_0$. Choose $i \in \mathbb{N}$ minimal such that $u = v_i$ does not occur in $\forall x \varphi$, $t_0, \ldots, t_{r-1}$ and $x_0, \ldots, x_{r-1}$. Then set*

$$(\forall x \varphi) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = \forall u (\varphi \frac{t_1 \ldots t_{r-1}\, u}{x_1 \ldots x_{r-1}\, x}).$$

– *if $x \notin \{x_0, \ldots, x_{r-1}\}$, choose $i \in \mathbb{N}$ minimal such that $u = v_i$ does not occur in $\forall x \varphi$, $t_0, \ldots, t_{r-1}$ and $x_0, \ldots, x_{r-1}$ and set*

$$(\forall x \varphi) \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}} = \forall u (\varphi \frac{t_0 \ldots t_{r-1}\, u}{x_0 \ldots x_{r-1}\, x}).$$

The following substitution theorem shows that syntactic substitution corresponds semantically to a (simultaneous) modification of assignments by interpreted terms.

Definition 29. *Consider an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$, pairwise distinct variables $x_0, \ldots, x_{r-1}$ and $a_0, \ldots, a_{r-1} \in A$. Define a modified assignment and interpretation by*

$$\beta \frac{a_0 \ldots a_{r-1}}{x_0 \ldots x_{r-1}} = (\beta \setminus \{(x_0, \beta(x_0)), \ldots, (x_{r-1}, \beta(x_{r-1}))\}) \cup \{(x_0, a_0), \ldots, (x_{r-1}, a_{r-1})\}$$

and

$$\mathfrak{I} \frac{a_0 \ldots a_{r-1}}{x_0 \ldots x_{r-1}} = (\mathfrak{A}, \beta \frac{a_0 \ldots a_{r-1}}{x_0 \ldots x_{r-1}}).$$

Theorem 30. *Consider an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$, pairwise distinct variables $x_0, \ldots, x_{r-1}$ and terms $t_0, \ldots, t_{r-1} \in T^S$. If $s \in T^S$ is a term,*

$$\mathfrak{I}(s \frac{t_0 \ldots t_{r-1}}{x_0 \ldots x_{r-1}}) = \mathfrak{I} \frac{\mathfrak{I}(t_0) \ldots \mathfrak{I}(t_{r-1})}{x_0 \ldots x_{r-1}}(s).$$

If $\varphi \in L^S$ is a formula,

$$\mathfrak{I} \vDash \varphi \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}} \; gdw. \; \mathfrak{I} \frac{\mathfrak{I}(t_0) .... \mathfrak{I}(t_{r-1})}{x_0 ... x_{r-1}} \vDash \varphi.$$

Proof. By induction on the complexities of s and φ.
Case 1: $s = x$.
Case 1.1: $x \notin \{x_0, ..., x_{r-1}\}$. Then

$$\mathfrak{I}(x \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}}) = \mathfrak{I}(x) = \beta(x) = \mathfrak{I} \frac{\mathfrak{I}(t_0) .... \mathfrak{I}(t_{r-1})}{x_0 ... x_{r-1}}(x).$$

Case 1.2: $x = x_i$. Then

$$\mathfrak{I}(x \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}}) = \mathfrak{I}(t_i) = \beta \frac{\mathfrak{I}(t_0) .... \mathfrak{I}(t_{r-1})}{x_0 ... x_{r-1}}(x_i) = \mathfrak{I} \frac{\mathfrak{I}(t_0) .... \mathfrak{I}(t_{r-1})}{x_0 ... x_{r-1}}(x).$$

Case 2: $s = c$ is a constant symbol. Then

$$\mathfrak{I}(c \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}}) = \mathfrak{I}(c) = c^{\mathfrak{A}} = \mathfrak{I} \frac{\mathfrak{I}(t_0) .... \mathfrak{I}(t_{r-1})}{x_0 ... x_{r-1}}(c).$$

Case 3: $s = f s_0 ... s_{n-1}$ where f is an n-ary function symbol and the terms $s_0, ..., s_{n-1} \in T^S$ satisfy the theorem. Then

$$\begin{aligned}
\mathfrak{I}((f s_0 ... s_{n-1}) \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}}) &= \mathfrak{I}(f s_0 \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}} ... s_{n-1} \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}}) \\
&= f^{\mathfrak{A}}(\mathfrak{I}(s_0 \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}}), ..., \mathfrak{I}(s_{n-1} \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}})) \\
&= f^{\mathfrak{A}}(\mathfrak{I} \frac{\mathfrak{I}(t_0) .... \mathfrak{I}(t_{r-1})}{x_0 ... x_{r-1}}(s_0), ..., \mathfrak{I} \frac{\mathfrak{I}(t_0) .... \mathfrak{I}(t_{r-1})}{x_0 ... x_{r-1}}(s_{n-1})) \\
&= \mathfrak{I} \frac{\mathfrak{I}(t_0) .... \mathfrak{I}(t_{r-1})}{x_0 ... x_{r-1}}(f s_0 ... s_{n-1}).
\end{aligned}$$

