Simplified Constructibility Theory*

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Prologue

These lecture notes were written for a lecture course on constructibility and fine structure theory at the University of Bonn in the summer term of 2005.

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Chapter 1
Introduction

The scope of these lecture notes roughly correspond to the contents of the monograph *Constructibility* by Keith J. Devlin. Some mayor simplifications are obtained by using the hyperfine structure theory of Sy D. Friedman and the present author.

Kurt Gödel proved the unprovability of the negation of the generalized continuum hypothesis (GCH), i.e., its (relative) consistency, in notes and articles published between 1938 and 1940 [1], [3], [2], [4]. He presented his results in various forms which we can subsume as follows: there is an $\in$-term L such that

$$\mathrm{ZF} \vdash \text{``}(L, \in) \vDash \mathrm{ZF} + \text{the axiom of choice (AC)} + \mathrm{GCH}\text{''}.$$

So ZF sees a model for the stronger theory ZF + AC + GCH. If the system ZF is consistent, then so is ZF+AC+GCH. In ZF, the term L has a host of special properties; L is the $\subseteq$-minimal inner model of ZF, i.e., the $\subseteq$-smallest model of ZF which is transitive and contains the class Ord of ordinals.

The model L will be the central object of study in this lecture course.

The construction of L is motivated by the idea of recursively constructing a minimal model of ZF. The archetypical ZF-axiom is Zermelo's *comprehension schema* (*axiom of subsets*): for every $\in$-formula $\varphi(v, \vec{w})$ postulate

$$\forall x \forall \vec{p}\ \{v \in x | \varphi(v, \vec{p})\} \in V.$$

The term V denotes the abstraction term $\{v | v = v\}$, i.e., the set theoretic *universe*; formulas with abstraction terms are abbreviations for pure $\in$-formulas. E.g., the above instance of the comprehension schema abbreviates the formula

$$\forall x \forall \vec{p}\ \exists y \forall v\, (v \in y \leftrightarrow v \in x \wedge \varphi(v, \vec{p})).$$

The basic idea for building a (minimal) model of set theory is to form some kind of closure under the operations

$$(x, \vec{p}) \longmapsto \{v \in x | \varphi(v, \vec{p})\}.$$

There is a difficulty where to evaluate the formula φ. The comprehension instance should be satisfied in the model to be built eventually, i.e., the quantifiers of φ may have to range about sets which have not yet been included in the construction. To avoid this one only lets the evaluation of the formula refer to sets already constructed and considers the modified definability operations

$$(x, \vec{p}) \longmapsto \{v \in x | (x, \in) \vDash \varphi(v, \vec{p})\}.$$

These could be termed *predicative* operations whereas the strong operation would be *impredicative*. The set $\{v \in x | (x, \in) \vDash \varphi(v, \vec{p})\}$ is determined by the parameters $x, \varphi, \vec{p}$. One can thus view $\{v \in x | (x, \in) \vDash \varphi(v, \vec{p})\}$ as an *interpretation* of a *name* $(x, \varphi, \vec{p})$. These ideas will be essential in the definition of the *constructible hierarchy*.

Chapter 2

The Language of Set Theory

The intuitive notion of set is usually described by GEORG CANTOR's dictum

> Unter einer Menge verstehen wir jede Zusammenfassung M von bestimmten, wohlunterschiedenen Objekten m unsrer Anschauung oder unseres Denkens (welche die „Elemente" von M genannt werden) zu einem Ganzen. [CANTOR, S. 282; *By a set we understand every collection M of definite, distinguished objects m of our perceptions or thoughts (which are called the "elements" of M) into a whole.*]

This idea may be formalized by *class terms*:

$$M = \{m \mid \varphi(m)\}.$$

M is the *class* of all m which satisfy the (mathematical) property φ. Class terms are common in modern mathematical practice. The transfer from the definining property φ to the corresponding collection $M = \{m \mid \varphi(m)\}$ supports the view that one is working with abstract "objects", namely *classes*, instead of "immaterial" *properties*. How such classes can reasonably and consistently be treated as objects is a matter of set theoretical and foundational concern. It is partially answered by the ZERMELO-FRAENKEL axioms of set theory which we shall introduce in the next chapter.

Even without set theory, classes can be treated intuitively. One can describe properties of class terms and define complex terms from given ones, thus developing a *class theory*. We shall take the view that sets are "small" classes. The language of class terms is thus also the language of set theory — or even of mathematics, if we think of all of mathematics as formalized within set theory.

2.1 Class Terms

Classes or collections may be queried for certain elements: m is an element of $M = \{m \mid \varphi(m)\}$ if it satisfies the defining property φ. In symbols:

$$m \in M \text{ if and only if } \varphi(m).$$

So in $m \in \{m \mid \varphi(m)\}$ the class term may be eliminated by just writing the property or *formula* φ. Carrying out this kind of elimination throughout mathematics shows that all mathematical terms and properties may be reduced to basic formulas without class terms. The basic language can be chosen extremely small, but we may also work in a very rich language employing class terms.

The set theoretic analysis of mathematics shows that the following basic language is indeed sufficient:

Definition 2.1. *The (basic)* ***language of set theory*** *has variables v_0, v_1, ... The* ***atomic formulas*** *of the language are the formulas $x = y$ ("x equals y") and $x \in y$ ("x is an element of y") where x and y are variables. The collection of* ***formulas*** *of the language is the smallest collection $L(\in)$ which contains the atomic formulas and is closed under the following rules:*

- *if φ is a formula then $\neg\varphi$ ("not φ") is a formula;*
- *if φ and ψ are formulas then $\varphi \vee \psi$ ("φ or ψ") is a formula;*
- *if φ is a formula and x is a variable then $\exists x \varphi$ ("there is x such that φ") is a formula.*

A formula is also called an $\in$-formula. As usual we understand other propositional operators or quantifiers as abbreviations. So $\varphi \wedge \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$ and $\forall x \varphi$ stand for $\neg(\neg\varphi \vee \neg\psi)$, $\neg\varphi \vee \psi$, $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ and $\neg\exists x \neg\varphi$ respectively. Also the formula $\varphi\frac{\vec{y}}{\vec{x}}$ is obtained from φ by ***substituting*** *the variables $\vec{x}$ by $\vec{y}$.*

We now introduce the rich language involving class terms.

Definition 2.2. *A* ***class term*** *is a symbol sequence of the form*

$$\{x \mid \varphi\} \text{ (``the class of } x \text{ such that } \varphi\text{'')}$$

where x is one of the variables v_0, v_1, ... and φ is an $\in$-formula. A ***term*** *is a variable or a class term. We now allow arbitrary terms to be used in (atomic) formulas. A* ***generalized atomic formula*** *is a formula of the form $s = t$ or $s \in t$ where s and t are terms. Form the* ***generalized formulas*** *from the generalized atomic formulas by the same rules as in the previous definition.*

Generalized formulas can be translated into strict $\in$-formulas according to the above intuition of class and collection. It suffices to define the elimination of class terms for generalized atomic formulas. So we recursively translate

$$\begin{aligned}
y \in \{x \mid \varphi\} &\text{ into } \varphi\frac{y}{x}, \\
\{x \mid \varphi\} = \{y \mid \psi\} &\text{ into } \forall z (z \in \{x \mid \varphi\} \leftrightarrow z \in \{y \mid \psi\}), \\
x = \{y \mid \psi\} &\text{ into } \forall z (z \in x \leftrightarrow z \in \{y \mid \psi\}), \\
\{y \mid \psi\} = x &\text{ into } \forall z (z \in \{y \mid \psi\} \leftrightarrow z \in x), \\
\{x \mid \varphi\} \in \{y \mid \psi\} &\text{ into } \exists z (\psi\frac{z}{y} \wedge z = \{x \mid \varphi\}), \\
\{x \mid \varphi\} \in y &\text{ into } \exists z (z \in y \wedge z = \{x \mid \varphi\}).
\end{aligned}$$

The translation of the equalities corresponds to the intuition that a class is determined by its extent rather by the specific formula defining it. If at least one of s and t is a class term, then by the elimination procedure

$$s=t \text{ iff } \forall z\,(z\in s\leftrightarrow z\in t).$$

We also have $x=\{y\,|\,y\in x\}$ where we assume a reasonale choice of variables. In this case this means that x and y are different variables. Under the natural assumption that our term calculus satisfies the usual laws of equality, we get

$$\begin{aligned} x=y \ \text{ iff } \ & \{v\,|\,v\in x\}=\{v\,|\,v\in y\} \\ \text{ iff } \ & \forall z\,(z\in\{v\,|\,v\in x\}\leftrightarrow z\in\{v\,|\,v\in y\}) \\ \text{ iff } \ & \forall z\,(z\in x\leftrightarrow z\in y). \end{aligned}$$

This is the *axiom of extensionality* for sets, which will later be part of the set-theoretical axioms. We have obtained it here assuming that $=$ for class terms is transitive. In our later development of set theory from the Zermelo-Fraenkel axioms one would rather have to show these axioms imply the equality laws for class terms.

2.2 Extending the Language

We introduce special names and symbols for important class terms and formulas. Naming and symbols follow traditions and natural intuitions. In principle, all mathematical notions could be interpreted this way, but we restrict our attention to set theoretical notions. We use also many usual notations and conventions, like $x\neq x$ instead of $\neg x=x$.

Definition 2.3. *Define the following class terms and formulas:*

a) $\emptyset:=\{x\,|\,x\neq x\}$ *is the* ***empty class****;*

b) $x\subseteq y:=\forall z\,(z\in x\rightarrow z\in y)$ *denotes that* x *is a* ***subclass*** *of* y*;*

c) $\{x\}:=\{y\,|\,y=x\}$ *is the* ***singleton*** *of* x*;*

d) $\{x,y\}:=\{z\,|\,z=x\vee z=y\}$ *is the* ***unordered pair*** *of* x *and* y*;*

e) $(x,y):=\{\{x\},\{x,y\}\}$ *is the* ***(ordered) pair*** *of* x *and* y*;*

f) $\{x_0,\ldots,x_{n-1}\}:=\{y\,|\,y=x_0\vee\ldots\vee y=x_{n-1}\}$*;*

g) $x\cap y:=\{z\,|\,z\in x\wedge z\in y\}$ *is the* ***intersection*** *of* x *and* y*;*

h) $x\cup y:=\{z\,|\,z\in x\vee z\in y\}$ *is the* ***union*** *of* x *and* y*;*

i) $x\setminus y:=\{z\,|\,z\in x\wedge z\notin y\}$ *is the* ***difference*** *of* x *and* y*;*

j) $\overline{x}:=\{y\,|\,y\in x\}$ *is the* ***complement*** *of* x*;*

k) $\bigcap x:=\{z\,|\,\forall y\,(y\in x\rightarrow z\in y)\}$ *is the* ***intersection*** *of* x*;*

l) $\bigcup x:=\{z\,|\,\exists y\,(y\in x\wedge z\in y)\}$ *is the* ***union*** *of* x*;*

m) $\mathcal{P}(x) := \{y \mid y \subseteq x\}$ *is the **power** of* x*;*

n) $V := \{x \mid x = x\}$ *is the **universe** or the **class of all sets**;*

o) x ***is a set*** $:= x \in V$.

Strictly speaking, these notions are just syntactical objects. Nevertheless they correspond to certain intuitive expectations, and the notation has been chosen accordingly. The axioms of Zermelo-Fraenkel set theory will later ensure, that the notions do have the expected properties.

Note that we have now formally introduced the notion of *set*. The variables of our language range over sets, terms which are equal to some variable are sets. If t is a term then

$$t \text{ is a set iff } t \in V \text{ iff } \exists x\,(x = x \wedge x = t) \text{ iff } \exists x\, x = t.$$

Here we have inserted the term t into the formula "x is a set". In general, the substitution of terms into formulas is understood as follows: the formula is translated into a basic $\in$-formula and then the term is substituted for the appropriate variable. In a similar way, terms $t_0, \ldots, t_{n-1}$ may be substituted into another terms $t(x_0, \ldots, x_{n-1})$: let $t(x_0, \ldots, x_{n-1})$ be the class term $\{x \mid \varphi(x, x_0, \ldots, x_{n-1})\}$; then

$$t(t_0, \ldots, t_{n-1}) = \{x \mid \varphi(x, t_0, \ldots, t_{n-1})\}$$

where the right-hand side substitution is carried out as before. This allows to work with complex terms and formulas like

$$\{\emptyset\}, \{\emptyset, \{\emptyset\}\}, x \cup (y \cup z), x \cap y \subseteq x \cup y, \emptyset \text{ is a set.}$$

A few natural properties can be checked already on the basis of the laws of first-order logic. We give some examples:

Theorem 2.4. *a*) *For terms* t *we have* $\emptyset \subseteq t$ *and* $t \subseteq V$.

b) *For terms* s, t, r *with* $s \subseteq t$ *and* $t \subseteq r$ *we have* $s \subseteq r$.

c) *For terms* s, t *we have* $s \cap t = t \cap s$ *and* $s \cup t = t \cup s$.

Proof. b) Assume $s \subseteq t$ and $t \subseteq r$. Let $z \in s$. Then $z \in t$, since $s \subseteq t$. $z \in r$, since $t \subseteq r$. Thus $\forall z\,(z \in s \rightarrow z \in r)$, i.e., $s \subseteq r$.

The other properties are just as easy. □

Russell's antinomy is also just a consequence of logic:

Theorem 2.5. *The class* $\{x \mid x \notin x\}$ *is not a set.*

Proof. Assume for a contradiction that $\{x \mid x \notin x\} \in V = \{x \mid x = x\}$. This translates into $\exists z\,(z = z \wedge z = \{x \mid x \notin x\})$. Take z such that $z = \{x \mid x \notin x\}$. Then

$$z \in z \leftrightarrow (x \notin x)\frac{z}{x} \leftrightarrow z \notin z.$$

Contradiction. □

2.3 Relations and Functions

Apart from sets, *relations* and *functions* are the main building blocks of mathematics. As usual, relations are construed as sets of ordered pairs. Again we note that the subsequent notions only attain all their intended properties under the assumption of sufficiently many set theoretical axioms.

Definition 2.6. *Let t be a term and φ be a formula, where $\vec{x}$ is the sequence of variables which are both free in t and in φ. Then write the generalized class term*

$$\{t\,|\,\varphi\} \textit{ instead of } \{y\,|\,\exists\vec{x}\,(y=t(\vec{x}\,)\wedge\varphi(\vec{x}\,))\}.$$

Definition 2.7. *a) $x\times y=\{(u,v)|u\in x\wedge v\in y\}$ is the* ***(cartesian) product*** *of x and y.*

b) x is a ***relation*** *$:= x\subseteq V\times V$.*

c) x is a ***relation on*** *y $:= x\subseteq y\times y$.*

d) $x r y := (x,y)\in r$ is the usual ***infix*** *notation for relations.*

e) $\operatorname{dom}(r):=\{x\,|\,\exists y\, x r y\}$ is the ***domain*** *of r.*

f) $\operatorname{ran}(r):=\{y\,|\,\exists x\, x r y\}$ is the ***range*** *of r.*

g) $\operatorname{field}(r):=\operatorname{dom}(r)\cup\operatorname{ran}(r)$ is the ***field*** *of r.*

h) $r\upharpoonright a:=\{(x,y)|(x,y)\in r\wedge x\in a\}$ is the ***restriction*** *of r to a.*

i) $r[a]:=\{y\,|\,\exists x(x\in a\wedge(x,y)\in r\}$ is the ***image*** *of a under r.*

j) $r^{-1}[b]:=\{x\,|\,\exists y(y\in b\wedge(x,y)\in r\}$ is the ***preimage*** *of b under r.*

k) $r\circ s:=\{(x,z)\,|\,\exists y(x r y\wedge y s z)\}$ is the ***composition*** *of r and s.*

l) $r^{-1}:=\{(y,x)\,|\,(x,y)\in r\}$ is the ***inverse*** *of r.*

Definition 2.8. *a) f is a* ***function*** *$:=$ f is a relation $\wedge\,\forall x\forall y\forall z\,(x f y\wedge x f z\rightarrow y=z)$.*

b) $f(x)=\bigcup\,\{y\,|\,x f y\}$ is the ***value*** *of f at x.*

c) f is a ***function*** *from a into b $:= f\colon a\rightarrow b :=$ f is a function $\wedge\operatorname{dom}(f)=a\wedge\operatorname{ran}(f)\subseteq b$.*

d) ${}^a b:=\{f\,|\,f\colon a\rightarrow b\}$ is the space of all functions from a into b.

e) $\bigtimes g:=\{f\,|\,\operatorname{dom}(f)=\operatorname{dom}(g)\wedge\forall x\,(x\in\operatorname{dom}(g)\rightarrow f(x)\in g(x))\}$ is the ***(cartesian) product*** *of g.*

Note that the product of g consists of *choice functions* f, where for every argument $x\in\operatorname{dom}(g)$ the value $f(x)$ chooses an element of $g(x)$.

Chapter 3
The Zermelo-Fraenkel Axioms

Russell's antinomy can be seen as a motivation for the axiomatization of set theory: not *all* classes can be sets, but we want *many* classes to be sets. We formulate axioms in the term language introduced above. Most of them are *set existence axioms* of the form $t \in V$. In writing the axioms we omit all initial universal quantifiers, i.e., φ stands for $\forall \vec{x}\, \varphi$ where $\{\vec{x}\,\}$ is the set of free variables of φ.

Definition 3.1.

1. ***Axiom of extensionality***: $x \subseteq y \wedge y \subseteq x \rightarrow x = y$.
2. ***Pairing axiom***: $\{x, y\} \in V$.
3. ***Union axiom***: $\bigcup x \in V$.
4. ***Axiom of infinity***: $\exists x\, (\emptyset \in x \wedge \forall y\, (y \in x \rightarrow y \cup \{y\} \in x))$.
5. ***Axiom (schema) of subsets***: *for all terms A postulate:* $x \cap A \in V$.
6. ***Axiom (schema) of replacement***: *for all terms F postulate:*
   F is a function $\rightarrow F[x] \in V$.
7. ***Axiom (schema) of foundation***: *for all terms A postulate:*
   $A \neq \emptyset \rightarrow \exists x\, (x \in A \wedge x \cap A = \emptyset)$.
8. ***Powerset axiom***: $\mathcal{P}(x) \in V$.
9. ***Axiom of choice*** *(AC)*:
   f is a function $\wedge \forall x\, (x \in \mathrm{dom}(f) \rightarrow f(x) \neq \emptyset) \rightarrow \bigtimes f \neq \emptyset$.
10. *The* **Zermelo-Fraenkel** ***axiom system*** ZF *consists of the axioms 1 - 8.*
11. *The axiom system* ZF^- *consists of the axioms 1 - 7.*
12. *The axiom system* ZFC *consists of the axioms 1 - 9.*

Remarkably, virtually all of mathematics can be developed naturally in the axiom system ZFC: one formalizes the systems of natural, integer, rational, and real numbers; all further notions of mathematics can be expressed by set operations and properties. This is usually presented in introductory texts on set theory.

Note that the set theoretical axioms possess very different characters. There are seemingly week axioms like the pairing or union axiom which postulate the existence of concretely specified sets. On the other hand, a powerset seems to be a vast object which is hard to specify other than by its general definition. The theory ZF^- avoids the problematic powerset axiom as well as the axiom of choice. We shall carry out most of our initial development within ZF^-.

Chapter 4
Induction, recursion, and ordinals

4.1 $\in$-induction

We work in the theory ZF^-. Let us first introduce some notation:

Definition 4.1. *Write*

$$\exists x \in s\,\varphi \text{ instead of } \exists x\,(x \in s \wedge \varphi),$$

$$\forall x \in s\,\varphi \text{ instead of } \forall x\,(x \in s \rightarrow \varphi),$$

and

$$\{x \in s \,|\, \varphi\} \text{ instead of } \{x \,|\, x \in s \wedge \varphi\}.$$

*These notations use x as a **bounded variable**, the quantifiers $\exists x \in s$ and $\forall x \in s$ are called **bounded quantifiers**.*

The axiom of foundation is equivalent to an induction schema for the $\in$-relation: if a property is inherited from the $\in$-predecessors, it holds everywhere.