Assuming that the substitution theorem is proved for terms, we turn to formulas:
Case 4: $\varphi = s_0 \equiv s_1$. Then

$$\begin{aligned}
\mathfrak{I} \vDash (s_0 \equiv s_1) \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}} \quad &\text{iff} \quad \mathfrak{I} \vDash (s_0 \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}} \equiv s_1 \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}}) \\
&\text{iff} \quad \mathfrak{I}(s_0 \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}}) = \mathfrak{I}(s_1 \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}}) \\
&\text{iff} \quad \mathfrak{I} \frac{\mathfrak{I}(t_0) .... \mathfrak{I}(t_{r-1})}{x_0 ... x_{r-1}}(s_0) = \mathfrak{I} \frac{\mathfrak{I}(t_0) .... \mathfrak{I}(t_{r-1})}{x_0 ... x_{r-1}}(s_1) \\
&\text{iff} \quad \mathfrak{I} \frac{\mathfrak{I}(t_0) .... \mathfrak{I}(t_{r-1})}{x_0 ... x_{r-1}} \vDash s_0 \equiv s_1.
\end{aligned}$$

Propositional connectives of formulas behave similar to terms, so we only consider the existential quantification case:
Case 5: $\varphi = (\exists x \psi) \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}}$, assuming that the theorem holds for ψ.
Case 5.1: $x = x_0$. Choose $i \in \mathbb{N}$ minimal such that $u = v_i$ does not occur in $\exists x \varphi$, $t_0, ..., t_{r-1}$ and $x_0, ..., x_{r-1}$. Then

$$(\exists x \varphi) \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}} = \exists u (\varphi \frac{t_1 .... t_{r-1} \, u}{x_1 ... x_{r-1} \, x}).$$

$$\begin{aligned}
\mathfrak{I} \vDash (\exists x \varphi) \frac{t_0 .... t_{r-1}}{x_0 ... x_{r-1}} \quad &\text{iff} \quad \mathfrak{I} \vDash \exists u (\varphi \frac{t_1 .... t_{r-1} \, u}{x_1 ... x_{r-1} \, x}) \\
&\text{iff} \quad \text{there exists } a \in A \text{ with } \mathfrak{I} \frac{a}{u} \vDash \varphi \frac{t_1 .... t_{r-1} \, u}{x_1 ... x_{r-1} \, x} \\
&\qquad \text{(definition of } \vDash \text{)} \\
&\text{iff} \quad \text{there exists } a \in A \text{ with} \\
&\qquad (\mathfrak{I} \frac{a}{u}) \frac{\mathfrak{I}\frac{a}{u}(t_1) .... \mathfrak{I}\frac{a}{u}(t_{r-1}) \, \mathfrak{I}\frac{a}{u}(u)}{x_1 ... x_{r-1} \, x} \vDash \varphi \\
&\qquad \text{(inductive hypothesis for } \varphi \text{)}
\end{aligned}$$

iff there exists $a \in A$ with
$$(\mathfrak{I}\frac{a}{u})\frac{\mathfrak{I}(t_1)....\mathfrak{I}(t_{r-1})\,a}{x_1...x_{r-1}\,x} \vDash \varphi$$
(since u does not occur in t_i)

iff there exists $a \in A$ with
$$\mathfrak{I}\,\frac{\mathfrak{I}(t_1)....\mathfrak{I}(t_{r-1})\,a}{x_1...x_{r-1}\,x} \vDash \varphi$$
(since u does not occur in φ)

iff there exists $a \in A$ with
$$(\mathfrak{I}\,\frac{\mathfrak{I}(t_1)....\mathfrak{I}(t_{r-1})}{x_1...x_{r-1}})\,\frac{a}{x} \vDash \varphi$$
(by simple properties of assignments)

iff $(\mathfrak{I}\,\frac{\mathfrak{I}(t_1)....\mathfrak{I}(t_{r-1})}{x_1...x_{r-1}}) \vDash \exists x\varphi$
(definition of $\vDash$)

iff $(\mathfrak{I}\,\frac{\mathfrak{I}(t_0)\,\mathfrak{I}(t_1)....\mathfrak{I}(t_{r-1})}{x_0\,x_1...x_{r-1}}) \vDash \exists x\varphi$
(since $x = x_0$ is not free in $\exists x\varphi$).