Theorem 4.2. *Let $\varphi(x, \vec{y}\,)$ be an $\in$-formula such that*

$$\forall x\,(\forall z \in x\,\varphi(z, \vec{y}\,) \rightarrow \varphi(x, \vec{y}\,)).$$

Then

$$\forall x\,\varphi(x, \vec{y}\,).$$

Proof. Assume not. Then $A := \{x \,|\, \neg\varphi(x, \vec{y}\,)\} \neq \emptyset$. By the foundation schema for A take some $x \in A$ such that $x \cap A = \emptyset$, i.e., $\forall z \in x\, x \notin A$. By the definition of A

$$\neg\varphi(x, \vec{y}\,) \text{ and } \forall z \in x\,\varphi(z, \vec{y}\,).$$

This contradicts the assumption of the theorem. $\square$

4.2 Transitive Sets and Classes

Definition 4.3. *The class s is **transitive** iff $\forall x \in s \forall y \in x\; y \in s$. We write* $\mathrm{Trans}(s)$ *if s is transitive.*

Theorem 4.4. *s is transitive iff $\forall x\in s\; x\subseteq s$ iff $\forall x\in s\; x=x\cap s$.*

A transitive class is an $\in$-initial segment of the class of all sets.

Theorem 4.5. *a) $\emptyset$ and V are transitive.*

b) If $\forall x\in A\,\mathrm{Trans}(x)$ then $\bigcap A$ and $\bigcup A$ are transitive.

c) If x is transitive then $x\cup\{x\}$ is transitive.

Proof. *Exercise.* □

4.3 $\in$-recursion

We prove a recursion principle which corresponds to the principle of $\in$-induction.

Theorem 4.6. *Let $G: V\to V$. Then there is a class term F such that*

$$F: V\to V \text{ and } \forall x\, F(x)=G(F\upharpoonright x).$$

The function F is uniquely determined: if F': $V\to F$ and $\forall x\, F'(x)=G(F'\upharpoonright x)$. Then

$$F=F'.$$

The term F is defined explicitely in the subsequent proof and is called the ***canonical term defined by $\in$-recursion by*** *$F(x)=G(F\upharpoonright x)$.*

Proof. We construct F as a union of approximations to F. Call a function $f\in V$ a *G-approximation* if

- f: $\mathrm{dom}(f)\to V$;
- $\mathrm{dom}(f)$ is transitive;
- $\forall x\, f(x)=G(f\upharpoonright x)$.

We prove some structural properties for the class of G-approximations:
(1) If f and f' are G-approximations then $\forall x\in\mathrm{dom}(f)\cap\mathrm{dom}(f')\; f(x)=f'(x)$.
Proof. Assume not and let $x\in\mathrm{dom}(f)\cap\mathrm{dom}(f')$ be $\in$-minimal with $f(x)\neq f'(x)$. Since $\mathrm{dom}(f)\cap\mathrm{dom}(f')$ is transitive, $x\subseteq\mathrm{dom}(f)\cap\mathrm{dom}(f')$. By the $\in$-minimality of x, $f\upharpoonright x=f'\upharpoonright x$. Then

$$f(x)=G(f\upharpoonright x)=G(f'\upharpoonright x)=f'(x),$$

contradiction. *qed*(1)
(2) $\forall x\exists f\,(f$ is a G-approximation $\wedge\, x\in\mathrm{dom}(f))$.
Proof. Assume not and let x be an $\in$-minimal counterexample. For $y\in x$ define

$$f_y=\bigcap\,\{f\,|\,f \text{ is a } G\text{-approximation} \wedge y\in\mathrm{dom}(f)\}.$$

By the minimality of x, there at least one f such that

$$f \text{ is a } G\text{-approximation} \wedge y\in\mathrm{dom}(f).$$

The intersection of such approximations is an approximation itself, so that

$$f_y \text{ is a } G\text{-approximation} \wedge y \in \mathrm{dom}(f_y).$$

Then define

$$f_x = (\bigcup_{y \in x} f_y) \cup \{(x, G((\bigcup_{y \in x} f_y) \restriction x)).$$

One can now check that f_x is a G-approximation with $x \in \mathrm{dom}(f_x)$. Contradiction. *qed*(2)

Now set

$$F = \bigcup \{f \mid f \text{ is a } G\text{-approximation}\}.$$

Then F satisfies the theorem. □

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Definition 4.7. *Let* TC *be the canonical term defined by* $\in$*-recursion by*

$$\mathrm{TC}(x) = x \cup \bigcup_{y \in x} \mathrm{TC}(y).$$

$\mathrm{TC}(x)$ *is called the* ***transitive closure of*** x*.*

Theorem 4.8. *For all* $x \in V$*:*

a) $\mathrm{TC}(x)$ *is transitive and* $\mathrm{TC}(x) \supseteq x$*;*

b) $\mathrm{TC}(x)$ *is the* $\subseteq$*-smallest transitive superset of* x*.*

Proof. By $\in$-induction. Let $x \in V$ and assume that a) and b) hold for all $z \in x$. Then
(1) $\mathrm{TC}(x) \supseteq x$ is obvious from the recursive equation for TC.
(2) $\mathrm{TC}(x)$ is transitive.
Proof. Let $u \in v \in \mathrm{TC}(x)$.
Case 1: $v \in x$. Then

$$u \in v \subseteq \mathrm{TC}(v) \subseteq \bigcup_{y \in x} \mathrm{TC}(y) \subseteq \mathrm{TC}(x).$$

Case 2: $v \notin x$. Then take $y \in x$ such that $v \in \mathrm{TC}(y)$. $\mathrm{TC}(y)$ is transitive by hypothesis, hence

$$u \in \mathrm{TC}(y) \subseteq \bigcup_{y \in x} \mathrm{TC}(y) \subseteq \mathrm{TC}(x).$$

qed(2)
b) Let $w \supseteq x$ be transitive. Let $y \in x$. Then $y \in w$, $y \subseteq w$. By hypothesis, $\mathrm{TC}(y)$ is the $\subseteq$-minimal superset of y, hence $\mathrm{TC}(y) \subseteq w$. Thus

$$\bigcup_{y \in x} \mathrm{TC}(y) \subseteq w$$

and

$$\mathrm{TC}(x) = x \cup \bigcup_{y \in x} \mathrm{TC}(y) \subseteq w$$

□

4.4 Ordinals

The number system of *ordinal numbers* is particularly adequate for the study of the infinite. We present the theory of von Neumann-ordinals based on the notion of transitivity.

Definition 4.9. *A set x is an* ***ordinal*** *if* $\mathrm{Trans}(x) \wedge \forall y \in x\, \mathrm{Trans}(y)$. *Let*

$$\mathrm{Ord} = \{x \,|\, x \textit{ is an ordinal}\}$$

be the class of all ordinals.

We show that the ordinals are a generalization of the natural numbers into the transfinite.

Theorem 4.10. *The class* Ord *is strictly well-ordered by* $\in$.

Proof. (1) $\in$ is a transitive relation on Ord.
Proof. Let $x, y, z \in \mathrm{Ord}$, $x \in y$, and $y \in z$. Since z is a transitive set, $x \in z$. *qed*(1)
(2) $\in$ is a linear relation on Ord, i.e., $\forall x, y \in \mathrm{Ord}\, (x \in y \vee x = y \vee y \in x)$.
Proof. Assume not. Let x be $\in$-minimal such that

$$\exists y\, (x \notin y \wedge x \neq y \wedge y \notin x).$$

Let y be $\in$-minimal such that

$$x \notin y \wedge x \neq y \wedge y \notin x. \tag{4.1}$$

Let $x' \in x$. Then by the minimality of x we have

$$x' \in y \vee x' = y \vee y \in x'.$$

If $x' = y$ then $y = x' \in x$, contradicting (4.1). If $y \in x'$ then $y \in x' \in x$ and $y \in x$, contradicting (4.1). Thus $x' \in y$. This shows $x \subseteq y$.

Conversely let $y' \in y$. Then by the minimality of y we have

$$x \in y' \vee x = y' \vee y' \in x.$$

If $x \in y'$ then $x \in y' \in y$ and $x \in y$, contradicting (4.1). If $x = y'$ then $x = y' \in y$, contradicting (4.1). Thus $y' \in x$. This shows $y \subseteq x$.

Hence $x = y$, contradicting (4.1). *qed*(2)
(3) $\in$ is an irreflexive relation on Ord, i.e., $\forall x \in \mathrm{Ord}\; x \notin x$.
Proof. Assume for a contradiction that $x \in x$. By the foundation scheme applied to the term $A = \{x\} \neq \emptyset$ let $y \in \{x\}$ with $y \cap \{x\} = \emptyset$. Then $y = x$, $x \in x = y$, $x \in y \cap \{x\}$ which contradicts the choice of y. *qed*(3)
(4) $\in$ is a well-order on Ord, i.e., for every non-empty $A \subseteq \mathrm{Ord}$ there exists $\alpha \in A$ such that $\forall \beta \in \alpha\, \beta \notin A$.
Proof. By the foundation scheme applied to A let $\alpha \in A$ with $\alpha \cap A = \emptyset$. Then $\forall \beta \in \alpha\, \beta \notin A$. □

By this theorem, $\in$ is the canonical order on the ordinal numbers. We use greek letters $\alpha, \beta, \gamma, \ldots$ as variables for ordinals and write $\alpha < \beta$ instead of $\alpha \in \beta$. When we talk about smallest or largest ordinals this is meant with respect to the ordering $<$.

Theorem 4.11. *a)* $\emptyset$ *is the smallest element of* Ord. *We write* 0 *instead of* $\emptyset$ *when* $\emptyset$ *is used as an ordinal.*

b) *If* $\alpha \in \mathrm{Ord}$ *then* $\alpha \cup \{\alpha\}$ *is the smallest element of* Ord *which is larger than* α*, i.e.,* $\alpha \cup \{\alpha\}$ *is the* ***successor*** *of* α. *We write* $\alpha + 1$ *instead of* $\alpha \cup \{\alpha\}$. *Every ordinal of the form* $\alpha + 1$ *is called a* ***successor ordinal***.

Proof. b) Let $\alpha \in \mathrm{Ord}$.
(1) $\alpha \cup \{\alpha\}$ is transitive.
Proof. Let $u \in v \in \alpha \cup \{\alpha\}$.
Case 1. $v \in \alpha$. Then $u \in \alpha \subseteq \alpha \cup \{\alpha\}$ since α is transitive.
Case 2. $v \in \{\alpha\}$. Then $u \in v = \alpha \subseteq \alpha \cup \{\alpha\}$. *qed*(1)
(2) $\forall y \in \alpha \cup \{\alpha\}\ \mathrm{Trans}(y)$.
Proof. Let $y \in \alpha \cup \{\alpha\}$.
Case 1. $y \in \alpha$. Then $\mathrm{Trans}(y)$, since α is an ordinal.
Case 2. $y \in \{\alpha\}$. Then $y = \alpha$, and $\mathrm{Trans}(y)$, since α is an ordinal. *qed*(2)
So $\alpha \cup \{\alpha\}$ is an ordinal, and $\alpha \cup \{\alpha\} > \alpha$.
(3) $\alpha \cup \{\alpha\}$ is the smallest ordinal $> \alpha$.
Proof. Let $\beta < \alpha \cup \{\alpha\}$. Then $\beta \in \alpha$ or $\beta = \alpha$. Hence $\beta \leqslant \alpha$ and $\beta \ngtr \alpha$. ☐

Theorem 4.12. *a)* Ord *is transitive.*

b) $\forall x \in \mathrm{Ord}\ \mathrm{Trans}(x)$.

c) $\mathrm{Ord} \notin V$*, i.e.,* Ord *is a proper class.*

Proof. a) Let $x \in y \in \mathrm{Ord}$.
(1) $\mathrm{Trans}(x)$, since every element of the ordinal y is transitive.
(2) $\forall u \in x\ \mathrm{Trans}(u)$.
Proof. Let $u \in x$. Since y is transitive, $u \in y$. Since every element of y is transitive, $\mathrm{Trans}(u)$. *qed*(2)
Thus $x \in \mathrm{Ord}$.
b) is part of the definition of ordinal.
c) Assume $\mathrm{Ord} \in V$. By a) and b), Ord satisfies the definition of an ordinal, and so $\mathrm{Ord} \in \mathrm{Ord}$. This contradicts the foundation scheme. ☐

4.5 Natural numbers

One can construe the common natural numbers as those ordinal numbers which can be reached from 0 by the $+1$-operation. Consider the following term:

Definition 4.13. $\omega = \{\alpha \in \mathrm{Ord} \mid \forall \beta \in \alpha + 1\, (\beta = 0 \vee \beta \text{ is a successor ordinal})\}$ *is the class of* ***natural numbers***.

Theorem 4.14. *ω is transitive.*

Proof. Let $x \in \alpha \in \omega$.
(1) $x \in \mathrm{Ord}$, since Ord is transitive.
(2) $x \subseteq \alpha$, since α is transitive.
(3) $x+1 \subseteq \alpha \subseteq \alpha+1$.
(4) $\forall \beta \in x+1\,(\beta = 0 \vee \beta$ is a successor ordinal), since $\alpha \in \omega$ and $x+1 \subseteq \alpha+1$.
Then (1) and (4) imply that $x \in \omega$. □

Theorem 4.15. *$\omega \in V$, i.e., ω is the* ***set of natural numbers****.*

Proof. By the axiom of infinity, take a set x such that

$$(0 \in x \wedge \forall y\,(y \in x \rightarrow y \cup \{y\} \in x)).$$

(1) $\omega \subseteq x$.
Proof. Assume for a contradiction that $\omega \nsubseteq x$. By foundation take $z \in \omega$ $\in$-minimal such that $z \notin x$. By the definition of ω we have $z = 0$ or z is a successor ordinal. The case $z = 0$ is impossible by the choice of x. Hence z is a successor ordinal. Take $y \in \mathrm{Ord}$ such that $z = y+1$. Then $y \in z \in \omega$ and $y \in \omega$ by the transitivity of ω. By the $\in$-minimal choice of z we have $y \in x$. By the choice of x we have $z = y+1 = y \cup \{y\} \in x$. This contradicts the choice of z. *qed*(1)

The subset schema implies that $\omega = x \cap \omega \in V$. □

Theorem 4.16. *ω is a* ***limit ordinal****, i.e., an ordinal $\neq 0$ which is not a successor ordinal. Indeed, ω is the smallest limit ordinal:*

$$\omega = \bigcap \{\alpha \mid \alpha \text{ is a limit ordinal}\}.$$

Proof. First note that ω is an ordinal, since it is a transitive set and each of its elements is transitive.

Obviously $0 \in \omega$, hence $\omega \neq 0$. Assume for a contradiction that ω is a successor ordinal. Take some $\alpha \in \mathrm{Ord}$ such that $\omega = \alpha + 1$. Then $\alpha \in \omega$ and

$$\forall \beta \in \alpha+1\,(\beta = 0 \vee \beta \text{ is a successor ordinal}).$$

$\omega + 1 = (\alpha+1) \cup \{\omega\}$. Since ω is assumed to be a successor ordinal

$$\forall \beta \in \omega+1\,(\beta = 0 \vee \beta \text{ is a successor ordinal}).$$

Hence $\omega \in \omega$. But this contradicts the foundation schema.

Thus ω is a limit ordinal.

Let γ be (another) limit ordinal. Since all elements of ω are 0 or successor ordinals, we cannot have $\gamma < \omega$. Therefore $\omega \leqslant \gamma$. □

Let us justify this formalization of the set of natural numbers by

Theorem 4.17. *The structure $(\omega, +1, 0)$ satisfies the* Peano *axioms:*

a) $0 \in \omega$*;*

b) $\forall n \in \omega\; n+1 \in \omega$*;*

c) $\forall n \in \omega\, n+1 \neq 0$*;*

d) $\forall m, n \in \omega\, (m+1 = n+1 \rightarrow m = n)$*;*

e) $\forall x \subseteq \omega\, ((0 \in x \wedge \forall m \in x\, m+1 \in x) \rightarrow x = \omega)$*.*

Proof. Axioms a) to d) are immediate from the definition of ω or from the general properties of ordinals. For e) consider a set $x \subseteq \omega$ such that

$$0 \in x \wedge \forall m \in x\, m+1 \in x.$$

Assume for a contradiction that $x \neq \omega$. By foundation take $z \in \omega$ $\in$-minimal such that $z \notin x$. By the definition of ω we have $z = 0$ or z is a successor ordinal. The case $z = 0$ is impossible by the properties of x. Hence z is a successor ordinal. Take $y \in \mathrm{Ord}$ such that $z = y+1$. Then $y \in z \in \omega$ and $y \in \omega$ by the transitivity of ω. By the $\in$-minimal choice of z we have $y \in x$. By the inductive property of x we have $z = y+1 = y \cup \{y\} \in x$. This contradicts the choice of z. $\square$

Chapter 5

Transitive $\in$-models

Axiomatic set theory studies the axiom systems ZF and ZFC. By the GÖDEL incompleteness theorem, these systems are incomplete. So one is lead to consider extensions of these systems of the form ZF + φ or ZFC + φ for various φ. Even some simple questions of the arithmetic of infinite cardinals like CANTOR's *continuum hypothesis* are not decided by ZFC and present an ongoing challenge to set theoretical research.

To show that a theory like ZFC + φ is consistent one constructs *models* of that theory (making some initial assumptions). Usually these models will be an $\in$-*model* of the form $(M, \in)$, where M is some class.

5.1 Relativizations of Formulas and Terms

Evaluating an $\in$-formula φ in a model $(M, \in)$ amount to bounding the range of quantifiers in φ to M.

Definition 5.1. *Let M be a term. For φ an $\in$-formula define the **relativization** φ^M of φ to M by recursion on the complexity of φ:*

- $(x \in y)^M := x \in y$
- $(x = y)^M := x = y$
- $(\neg\varphi)^M := \neg(\varphi^M)$
- $(\varphi \vee \psi)^M := \varphi^M \vee \psi^M$
- $(\exists x \varphi)^M := \exists x \in M \varphi^M$

Definition 5.2. *Let M be a term and let Φ be a (metatheoretical) set of formulas. Then the (metatheoretical) set*

$$\Phi^M = \{\varphi^M \mid \varphi \in \Phi\}$$

*is the **relativization** of Φ to M.*

The relativizations φ^M and Φ^M correspond to the model-theoretic satisfaction relations $(M, \in) \vDash \varphi$ and $(M, \in) \vDash \Phi$. This is illustrated by

Theorem 5.3. *Let Φ be a finite set of $\in$-formulas and let φ be an $\in$-formula such that $\Phi\vdash\varphi$ in the calculus of first-order logic. Let M be a transitive term, $M\neq\emptyset$, which has no common free variables with Φ or φ. Then*

$$\forall\vec{x}\in M\,((\bigwedge\Phi)^M\rightarrow\varphi^M),$$

where $\vec{x}$ includes all the free variables of Φ and φ.

Proof. By induction on the lengths of derivations it suffices to prove the theorem for the case that $\Phi\vdash\varphi$ is derivable by a single application of a rule of the first-order calculus. We check this for the various rules.

The theorem is obvious in case φ is an element of Φ.

In case $\varphi=(x=x)$, the relativization $(x=x)^M=(x=x)$ holds in any case.

The theorem is easy to show for all propositional rules and the substitution rule.

So let us now consider the quantifier rules. Assume that $\varphi(x,\vec{y}\,)^M$ where x, $\vec{y}\in M$. Then $\exists x\,(x\in M\wedge\varphi(x,\vec{y}\,)^M)$ and

$$(\exists x\,\varphi(x,\vec{y}\,))^M$$

as required.