Case 5.2: $x \notin \{x_0, ..., x_{r-1}\}$. Then proceed similarly. Choose $i \in \mathbb{N}$ minimal such that $u = v_i$ does not occur in $\exists x\varphi$, $t_0, ..., t_{r-1}$ and $x_0, ..., x_{r-1}$. Then

$$(\exists x\varphi)\,\frac{t_0....t_{r-1}}{x_0...x_{r-1}} = \exists u\,(\varphi\,\frac{t_0....t_{r-1}\,u}{x_0...x_{r-1}\,x}).$$

$\mathfrak{I} \vDash (\exists x\varphi)\,\frac{t_0....t_{r-1}}{x_0...x_{r-1}}$ iff $\mathfrak{I} \vDash \exists u\,(\varphi\,\frac{t_0....t_{r-1}\,u}{x_0...x_{r-1}\,x})$

iff there exists $a \in A$ with $\mathfrak{I}\frac{a}{u} \vDash \varphi\,\frac{t_0....t_{r-1}\,u}{x_0...x_{r-1}\,x}$
(definition of $\vDash$)

iff there exists $a \in A$ with
$$(\mathfrak{I}\frac{a}{u})\frac{\mathfrak{I}\frac{a}{u}(t_0)....\mathfrak{I}\frac{a}{u}(t_{r-1})\,\mathfrak{I}\frac{a}{u}(u)}{x_0...x_{r-1}\,x} \vDash \varphi$$
(inductive hypothesis for φ)

iff there exists $a \in A$ with
$$(\mathfrak{I}\frac{a}{u})\frac{\mathfrak{I}(t_0)....\mathfrak{I}(t_{r-1})\,a}{x_0...x_{r-1}\,x} \vDash \varphi$$
(since u does not occur in t_i)

iff there exists $a \in A$ with
$$\mathfrak{I}\,\frac{\mathfrak{I}(t_0)....\mathfrak{I}(t_{r-1})\,a}{x_0...x_{r-1}\,x} \vDash \varphi$$
(since u does not occur in φ)

iff there exists $a \in A$ with
$$(\mathfrak{I}\,\frac{\mathfrak{I}(t_0)....\mathfrak{I}(t_{r-1})}{x_0...x_{r-1}})\,\frac{a}{x} \vDash \varphi$$
(by simple properties of assignments)

iff $(\mathfrak{I}\,\frac{\mathfrak{I}(t_0)....\mathfrak{I}(t_{r-1})}{x_0...x_{r-1}}) \vDash \exists x\varphi$
(definition of $\vDash$)

□

We can now formulate further properties of the $\vDash$ relation.

Theorem 31. *Let S be a language. Let $\Phi \subseteq L^S$, $t, t' \in T^S$ and $\varphi, \psi \in L^S$. Then:*

a) *if* $\Phi \vDash \forall x\varphi$, *then* $\Phi \vDash \varphi\,\frac{t}{x}$*;*

b) *if* $\Phi \vDash \varphi\,\frac{t}{x}$, *then* $\Phi \vDash \exists x\varphi$*;*

c) *if* $\Phi \vDash \varphi \frac{y}{x}$ *and* $y \notin \mathrm{free}(\Phi \cup \{\forall x \varphi\})$, *then* $\Phi \vDash \forall x \varphi$ *;*

d) *if* $\Phi \cup \{\varphi \frac{y}{x}\} \vDash \psi$ *and* $y \notin \mathrm{free}(\Phi \cup \{\exists x \varphi, \psi\})$, *then* $\Phi \cup \{\exists x \varphi\} \vDash \psi$ *;*

e) *if* $\Phi \vDash \varphi \frac{t}{x}$, *then* $\Phi \cup \{t \equiv t'\} \vDash \varphi \frac{t'}{x}$ *.*

Proof. a) Let $\Phi \vDash \forall x \varphi$. Consider an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ with $\mathfrak{I} \vDash \Phi$. For all $a \in A$ holds $\mathfrak{I} \frac{a}{x} \vDash \varphi$. In particular $\mathfrak{I} \frac{\mathfrak{I}(t)}{x} \vDash \varphi$. By the substitution theorem, $\mathfrak{I} \vDash \varphi \frac{t}{x}$. Thus $\Phi \vDash \varphi \frac{t}{x}$.
b) Let $\Phi \vDash \varphi \frac{t}{x}$. Consider an S-interpretation $\mathfrak{I}$ with $\mathfrak{I} \vDash \Phi$. Then $\mathfrak{I} \vDash \varphi \frac{t}{x}$. By the substitution theorem $\mathfrak{I} \frac{\mathfrak{I}(t)}{x} \vDash \varphi$. Hence $\mathfrak{I} \vDash \exists x \varphi$. Thus $\Phi \vDash \exists x \varphi$.
c) Let $\Phi \vDash \varphi \frac{y}{x}$ and $y \notin \mathrm{free}(\Phi \cup \{\forall x \varphi\})$. Consider an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ with $\mathfrak{I} \vDash \Phi$. Let $a \in A$. Since $y \notin \mathrm{free}(\Phi)$, $\mathfrak{I} \frac{a}{y} \vDash \Phi$. By assumption, $\mathfrak{I} \frac{a}{y} \vDash \varphi \frac{y}{x}$. By the substitution theorem,

$$(\mathfrak{I} \frac{a}{y}) \frac{\mathfrak{I} \frac{a}{y}(y)}{x} \vDash \varphi \text{ and so } (\mathfrak{I} \frac{a}{y}) \frac{a}{x} = \mathfrak{I} \frac{a\,a}{y\,x} \vDash \varphi$$

Case 1: $x = y$. Then $\mathfrak{I} \frac{a}{x} \vDash \varphi$.
Case 2: $x \neq y$. Then $y \notin \mathrm{free}(\varphi)$ and so $\mathfrak{I} \frac{a}{x} \vDash \varphi$.
Thus $\mathfrak{I} \vDash \forall x \varphi$. Thus $\Phi \vDash \forall x \varphi$.
d) Let $\Phi \cup \{\varphi \frac{y}{x}\} \vDash \psi$ and $y \notin \mathrm{free}(\Phi \cup \{\exists x \varphi, \psi\})$. Consider an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ with $\mathfrak{I} \vDash \Phi \cup \{\exists x \varphi\}$. Take $a \in A$ with $\mathfrak{I} \frac{a}{x} \vDash \varphi$.
Case 1: $x = y$. Then $(\mathfrak{I} \frac{a}{y}) \frac{a}{x} = \mathfrak{I} \frac{a}{x}$ und $(\mathfrak{I} \frac{a}{y}) \frac{a}{x} \vDash \varphi$.
Case 2: $x \neq y$. Then $y \notin \mathrm{free}(\varphi)$ and $(\mathfrak{I} \frac{a}{y}) \frac{a}{x} \vDash \varphi$.
Obviously $a = \mathfrak{I} \frac{a}{y}(y)$ and so

$$(\mathfrak{I} \frac{a}{y}) \frac{\mathfrak{I} \frac{a}{y}(y)}{x} \vDash \varphi.$$

By the substitution theorem

$$(\mathfrak{I} \frac{a}{y}) \vDash \varphi \frac{y}{x}.$$

Since $y \notin \mathrm{free}(\Phi)$

$$(\mathfrak{I} \frac{a}{y}) \vDash \Phi.$$

By assumption $(\mathfrak{I} \frac{a}{y}) \vDash \psi$ and since $y \notin \mathrm{free}(\psi)$ we get $\mathfrak{I} \vDash \psi$. Thus $\Phi \cup \{\exists x \varphi\} \vDash \psi$.
e) Let $\Phi \vDash \varphi \frac{t}{x}$. Consider an S-Interpretation $\mathfrak{I}$ mit $\mathfrak{I} \vDash \Phi \cup \{t \equiv t'\}$. Then $\mathfrak{I}(t) = \mathfrak{I}(t')$. By assumption $\mathfrak{I} \vDash \varphi \frac{t}{x}$. By the substitution theorem

$$\mathfrak{I} \frac{\mathfrak{I}(t)}{x} \vDash \varphi.$$

Then

$$\mathfrak{I} \frac{\mathfrak{I}(t')}{x} \vDash \varphi$$

and again by the substitution theorem

$$\mathfrak{I} \vDash \varphi \frac{t'}{x}.$$

Thus $\Phi \cup \{t \equiv t'\} \vDash \varphi \frac{t'}{x}$. □

Note that in proving these proof rules we have obviously used the corresponding figures of argument in the language of our discourse.

10 A sequent calculus

We can list the above rules of implication established in the previous two sections in the form of a calculus which leads from correct implications $\Phi \vDash \varphi$ to further correct implications $\Phi' \vDash \varphi'$. We shall later show in the GÖDEL completeness theorem that these rules actually generate the implication relation $\vDash$. Fix a language S for this section.