For the $\exists$-introduction in the *antecedens* suppose that

$$\forall x,\vec{y}\in M\,((\bigwedge\Phi)^M\wedge\psi^M(x,\vec{y}\,)\rightarrow\varphi^M(\vec{y}\,)),\tag{5.1}$$

where the variable x does not occur in Φ or φ. Now let $\vec{y}\in M$ and assume that $(\bigwedge\Phi)^M\wedge(\exists x\psi)^M(\vec{y})$. Then $\exists x\in M\psi^M(x,\vec{y}\,)$. Take $x\in M$ such that $\psi^M(x,\vec{y}\,)$. By (4.1) we get $\varphi^M(\vec{y}\,)$. Hence

$$\forall\vec{y}\in M\,((\bigwedge\Phi)^M\wedge(\exists x\psi)^M(\vec{y}\,)\rightarrow\varphi^M(\vec{y}\,)).\qquad\square$$

We shall later construct models M such that ZFC^M holds.

Definition 5.4. *Let M be a term. For a class term $s=\{x\,|\,\varphi\}$ define the **relativization** s^M of s to M by:*

$$s^M:=\{x\in M\,|\,\varphi^M\}.$$

If s is a variable, $s=x$, then let $s^M=s$.

s^M is the term s as evaluated in M. We show that evaluating a formula with class terms (a *generalized* formula) in a *transitive* class M is the same as relativizing the basic formula without class terms and then inserting the relatived class terms. This will make many notions *absolute* between M and V.

Note that the relativization of a *bounded* quantifier $\exists x\in y$ to a transitive class M with $y\in M$ has no effect:

$$\exists x\in y\varphi\leftrightarrow\exists x\in y\cap M\,\varphi.$$

Theorem 5.5. *Let M be a transitive class. Let $\varphi(x_0,\ldots,x_{n-1})$ be a basic formula and $t_0,\ldots,t_{n-1}$ be terms. Then*

$$\forall\vec{w}\in M[(\chi(t_0,\ldots,t_{n-1}))^M\leftrightarrow\chi^M(t_0^M,\ldots,t_{n-1}^M)],$$

where $\{\vec{w}\}$ *is the set of free variables of* $\chi(t_0, \ldots, t_{n-1})$.

Proof. By induction on the complexity of χ. Let $\vec{w} \in M$.

Let χ be an atomic formula of the form $u \in v$ or $u = v$. If t_0 and t_1 are variables there is nothing to show. The other cases correspond to the following equivalences:

$$\begin{aligned}
(y \in \{x \mid \varphi\})^M &\leftrightarrow (\varphi\frac{y}{x})^M \\
&\leftrightarrow \varphi^M\frac{y}{x} \\
&\leftrightarrow (x \in M \wedge \varphi^M)\frac{y}{x} \\
&\leftrightarrow y \in \{x \mid x \in M \wedge \varphi^M\} = \{x \in M \mid \varphi^M\} \\
&\leftrightarrow y^M \in \{x \mid \varphi\}^M.
\end{aligned}$$

This equivalence is already used in:

$$\begin{aligned}
(\{x \mid \varphi\} = \{y \mid \psi\})^M &\leftrightarrow (\forall z (z \in \{x \mid \varphi\} \leftrightarrow z \in \{y \mid \psi\}))^M \\
&\leftrightarrow \forall z \in M (z \in \{x \mid \varphi\}^M \leftrightarrow z \in \{y \mid \psi\}^M) \\
&\leftrightarrow \forall z (z \in \{x \mid \varphi\}^M \leftrightarrow z \in \{y \mid \psi\}^M), \text{ since } \{x|\varphi\}^M \subseteq M, \\
&\leftrightarrow \{x \mid \varphi\}^M = \{y \mid \psi\}^M.
\end{aligned}$$

Note, that $x \subseteq M$ by the transitivity of M:

$$\begin{aligned}
(x = \{y \mid \psi\})^M &\leftrightarrow (\forall z (z \in x \leftrightarrow z \in \{y \mid \psi\}))^M \\
&\leftrightarrow \forall z \in M (z \in x \leftrightarrow z \in \{y \mid \psi\}^M) \\
&\leftrightarrow \forall z (z \in x \leftrightarrow z \in \{y \mid \psi\}^M), \text{ since } x \subseteq M, \\
&\leftrightarrow x^M = \{y \mid \psi\}^M.
\end{aligned}$$

$$\begin{aligned}
(\{x \mid \varphi\} \in \{y \mid \psi\})^M &\leftrightarrow (\exists z (\psi\frac{z}{y} \wedge z = \{x \mid \varphi\}))^M \\
&\leftrightarrow \exists z \in M (\psi^M\frac{z}{y} \wedge z = \{x \mid \varphi\}^M) \\
&\leftrightarrow \exists z (z \in M \wedge \psi^M\frac{z}{y} \wedge z = \{x \mid \varphi\}^M) \\
&\leftrightarrow \{x \mid \varphi\}^M \in \{y \mid y \in M \wedge \psi^M\} = \{y|\psi\}^M.
\end{aligned}$$

$$\begin{aligned}
(\{x \mid \varphi\} \in y)^M &\leftrightarrow (\exists z (z \in y \wedge z = \{x \mid \varphi\}))^M \\
&\leftrightarrow \exists z \in M (z \in y \wedge z = \{x \mid \varphi\}^M) \\
&\leftrightarrow \exists z (z \in y \wedge z = \{x \mid \varphi\}^M), \text{ since } y \subseteq M, \\
&\leftrightarrow \{x \mid \varphi\}^M \in y = y^M.
\end{aligned}$$

Now assume that χ is a complex formula and the theorem holds for all proper subformulas. If $\chi = \neg\psi$ and $\vec{w} \in M$ then

$$(\chi(t_0, \ldots, t_{n-1}))^M \leftrightarrow \neg(\psi(t_0, \ldots, t_{n-1}))^M \leftrightarrow \neg\psi^M(t_0^M, \ldots, t_{n-1}^M) \leftrightarrow \chi^M(t_0^M, \ldots, t_{n-1}^M).$$

If $\chi = \varphi \vee \psi$ and $\vec{w} \in M$ then

$$\begin{aligned}(\chi(t_0, \ldots, t_{n-1}))^M &\leftrightarrow (\varphi(t_0, \ldots, t_{n-1}))^M \vee (\psi(t_0, \ldots, t_{n-1}))^M \\ &\leftrightarrow \varphi^M(t_0^M, \ldots, t_{n-1}^M) \vee \psi^M(t_0^M, \ldots, t_{n-1}^M) \\ &\leftrightarrow \chi^M(t_0^M, \ldots, t_{n-1}^M).\end{aligned}$$

If $\chi = \exists x \varphi$ and $\vec{w} \in M$ then

$$\begin{aligned}(\chi(t_0, \ldots, t_{n-1}))^M &\leftrightarrow \exists x \in M\, (\varphi(x, t_0, \ldots, t_{n-1}))^M \\ &\leftrightarrow \exists x \in M \varphi^M(x, t_0^M, \ldots, t_{n-1}^M) \\ &\leftrightarrow \chi^M(t_0^M, \ldots, t_{n-1}^M).\end{aligned}$$

□

5.2 Transitive Models of set theory

Theorem 5.6. *Let M be a non-empty transitive term. Assume that M satisfies the following closure properties:*

a) $\forall x, y \in M\ \{x, y\} \in M$;

b) $\forall x \in M\ \bigcup x \in M$;

c) $\omega \in M$;

d) *for all terms A:* $\forall x \in M\, x \cap A^M \in M$;

e) *for all terms F: if F^M is a function then* $\forall x\, F^M[x] \in M$.

Then ZF^- *holds in M.*

Proof. (1) The axiom of extensionality holds in M.
Proof. Consider $x, y \in M$. By the axiom of extensionality in V

$$x \subseteq y \wedge y \subseteq x \rightarrow x = y.$$

Since M is transitive, $x \cap M = x$, $y \cap M = y$ and

$$x \cap M \subseteq y \wedge y \cap M \subseteq x \rightarrow x = y.$$

This is equivalent to

$$(\forall z \in M\, (z \in x \rightarrow z \in y) \wedge \forall z \in M\, (z \in y \rightarrow z \in x)) \rightarrow x = y$$

and

$$(x \subseteq y \wedge y \subseteq x \rightarrow x = y)^M.$$

Thus

$$(\forall x, y\, (x \subseteq y \wedge y \subseteq x \rightarrow x = y))^M.$$

(2) The pairing axiom holds in M.
Proof. Observe that for $x, y \in M$

$$\{x, y\}^M = \{z \in M \mid z = x \vee z = y\} = \{x, y\}.$$

Moreover $V^M = \{x \in M \mid x = x\} = M$. By assumption a),

$$\begin{array}{l} \forall x, y \in M \ \{x, y\} \in M \\ \forall x, y \in M \ \{x, y\}^M \in V^M \\ (\forall x, y \ \{x, y\} \in V)^M, \end{array}$$

i.e., the pairing axiom holds in M.
(3) The union axiom holds in M.
Proof. Observe that for $x \in M$,

$$\begin{array}{rcl} (\bigcup x)^M & = & \{z \in M \mid \exists y \in M \, (y \in x \wedge z \in y)\} \\ & = & \{z \in M \mid \exists y (y \in x \wedge z \in y)\}, \text{ since } x \subseteq M, \\ & = & \{z \mid \exists y (y \in x \wedge z \in y)\}, \text{ since } \forall y \in x \forall z \in y \, z \in M, \\ & = & \bigcup x \end{array}$$

By assumption b),

$$\begin{array}{l} \forall x \in M \ \bigcup x \in M \\ \forall x \in M \ (\bigcup x)^M \in V^M \\ (\forall x \ \bigcup x \in V)^M, \end{array}$$

i.e., the union axiom holds in M.
(4) The axiom of infinity holds in M.
Proof. Let $x = \omega \in M$. Then

$$\emptyset \in x \wedge \forall y \, (y \in x \rightarrow y \cup \{y\} \in x).$$

The universal quantifier may be restricted to M:

$$\emptyset \in x \wedge \forall y \in M (y \in x \rightarrow y \cup \{y\} \in x).$$

Since $(y \cup \{y\})^M = y \cup \{y\}$ this formula is equivalent to

$$(\emptyset \in x \wedge \forall y (y \in x \rightarrow y \cup \{y\} \in x))^M.$$

Then

$$\exists x \in M \, (\emptyset \in x \wedge \forall y (y \in x \rightarrow y \cup \{y\} \in x))^M,$$

i.e., the axiom of infinity holds in M.
(5) The axiom schema of subsets holds in M.
Proof. Let $A(\vec{y})$ be a term and $x, \vec{y} \in M$. By assumption,

$$x \cap A^M(\vec{y}) \in M.$$

Note that

$$\begin{array}{rcl} x \cap A^M(\vec{y}) & = & \{v \mid v \in x \wedge v \in A^M(\vec{y})\} \\ & = & \{v \mid (v \in x \wedge v \in A(\vec{y}))^M\} \\ & = & \{v \in M \mid (v \in x \wedge v \in A(\vec{y}))^M\}, \text{ since } x \subseteq M, \\ & = & \{v \mid v \in x \wedge v \in A(\vec{y})\}^M \\ & = & (x \cap A)^M. \end{array}$$

So

$$(x\cap A)^M \in M = V^M.$$

This proves

$$\forall x, \vec{y} \in M\,(x\cap A\in V)^M,$$

i.e., the axiom scheme of subsets holds relativized to M.
(6) The axiom scheme of replacement holds relativized to M.
Proof. Let $F(\vec{y})$ be a term, and let $x, \vec{y} \in M$ such that (F is a function)^M. Note that

$$\begin{aligned} F^M[x] &= \{v \mid \exists u\in x\,(u,v)\in F^M\} \\ &= \{v\in M \mid \exists u (u\in x\wedge (u,v)\in F^M)\}, \text{ since } F^M\subseteq M \text{ and } M \text{ is transitive,} \\ &= \{v\in M \mid \exists u\in M\,(u\in x\wedge (u,v)\in F)^M \\ &= \{v \mid \exists u (u\in x\wedge (u,v)\in F)\}^M \\ &= (F[x])^M. \end{aligned}$$

The assumption implies

$$F^M[x]\in M$$
$$F^M[x]\in M = V^M$$
$$(F[x]\in V)^M$$

Thus

$$\forall x, \vec{y}\in M\,(F \text{ is a function } \rightarrow F[x]\in V)^M, \text{ and}$$
$$(\forall x, \vec{y}\,(F \text{ is a function } \rightarrow F[x]\in V))^M,$$

as required.
(7) The axiom schema of foundation holds in M.
Proof. Let $A(\vec{y})$ be a term and let $\vec{y}\in M$ such that $(A\neq\emptyset)^M$. Then $A^M\neq\emptyset$. By the replacement schema in V, take $x\in A^M$ such that $x\cap A^M=\emptyset$. We have seen before that $x\cap A^M=(x\cap A)^M$. So $(x\cap A)^M=\emptyset$ and $(x\cap A=\emptyset)^M$. Hence

$$\exists x\in M\,(x\in A^M\wedge (x\cap A=\emptyset)^M)$$
$$(\exists x\,(x\in A\wedge x\cap A=\emptyset)^M$$

Thus

$$(A\neq\emptyset\rightarrow \exists x\,(x\in A\wedge x\cap A=\emptyset)^M,$$

i.e., the foundation schema holds in M. □

The converse to this theorem will be shown later.

Theorem 5.7. *Let M be a non-empty transitive term such that*

$$\forall x\in M\,\mathcal{P}(x)\cap M\in M.$$

Then the power set axiom holds in M.

Proof. Note that for $x \in M$

$$\begin{aligned}
(\mathcal{P}(x))^M &= \{y \mid y \subseteq x\}^M \\
&= \{y \in M \mid (y \subseteq x)^M\} \\
&= \{y \in M \mid (y \subseteq x)^M\} \\
&= \{y \in M \mid (\forall z\, (z \in y \rightarrow z \in x))^M\} \\
&= \{y \in M \mid \forall z \in M\, (z \in y \rightarrow z \in x)\} \\
&= \{y \in M \mid \forall z (z \in y \rightarrow z \in x)\}, \text{ da } y \subseteq M, \\
&= \{y \in M \mid y \subseteq x\} = \mathcal{P}(x) \cap M.
\end{aligned}$$

The assumption yields

$$\begin{gathered}
\forall x \in M\, \mathcal{P}(x) \cap M \in M \\
\forall x \in M\, \mathcal{P}(x)^M \in V^M \\
(\forall x\, \mathcal{P}(x) \in V)^M,
\end{gathered}$$

i.e., the power set axiom holds in M. $\square$

Chapter 6
Definite formulas and terms

6.1 Definiteness

In the set existence axioms of the theory ZF^- every element of a term whose existence is postulated is determined by some parameters of the axiom. In the replacement scheme, e.g., every element

$$v \in \{F(x) \mid x \in z\}$$

is of the form $v = F(x)$ and is thus definable from the "simpler" parameter x by the term F. In contrast, there is no way to define an arbitrary element of an infinite power set from simple parameters; this impression can be made more formal by using CANTOR's diagonal argument. The axiom of choice also is a pure existence statement. There exists a choice functions, but it is in general not definable from the parameters of the situation at hand.

The notion of *defineteness* aims to capture the concrete nature of ZF^- as compared to full ZFC. It will be seen that most basic notions of set theory are definite and that these notions can be decided in ZF^- independantly of the specific transitive model of ZF^-. The definition of *definite term* tries to capture the "absolute" part of the theory ZF^-.

Definition 6.1. *Define the collections of* ***definite formulas*** *and* ***definite terms*** *by a common recursion on syntactic complexities:*

a) the atomic formulas $x \in y$ and $x = y$ are definite;

b) if φ and ψ are definite formulas then $\varphi \vee \psi$ and $\neg\varphi$ are definite;

c) if φ is a definite formula then $\forall x \in y\, \varphi$ and $\exists x \in y\, \varphi$ are definite formulas;

d) x, $\{x, y\}$, $\bigcup x$ and ω are definite terms;

e) if $s(x_0, \ldots, x_{n-1})$ and $t_0, \ldots, t_{n-1}$ are definite terms then $s(t_0, \ldots, t_{n-1})$ is a definite term;

f) if $\varphi(x_0, \ldots, x_{n-1})$ is a definite formula and $t_0, \ldots, t_{n-1}$ are definite terms then $\varphi(t_0, \ldots, t_{n-1})$ is a definite formula;

g) if φ is a definite formula then $\{x \in y \mid \varphi(x, \vec{z}\,)\}$ is a definite term;

h) if $t(x, \vec{z}\,)$ is a definite term then $\{t(x, \vec{z}\,) \mid x \in y\}$ is a definite term;

i) if G is a definite term then the canonical term F defined by $\in$-recursion with $F(x) = G(F \upharpoonright x)$ is definite.

The majority of basic notions of set theory (and of mathematics) are definite. The following theorems list some representative examples.

Theorem 6.2. *The following terms are definite:*

a) $x \setminus y$

b) (x, y)

c) $x \times y$

d)

$$\begin{cases} x, \text{ if } \varphi \\ y, \text{ if } \neg\varphi \end{cases},$$

where φ *is a definite formula ("definition by cases")*

Proof. a) $x \setminus y = \{z \in x \mid z \notin y\}$.
b) $(x, y) = \{\{x\}, \{x, y\}\}$.
c) $x \times y = \bigcup \{x \times \{v\} \mid v \in y\} = \bigcup \{\{(u, v) \mid u \in x\} \mid v \in y\}$.
d) $\left\{\begin{smallmatrix} x, \text{ if } \varphi \\ y, \text{ if } \neg\varphi \end{smallmatrix}\right.$ can be defined definitely by

$$\{u \in x \mid \varphi\} \cup \{u \in y \mid \neg\varphi\}.$$

□

Theorem 6.3. *The following formulas are definite:*

a) x *is transitive*

b) x *is an ordinal*

c) x *is a successor ordinal*

d) x *is a limit ordinal*

e) x *is a natural number*

Proof. All these formulas are equivalent to Σ_0-formulas. □

Recursion on the ordinals is a special case of $\in$-recursion which also leads to definite terms.

Theorem 6.4. *Let* G_0*,* G_{succ} *and* G_{limit} *be definite terms defining a term* F: $\mathrm{Ord} \to V$ *by the following recursion:*

– $F(0) = G_0$ *;*

– $F(\alpha + 1) = G_{\mathrm{succ}}(F \upharpoonright (\alpha + 1))$ *;*

– $F(\lambda) = G_{\mathrm{limit}}(F \upharpoonright \lambda)$ *for limit ordinals* λ *.*

Then the term $F(\alpha)$ *is definite.*

Proof. Let F' be the canonical term defined by the $\in$-recursion

$$F'(x) = \begin{cases} 0, \text{ if } x = 0, \\ G_{\mathrm{succ}}(F' \upharpoonright x), \text{ if } x \text{ is a successor ordinal}, \\ G_{\mathrm{limit}}(F' \upharpoonright x), \text{ if } x \text{ is a limit ordinal}, \\ 0, \text{ if } x \notin \mathrm{Ord}. \end{cases}$$

By (an extension of) Theorem 6.2 d) on definition by cases, the recursion condition is definite and so is $F'(x)$. Then $F = F' \upharpoonright \mathrm{Ord}$. $\square$

${}^n x$, V_n, V_ω

6.2 Absoluteness

Definition 6.5. *Let W be a transitive non-empty class. Let $\varphi(\vec{x}\,)$ be an $\in$-formula and $t(\vec{x}\,)$ be a term. Then*

a) *φ is W-**absolute** iff $\forall \vec{x} \in W\, (\varphi^W(\vec{x}\,) \leftrightarrow \varphi(\vec{x}\,))$;*

b) *t is W-**absolute** iff $\forall \vec{x} \in W\, (t^W(\vec{x}\,) \in W \leftrightarrow t(\vec{x}\,) \in V)$ and $\forall \vec{x} \in W\, t^W(\vec{x}\,) = t(\vec{x}\,)$.*

Theorem 6.6. *Let W be a transitive model of* ZF^-. *Then*

a) *if $t(\vec{x}\,)$ is a definite term then $\forall \vec{x}\, t(\vec{x}\,) \in V$;*

b) *every definite formula is W-absolute;*

c) *every definite term is W-absolute.*

Proof. a) may be proved by induction on the complexity of the definite term t. Most cases are immediate from the ZF^--axioms; if t is a canonical term defined by recursion with a definite recursion rule then the existence of $t(\vec{x}\,)$ follows from the recursion principle.