We only

Definition 32. *An ordered pair* (Φ, φ) *is a* sequent *if* $\Phi \subseteq L^S$ *and* $\varphi \in L^S$. *Let* $\mathrm{Seq}(S)$ *be the set of all sequents for the language* S. *We write* $\Phi\,\varphi$ *instead of* (Φ, φ). Φ *and* φ *are* the antecedent *and the* succedent *of the sequent* $\Phi\,\varphi$. *We can also write the antecedent as a concatenation of sets of formulas and single formulas:*

$$\Phi\,\psi_o \ldots \psi_{k-1}\,\varphi \text{ instead of } \Phi \cup \{\psi_0, \ldots, \psi_{k-1}\}\,\varphi$$

und

$$\psi_o \ldots \psi_{k-1}\,\varphi \text{ instead of } \{\psi_0, \ldots, \psi_{k-1}\}\,\varphi\,.$$

A sequent $\Phi\,\varphi$ *is* correct *if* $\Phi \vDash \varphi$.

Definition 33. *The* sequent calculus *consists of the following (sequent-)rules:*

- *monotonicity rule* (MR) $\dfrac{\Phi \quad \varphi}{\Phi \quad \psi \quad \varphi}$
- *assumption rule* (AR) $\dfrac{}{\Phi \quad \varphi \quad \varphi}$
- $\to$*-introduction* $(\to I)$ $\dfrac{\Phi \quad \varphi \quad \psi}{\Phi \quad \varphi \to \psi}$
- $\to$*-elimination* $(\to E)$ $\dfrac{\begin{array}{ll}\Phi & \varphi \\ \Phi & \varphi \to \psi\end{array}}{\Phi \quad \psi}$
- $\vee$*-introduction* $(\vee E)$ $\dfrac{\Phi \quad \varphi}{\Phi \quad \varphi \vee \psi}$
- $\vee$*-introduction* $(\vee E)$ $\dfrac{\Phi \quad \varphi}{\Phi \quad \psi \vee \varphi}$
- $\vee$*-elimination* $(\vee E)$ $\dfrac{\begin{array}{ll}\Phi & \varphi \vee \psi \\ \Phi & \varphi \to \chi \\ \Phi & \psi \to \chi\end{array}}{\Phi \quad \chi}$
- $\bot$*-introduction* $(\bot I)$ $\dfrac{\begin{array}{ll}\Phi & \varphi \\ \Phi & \neg\varphi\end{array}}{\Phi \quad \bot}$
- $\bot$*-elimination* $(\bot E)$ $\dfrac{\Phi \quad \bot}{\Phi \quad \varphi}$
- $\forall$*-introduction* $(\forall I)$ $\dfrac{\Phi \quad \varphi\frac{y}{x}}{\Phi \quad \forall x\varphi}$, *if* $y \notin \mathrm{free}(\Phi \cup \{\forall x\varphi\})$
- $\forall$*-elimination* $(\forall E)$ $\dfrac{\Phi \quad \forall x\varphi}{\Phi \quad \varphi\frac{t}{x}}$, *if* $t \in T^S$
- $\exists$*-introduction* $(\exists I)$ $\dfrac{\Phi \quad \varphi\frac{t}{x}}{\Phi \quad \exists x\varphi}$, *if* $t \in T^S$
- $\exists$*-elimination* $(\exists E)$ $\dfrac{\begin{array}{lll}\Phi & & \exists x\varphi \\ \Phi & \varphi\frac{y}{x} & \psi\end{array}}{\Phi \quad \psi}$, *if* $y \notin \mathrm{free}(\Phi \cup \{\exists x\varphi, \psi\})$
- $\equiv$*-introduction* $(\equiv I)$ $\dfrac{}{\Phi \quad t \equiv t}$, *if* $t \in T^S$

– $\equiv$-*elimination* ($\equiv E$) $\begin{array}{ll} \Phi & \varphi\frac{t}{x} \\ \Phi & t\equiv t' \\ \hline \Phi & \varphi\frac{t'}{x} \end{array}$

The deduction relation *is the smallest subset* $\vdash\subseteq \mathrm{Seq}(S)$ *of the set of sequents which is closed under these rules. We write* $\Phi\vdash\varphi$ *instead of* $(\Phi,\,\varphi)\in\vdash$ *and say that* φ *can be* deduced *or* derived *from* Φ.