The properties b) and c) are proved by a common induction along the generation rules of Definition 6.1 for definite formulas and terms. If $t(\vec{x}\,)$ is a definite term, then by a)

$$(\forall \vec{x}\, t(\vec{x}\,) \in V)^W \rightarrow \forall \vec{x} \in W\, t^W(\vec{x}\,) \in W$$

so that always

$$\forall \vec{x} \in W\, (t^W(\vec{x}\,) \in W \leftrightarrow t(\vec{x}\,) \in V).$$

Thus for the W-absoluteness of t one only has to check

$$\forall \vec{x} \in W\, t^W(\vec{x}\,) = t(\vec{x}\,).$$

We now begin the induction. The cases 6.1 a) and b) are trivial.
6.1 c): Let $\varphi(x, \vec{z}\,)$ be definite and assume that $\varphi(x, \vec{z}\,)$ is W-absolute. Let y, $\vec{z} \in W$. Then $y \subseteq W$ and $y \cap W = y$, since W is transitive.

$$\begin{aligned}
(\forall x \in y\, \varphi(x, \vec{z}\,))^W &\leftrightarrow (\forall x\, (x \in y \rightarrow \varphi(x, \vec{z}\,)))^W \\
&\leftrightarrow \forall x \in W\, (x \in y \rightarrow \varphi^W(x, \vec{z}\,)) \\
&\leftrightarrow \forall x\, (x \in y \cap W \rightarrow \varphi^W(x, \vec{z}\,)) \\
&\leftrightarrow \forall x\, (x \in y \rightarrow \varphi(x, \vec{z}\,)), \text{ since } \varphi \text{ is } W\text{-absolute}, \\
&\leftrightarrow \forall x \in y\, \varphi(x, \vec{z}\,).
\end{aligned}$$

Thus $\forall x \in y\, \varphi(x, \vec{z}\,)$ is W-absolute. Similarly, $\exists x \in y\, \varphi(x, \vec{z}\,)$ is W-absolute.

Let us remark that cases 6.1 a) to c) imply that every $\in$-formula in which every quantifier is bounded is W-absolute. Such formulas are called Σ_0-formulas.
6.1 d): The only non-trivial case is the term

$$\omega = \{\alpha \in \mathrm{Ord} \,|\, \forall \beta \in \alpha + 1\, (\beta = 0 \vee \beta \text{ is a successor ordinal})\}.$$

(1) The formula $\alpha \in \omega$ is W-absolute.
Proof. By the remark above it suffices to see that the formula $\alpha \in \omega$ is equivalent to a Σ_0-formula.

$$\begin{aligned}
\alpha \in \omega \;&\leftrightarrow\; \alpha \in \mathrm{Ord} \wedge \forall \beta \in \alpha + 1\, (\beta = 0 \vee \beta \text{ is a successor ordinal}) \\
&\leftrightarrow\; \mathrm{Trans}(\alpha) \wedge \forall y \in \alpha\, \mathrm{Trans}(y) \wedge \forall \beta \in \alpha\, (\forall x \in \beta\, x \neq x \vee \\
&\quad\;\; \exists \gamma \in \beta\ \beta = \gamma + 1) \wedge (\forall x \in \alpha\ x \neq x \vee \exists \gamma \in \alpha\ \alpha = \gamma + 1) \\
&\leftrightarrow\; \forall u \in \alpha \forall v \in u\, v \in \alpha \wedge \forall y \in \alpha \forall u \in y \forall v \in u\, v \in y \wedge \\
&\quad\;\; \forall \beta \in \alpha\, (\forall x \in \beta\, x \neq x \vee \exists \gamma \in \beta\, (\forall u \in \beta\, (u \in \gamma \vee u = \gamma) \wedge \\
&\quad\;\; \forall u \in \gamma\, u \in \beta \wedge \gamma \in \beta)) \wedge (\forall x \in \alpha\, x \neq x \vee \\
&\quad\;\; \exists \gamma \in \alpha\, (\forall u \in \alpha\, (u \in \gamma \vee u = \gamma) \wedge \forall u \in \gamma\, u \in \alpha \wedge \gamma \in \alpha))
\end{aligned}$$

qed(1)
(2) $\omega \subseteq W$.
Proof. By complete induction. $0 \in W$ since W is a non-empty transitive term. Assume that $n \in \omega$ and $n \in W$. Then, since $(\mathrm{ZF}^-)^W$, $(n \cup \{n\})^W \in W$.

$$\begin{aligned}
(n \cup \{n\})^W &= \{x \in W \,|\, (x \in n \vee x \in \{n\})^W\} \\
&= \{x \in W \,|\, (x \in n \vee x = n)^W\} \\
&= \{x \in W \,|\, x \in n \vee x = n\} \\
&= \{x \,|\, x \in n \vee x = n\}, \text{ since } n \cup \{n\} \subseteq W, \\
&= n \cup \{n\}.
\end{aligned}$$

Hence $n + 1 \in W$. *qed*(2)
(3) $\omega^M = \omega$.
Proof.

$$\begin{aligned}
\omega^M &= \{x \in M \,|\, (x \in \omega)^M\} \\
&= \{x \in M \,|\, x \in \omega\}, \text{ since } x \in \omega \text{ is } W\text{-absolute}, \\
&= \{x \,|\, x \in \omega\}, \text{ since } \omega \subseteq M, \\
&= \omega.
\end{aligned}$$

qed(3)

By our previous remarks this concludes case 6.1 d).
6.1 e): Let $\vec{y}$ be the free variables of the terms $t_0, \ldots, t_{n-1}$ and let $\vec{y} \in W$. Then by the inductive assumption

$$\begin{aligned}
(s(t_0, \ldots, t_{n-1}))^W(\vec{y}) &= s^W(t_0^W(\vec{y}), \ldots, t_{n-1}^W(\vec{y})) \\
&= s^W(t_0(\vec{y}), \ldots, t_{n-1}(\vec{y})) \\
&= s(t_0(\vec{y}), \ldots, t_{n-1}(\vec{y})) \\
&= s(t_0, \ldots, t_{n-1})(\vec{y}).
\end{aligned}$$

6.1 f): Let $\vec{y}$ be the free variables of the terms $t_0, \ldots, t_{n-1}$ and let $\vec{y} \in W$. Then by the inductive assumption

$$\begin{aligned}
(\varphi(t_0, \ldots, t_{n-1}))^W(\vec{y}) &\leftrightarrow \varphi^W(t_0^W(\vec{y}), \ldots, t_{n-1}^W(\vec{y})) \\
&\leftrightarrow \varphi^W(t_0(\vec{y}), \ldots, t_{n-1}(\vec{y})) \\
&\leftrightarrow \varphi(t_0(\vec{y}), \ldots, t_{n-1}(\vec{y})) \\
&\leftrightarrow \varphi(t_0, \ldots, t_{n-1})(\vec{y}).
\end{aligned}$$

6.1 g): Let $y, \vec{z} \in W$. Then $y \subseteq W$ since W is transitive. By the inductive assumption

$$\begin{aligned}
\{x \in y \mid \varphi(x, \vec{z})\}^W &= \{x \mid x \in y \wedge \varphi(x, \vec{z})\}^W \\
&= \{x \in W \mid x \in y \wedge \varphi^W(x, \vec{z})\} \\
&= \{x \in W \mid x \in y \wedge \varphi(x, \vec{z})\} \\
&= \{x \mid x \in y \wedge \varphi(x, \vec{z})\}, \text{ since } y \subseteq W, \\
&= \{x \in y \mid \varphi(x, \vec{z})\}.
\end{aligned}$$

6.1 h): Let $y, \vec{z} \in W$. Then $y \subseteq W$ since W is transitive, and

$$\begin{aligned}
\{t(x, \vec{z}) \mid x \in y\}^W &= \{z \mid \exists x \in y\ z = t(x, \vec{z})\}^W \\
&= \{z \mid \exists x (x \in y \wedge z = t(x, \vec{z})\}^W \\
&= \{z \in W \mid \exists x \in W (x \in y \wedge z = t^W(x, \vec{z})\} \\
&= \{z \mid \exists x \in W (x \in y \wedge z = t^W(x, \vec{z})\}, \text{ since } \forall x \in W t^W(x, \vec{z}) \in W, \\
&= \{z \mid \exists x \in W (x \in y \wedge z = t(x, \vec{z})\}, \text{ by inductive assumption}, \\
&= \{z \mid \exists x (x \in y \wedge z = t(x, \vec{z})\}, \text{ since } y \subseteq W, \\
&= \{t(x, \vec{z}) \mid x \in y\}.
\end{aligned}$$

6.1 i): Let $G = G(z, \vec{y})$ with all free variables displayed and let F be the canonical term with

$$F(x, \vec{y}) = G(F \upharpoonright x, \vec{y}).$$

Let $\vec{y} \in W$. We show that $\forall x \in W F^W(x, \vec{y}) = F(x, \vec{y})$. Assume the contrary and let $x \in W$ be $\in$-minimal such that $F^W(x, \vec{y}) \neq F(x, \vec{y})$. Then by the recursion theorem in W,

$$\begin{aligned}
F^W(x, \vec{y}) &= G^W(F^W \upharpoonright x, \vec{y}) \\
&= G(F^W \upharpoonright x, \vec{y}), \text{ since } F^W \upharpoonright x \in W \text{ and } G \text{ is definite}, \\
&= G(F \upharpoonright x, \vec{y}), \text{ by the minimality of } x, \\
&= F(x, \vec{y}), \text{ contradiction.}
\end{aligned}$$

□

Recursion can be used to show that certain terms involving *finiteness* are definite.