Theorem 34. *A formula* $\varphi\in L^S$ *is derivable from* $\Phi\subseteq L^S$ $(\Phi\vdash\varphi)$ *iff there is a* derivation *or a* formal proof

$$\Phi_0\varphi_0\ \Phi_1\varphi_1\ \ldots\ldots\Phi_{k-1}\varphi_{k-1}$$

of $\Phi\,\varphi=\Phi_{k-1}\varphi_{k-1}$ *, in which every sequent* $\Phi_i\varphi_i$ *is generated by a sequent rule from sequents* $\Phi_{i_0}\varphi_{i_0},\ldots,\Phi_{i_{n-1}}\varphi_{i_{n-1}}$ *with* $i_0,\ldots,i_{n-1}<i$ *.*

We usually write the derivation $\Phi_0\varphi_0\ \Phi_1\varphi_1\ \ldots\ldots\Phi_{k-1}\varphi_{k-1}$ *as a scheme*

$$\begin{array}{c} \Phi_0\varphi_0 \\ \Phi_1\varphi_1 \\ \vdots \\ \Phi_{k-1}\varphi_{k-1} \end{array}$$

where we may also mention the rules and other remarks along the course of the derivation.

In our theorems on the laws of implication we have already shown:

Theorem 35. *The sequent calculus is* correct*, i.e., every rule of the sequent calculus leads from correct sequents to correct sequents. Thus every derivable sequent is correct. This means that*

$$\vdash\subseteq\vDash.$$

The GÖDEL completeness theorem proves the opposite inclusion: $\vDash\subseteq\vdash$.

We also note the compactness theorem: finite subsets etc.

11 Examples of formal proofs

11.1 Properties of $\equiv$

We show that $\equiv$ as seen by the sequent calculus is an equivalence relation.
Symmetry:

$$\begin{array}{lll} x\equiv y & x\equiv y & \text{(assumption rule)} \\ x\equiv y & x\equiv x & (\equiv\text{-introduction}) \\ x\equiv y & (z\equiv x)\frac{x}{z} & (\text{where } z\notin\mathrm{var}(t_0)) \\ x\equiv y & (z\equiv x)\frac{y}{x} & (\equiv\text{-elimination}) \\ x\equiv y & y\equiv x & \\ \multicolumn{2}{c}{x\equiv y\rightarrow y\equiv x} & \\ \multicolumn{2}{c}{\forall y(x\equiv y\rightarrow y\equiv x)} & \\ \multicolumn{2}{c}{\forall x\forall y(x\equiv y\rightarrow y\equiv x)} & \end{array}$$

Transitivity:

$$\begin{array}{lllll} & t_0\equiv t_1 & t_0\equiv t_1 & \text{(VR)} \\ & t_0\equiv t_1 & (t_0\equiv x)\frac{t_1}{x} & (\text{where } x\notin\mathrm{var}(t_0)) \\ t_0\equiv t_1 & t_1\equiv t_2 & (t_0\equiv x)\frac{t_2}{x} & \text{(Sub)} \\ t_0\equiv t_1 & t_1\equiv t_2 & t_0\equiv t_2 & \end{array}$$

We show further that $\equiv$ is actually a *congruence relation* from the perspective of $\vdash$.

Theorem 36. *Let φ be an S-formula and $t_0, \ldots, t_{n-1}, t_0', \ldots, t_{n-1}' \in T^S$. Then*

$$\varphi \frac{t_0 \ldots t_{n-1}}{v_0 \ldots v_{n-1}}\ t_0 \equiv t_0' \ldots t_{n-1} \equiv t_{n-1}'\ \varphi \frac{t_0' \ldots t_{n-1}'}{v_0 \ldots v_{n-1}}$$

is derivable.

Proof. Choose pairwise distinct "new" variables $u_0, \ldots, u_{n-1}$. Then

$$\varphi \frac{t_0 \ldots t_{n-1}}{v_0 \ldots v_{n-1}} = \varphi \frac{u_0}{v_0} \frac{u_1}{v_1} \ldots \frac{u_{n-1}}{v_{n-1}} \frac{t_0}{u_0} \frac{t_1}{u_1} \ldots \frac{t_{n-1}}{u_{n-1}}$$

and

$$\varphi \frac{t_0' \ldots t_{n-1}'}{v_0 \ldots v_{n-1}} = \varphi \frac{u_0}{v_0} \frac{u_1}{v_1} \ldots \frac{u_{n-1}}{v_{n-1}} \frac{t_0'}{u_0} \frac{t_1'}{u_1} \ldots \frac{t_{n-1}'}{u_{n-1}}.$$

Thus the simultaneous substitutions can be seen as successive substitutions, and we may use the substitution rule repeatedly:

$$\begin{array}{l}
\varphi \frac{t_0 \ldots t_{n-1}}{v_0 \ldots v_{n-1}} \\
\varphi \frac{u_0}{v_0} \ldots \frac{u_{n-1}}{v_{n-1}} \frac{t_0}{u_0} \ldots \frac{t_{n-1}}{u_{n-1}} \\
\varphi \frac{u_0}{v_0} \ldots \frac{u_{n-1}}{v_{n-1}} \frac{t_0}{u_0} \ldots \frac{t_{n-1}}{u_{n-1}}\ t_{n-1} \equiv t_{n-1}' \\
\vdots \\
\varphi \frac{u_0}{v_0} \ldots \frac{u_{n-1}}{v_{n-1}} \frac{t_0}{u_0} \ldots \frac{t_{n-1}}{u_{n-1}}\ t_{n-1} \equiv t_{n-1}' \ldots t_0 \equiv t_0' \\
\varphi \frac{t_0 \ldots t_{n-1}}{v_0 \ldots v_{n-1}}\ t_0 \equiv t_0' \ldots t_{n-1} \equiv t_{n-1}'
\end{array}
\qquad
\begin{array}{l}
\varphi \frac{t_0 \ldots t_{n-1}}{v_0 \ldots v_{n-1}} \\
\varphi \frac{u_0}{v_0} \ldots \frac{u_{n-1}}{v_{n-1}} \frac{t_0}{u_0} \ldots \frac{t_{n-1}}{u_{n-1}} \\
\varphi \frac{u_0}{v_0} \ldots \frac{u_{n-1}}{v_{n-1}} \frac{t_0}{u_0} \ldots \frac{t_{n-1}'}{u_{n-1}} \\
\\
\varphi \frac{u_0}{v_0} \ldots \frac{u_{n-1}}{v_{n-1}} \frac{t_0'}{u_0} \ldots \frac{t_{n-1}'}{u_{n-1}} \\
\varphi \frac{t_0' \ldots t_{n-1}'}{v_0 \ldots v_{n-1}}.
\end{array}$$

□

11.2 Derivable rules

In proofs recurring combinations of elementary rules are combined into derived rules. This corresponds to the introduction of involved proof schemes in ordinary proofs.

Definition 37. *A sequent rule is* derivable *if its transformation from input to output sequents can be achieved by a composition of rules of the sequent calculus. By the correctness of the sequent calculus, every derivable rule is correct.*

Theorem 38. *For $\Gamma \subseteq L^S$, $\varphi, \psi \in L^S$, the following* cut rule *is derivable:*

$$\frac{\Gamma \varphi \quad \Gamma \varphi \psi}{\Gamma \psi}.$$

A rule with several input sequents can also be written in a vertical fashion:

$$\begin{array}{lll}
\Gamma & & \varphi \\
\Gamma & \varphi & \psi \\
\hline
\Gamma & & \psi
\end{array}$$

Proof.

1.	Γ			φ	input sequent
2.	Γ	φ		ψ	input sequent
3.	Γ	$\neg\varphi$	$\neg\psi$	φ	antecedent rule with 1
4.	Γ	$\neg\varphi$	$\neg\psi$	$\neg\varphi$	hypothesis rule
5.	Γ	$\neg\varphi$		ψ	contradiction rule with 3, 4
6.	Γ			ψ	case

□

We derive more rules which will be used to formalize “natural proofs”.

12 Natural proofs

The preceding example shows that formal proofs with sequents contain substantial redundancies. The surjectivity assumptions $\forall y \exists x\, y \equiv fx$ and $\forall y \exists x\, y \equiv gx$, e.g., have to be repeated in every antecedent. We introduce a notation for proofs of the above kind which basically consists of the succedents of the sequents. The antecedents can be determined from the succedents and the keywords “let” and “thus”. We work over a fixed language S.

Definition 39. *A* proof line *is a sequence z of symbols of one of the following forms:*

- *“φ.”*
- *“Let φ.”*
- *“Thus φ.”*

where $\varphi \in L^S$ is a formula.

If $z_0 \dots z_{l-1}$ is a sequence of proof lines, we define a corresponding expansion

$$A_0 \varphi_0\, A_1 \varphi_1 \dots\dots A_{l-1} \varphi_{l-1}$$

by recursion such that for all $i < l$, A_i is a finite sequence of S-formulas and φ_i is an S-formula: Let $A_{-1} = \emptyset$. Assume that A_{i-1} is defined; then

- *if z_i is of the form “φ.” then set $A_i = A_{i-1}$ and $\varphi_i = \varphi$;*
- *if z_i is of the form “Let φ.” then set $A_i = A_{i-1}\hat{}\,\varphi$ and $\varphi_i = \varphi$;*
- *if z_i is of the form “Thus φ.” then set $A_i = A_{i-1} \upharpoonright (\mathrm{length}(A_{i-1}) - 1)$ and $\varphi_i = \varphi$.*

For a finite sequence $A = (a_0, \dots, a_{n-1})$ define $\{\{A\}\} = \{a_0, \dots, a_{n-1}\}$. Then we say that the sequence $z_0 \dots z_{l-1}$ of proof lines is a natural proof*, if*

$$\{\{A_0\}\} \varphi_0\, \{\{A_1\}\} \varphi_1 \dots\dots \{\{A_{l-1}\}\} \varphi_{l-1}$$

is a derivation.

Note that the sequences $A_0, \dots, A_{l-1}$ can be seen as a “stack” of formulas; A_i consists of all hypotheses which are active at step i of the proof. The “command” “Let φ.” pushes the formula φ onto the stack, “Thus φ.” pops the top element from the stack.

12.1 Surjective functions

Consider the example of the introduction about surjective functions:

(1) Let $\forall y \exists x\, y = f(x)$.
(2) Let $\forall y \exists x\, y = g(x)$.
(Theorem. $\forall y \exists x\, y = g(f(x))$.
Proof.)
(3) Consider y.
(4) $\exists z\, y = g(z)$.
(5) Let $y = g(z)$.
(6) $\exists x\, z = f(x)$.
(7) Let $z = f(x)$.
(8) $y = g(f(x))$.
(9) $\exists x\, y = g(f(x))$
(10) Thus $\exists x\, y = g(f(x))$.
(11) Thus $\exists x\, y = g(f(x))$.
(12) Thus $\forall y \exists x\, y = g(f(x))$.
(Qed.)

We reformulate the argument as a formal proof, i.e., a sequence of sequents. The formulas of the argument occur as succedents of the formal proof. The antecedents list the assumptions which are locally available. The introduction of an assumption ("let", "consider") corresponds to adding the assumption to the antecedent; the withdrawal of an assumption ("thus") corresponds to taking the assumption off the antecedent. The sequents corresponding to a withdrawal are justified by the rules introduced in the last theorem.

	antecedent					succedent	comment
1.	$\forall y \exists x y \equiv fx$					$\forall y \exists x y \equiv fx$	HR
2.	$\forall y \exists x y \equiv fx$	$\forall y \exists x\, y \equiv gx$				$\forall y \exists x\, y \equiv gx$	HR
3.	$\forall y \exists x y \equiv fx$	$\forall y \exists x\, y \equiv gx$	$\top$			$\top$	HR
4.	$\forall y \exists x y \equiv fx$	$\forall y \exists x\, y \equiv gx$	$\top$			$\exists x\, y \equiv gx$	$\forall$E with 2
5.	$\forall y \exists x y \equiv fx$	$\forall y \exists x\, y \equiv gx$	$\top$	$y = gz$		$y = gz$	HR
6.	$\forall y \exists x y \equiv fx$	$\forall y \exists x\, y \equiv gx$	$\top$	$y = gz$		$\exists x z \equiv fx$	$\forall$E with 1
7.	$\forall y \exists x y \equiv fx$	$\forall y \exists x\, y \equiv gx$	$\top$	$y = gz$	$z \equiv fx$	$z \equiv fx$	HR
8.	$\forall y \exists x y \equiv fx$	$\forall y \exists x\, y \equiv gx$	$\top$	$y = gz$	$z \equiv fx$	$y \equiv gfx$	Sub
9.	$\forall y \exists x y \equiv fx$	$\forall y \exists x\, y \equiv gx$	$\top$	$y = gz$	$z \equiv fx$	$\exists x y \equiv gfx$	$\exists$I with 8
10.	$\forall y \exists x y \equiv fx$	$\forall y \exists x\, y \equiv gx$	$\top$	$y = gz$		$\exists x y \equiv gfx$	instantiation with 6,9
11.	$\forall y \exists x y \equiv fx$	$\forall y \exists x\, y \equiv gx$	$\top$			$\exists x y \equiv gfx$	instantiation with 4,10
12.	$\forall y \exists x y \equiv fx$	$\forall y \exists x\, y \equiv gx$				$\forall y \exists x y \equiv gfx$	universalization with 3,11