Definition 6.7. *Define* $\mathcal{P}_n(x) = \{y \subseteq x \mid \operatorname{card}(y) < n\}$ *for* $n \leqslant \omega$ *recursively by induction on* n*:*

- $\mathcal{P}_0(x) = \emptyset$;

- $\mathcal{P}_1(x) = \{\emptyset\}$;
- $\mathcal{P}_{n+1}(x) = \{y \cup \{z\} \mid y \in \mathcal{P}_n(x) \wedge z \in x\}$;
- $\mathcal{P}_\omega(x) = \bigcup_{n<\omega} \mathcal{P}_n(x)$.

Since this is an $\in$ *-recursion with a definite recursion rule the terms* $\mathcal{P}_n(x)$ *and* $\mathcal{P}_n(x)$ *are definite.*

We define a finitary version of the von Neumann-hierarchy which agrees with the usual V_α-hierarchy for $\alpha \leqslant \omega$.

Definition 6.8. *Define* V_α^{fin} *for* $\alpha \in \mathrm{Ord}$ *recursively:*

- $V_0^{\mathrm{fin}} = \emptyset$,
- $V_{\alpha+1}^{\mathrm{fin}} = \mathcal{P}_\omega(V_\alpha^{\mathrm{fin}})$,
- $V_\lambda^{\mathrm{fin}} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathrm{fin}}$ *for limit ordinals* λ.

Note that $V_\omega^{\mathrm{fin}} = V_\omega$ *and that the term* V_α^{fin} *is definite. Hence* V_ω *is a definite term.*

Definition 6.9. *Define a well-order* $<_n$ *of* V_n *for* $n \leqslant \omega$ *recursively by induction on* n*:*

- $<_0 = \emptyset$;
- $<_{n+1} = <_n \cup (V_n \times (V_{n+1} \setminus V_n)) \cup$
  $\cup \{(x, y) \in V_{n+1} \times V_{n+1} \mid \exists v \in y \setminus x \forall u \in V_n\, (u >_n v \rightarrow (u \in x \leftrightarrow u \in y))\}$;
- $<_\omega = \bigcup_{n<\omega} <_n$.

The terms $<_n$ *for* $n \leqslant \omega$ *are definite.*

We shall next give a definite definition of the set of finite sequences from a given set x which will later be used as the set of assignments in x.

Definition 6.10. *Define* ${}^n x = \{f \mid f: n \rightarrow x\}$ *for* $n \in \omega$ *by recursion on* n*:*

- ${}^0 x = \{\emptyset\}$;
- ${}^{n+1} x = \{f \cup \{(n, u)\} \mid f \in^n x \wedge u \in x\}$;
- ${}^{<\omega} x = \bigcup_{n<\omega} {}^n x$.

Call ${}^{<\omega} x$ *the set of* ***assigments*** *in* x.

There are natural operations on assignments:

Definition 6.11. *For* $f \in^{<\omega} x$, $a \in x$ *and* $k \in \mathrm{dom}(f)$ *let*

$$f\frac{a}{k} = (f \setminus \{(k, f(k))\}) \cup \{(k, a)\}$$

be the ***substitution*** *of* a *into* f *at* k.

Chapter 7
Formalizing the logic of set theory

7.1 First-order logic

The theory ZF^- is able to formalize most basic mathematical notions. This general *formalization principle* also applies to first-order logic. For the definition of the constructible universe we shall be particularly interested in formalizing the logic of set theory within ZF^-, i.e., the logic of syntax and semantics of the language $\{\in\}$. Given some experience with definite formalizations the definite formalizability of first-order logic is quite obvious. For the sake of completeness we shall employ a concrete formalization as described in the monograph *Set Theory* by Frank Drake.

Standard first-order logic can be embedded into its formalized counterpart. So for every formula φ of the language of set theory we shall have a term $\ulcorner\varphi\urcorner$ which is a formalization of φ. Let us motivate the intended formalization by defining $\ulcorner\varphi\urcorner$ inductively over the complexity of φ.

Definition 7.1. *For each concrete $\in$-formula φ define its* **Goedel set** $\ulcorner\varphi\urcorner$ *by induction on the complexity of φ:*

– $\ulcorner v_i = v_j\urcorner = (0, i, j)$;

– $\ulcorner v_i \in v_j\urcorner = (1, i, j)$;

– $\ulcorner\varphi \wedge \psi\urcorner = (2, \ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$;

– $\ulcorner\neg\varphi\urcorner = (3, \ulcorner\varphi\urcorner)$;

– $\ulcorner\exists v_i\,\varphi\urcorner = (4, i, \ulcorner\varphi\urcorner)$.

Definition 7.2. *The formula* $\mathrm{Fm}(u, s, n)$ *describes that a formula u is constructed along a finite sequence s of length $n+1$ according to the construction principles of the previous definition:*

$$\begin{aligned}\mathrm{Fm}(u,s,n) \ \leftrightarrow\ & n \in \omega \wedge s \in {}^{n+1}V_\omega \wedge u = s(n) \wedge \\ & \wedge \forall k < n+1 \\ & (\exists i, j < \omega\ s(k) = (0, i, j) \vee \\ & \vee \exists i, j < \omega\ s(k) = (1, i, j) \vee \\ & \vee \exists l, m < k\ s(k) = (2, s(l), s(m)) \vee \\ & \vee \exists l < k\ s(k) = (3, s(l)) \vee \\ & \vee \exists l < k\ \exists i < \omega\ s(k) = (4, i, s(l))).\end{aligned}$$

Inspection of this definition shows that $\mathrm{Fm}(u, s, n)$ is definite.

Definition 7.3. *The formula* $\mathrm{Fmla}(u)$ *describes that* u *is a formalized* $\in$ *-formula:*

$$\mathrm{Fmla}(u) \leftrightarrow \exists n < \omega \exists s \in V_\omega \ \mathrm{Fm}(u, s, n).$$

The formula Fmla is also definite.

We formalize the Tarskian satisfaction relation for the formulas u defined by Fmla. For each member of a construction sequence leading to u we consider the set of assignments in an $\in$-structure $(a, \in)$ which make the formula true.

Definition 7.4. *The formula* $S(s, a, r, t)$ *describes that* s *builds an* $\in$ *-formula as in Definition 7.2, and that* t *is a sequence of assignments of the variables* $v_0, \ldots, v_{r-1}$ *in the* $\in$ *-structure* $(a, \in)$ *which make the corresponding* $\in$ *-formula of the sequence* s *true:*

$$\begin{aligned}
S(s, a, r, t) \ \leftrightarrow \ & \exists u, n \in V_\omega \ \mathrm{Fm}(u, s, n) \wedge a \neq \emptyset \wedge r < \omega \wedge t \colon \mathrm{dom}(s) \to V_\omega \wedge \\
& \wedge \forall k \in \mathrm{dom}(s) \\
& \quad ((\exists i, j < \omega \ s(k) = (0, i, j) \wedge t(k) = \{b \in^r a \,|\, b(i) = b(j)\}) \vee \\
& \quad \vee (\exists i, j < \omega \ s(k) = (1, i, j) \wedge t(k) = \{b \in^r a \,|\, b(i) \in b(j)\}) \vee \\
& \quad \vee (\exists l, m < k \ s(k) = (2, s(l), s(m)) \wedge t(k) = t(l) \cap t(m)) \vee \\
& \quad \vee (\exists l < k \ s(k) = (3, s(l)) \wedge t(k) = {}^r a \setminus t(l)) \vee \\
& \quad \vee (\exists l < k \ \exists i < \omega \ s(k) = (4, i, s(l))) \wedge \\
& \qquad \wedge t(k) = \{b \in^r a \,|\, \exists x \in a \, (b \setminus \{(i, b(i))\}) \cup \{(i, x)\} \in t(l)\})).
\end{aligned}$$

Then define the ***satisfaction relation*** $a \vDash u[b]$ *by* b *belonging to the assignments satisfying* u*:*

$$\begin{aligned}
a \vDash u[b] \ \leftrightarrow \ & a \neq \emptyset \wedge \mathrm{Fmla}(u) \wedge b \in^{<\omega} a \wedge \\
& \wedge \exists s, r, t \in V_\omega \, (S(s, a, r, t) \wedge r = \mathrm{rk}(u) \wedge u = s(\mathrm{dom}(s) - 1) \wedge \\
& \wedge b \in t(\mathrm{dom}(s) - 1)).
\end{aligned}$$

Note that

Theorem 7.5. *For each* $\in$ *-formula* $\varphi(v_0, \ldots, v_{n-1})$*:*

$$\forall a \forall x_0, \ldots, x_{n-1} \in a \, (\varphi^a(x_0, \ldots, x_{n-1}) \leftrightarrow a \vDash \ulcorner \varphi \urcorner [(x_0, \ldots, x_{n-1})].$$

On the right-hand side, $(x_0, \ldots, x_{n-1})$ *is the term*

$$\{(0, x_0), \ldots, (n-1, x_{n-1})\}.$$

Proof. By induction on the formula complexity of φ. $\square$

7.2 Definable power sets

With these notions we can define a notion of definable power set crucial for the constructible hierarchy.

Definition 7.6. *a) For $x \in V$, $\varphi \in \mathrm{Fml}$, and $\vec{a} \in^{<\omega} x$ define the* ***interpretation*** *of $(x, \varphi, \vec{a}\,)$ by*

$$I(x, \varphi, \vec{a}\,) = \{v \in x | x \vDash \varphi[\vec{a}\,\frac{v}{0}]\}$$

b) $\mathrm{Def}(x) = \{I(x, \varphi, \vec{p}\,) | \varphi \in \mathrm{Fml}, \vec{p} \in x\}$ is the ***definable power set*** *of x.*

The terms $I(x, \varphi, \vec{a}\,)$ and $\mathrm{Def}(x)$ are definite.

Chapter 8
The constructible hierarchy

The constructible hierarchy is obtained by iterating the Def-operation along the ordinals.

Definition 8.1. *Define the* ***constructible hierarchy*** L_α , $\alpha \in \mathrm{Ord}$ *by recursion on* α*:*

$$\begin{aligned} L_0 &= \emptyset \\ L_{\alpha+1} &= \mathrm{Def}(L_\alpha) \\ L_\lambda &= \bigcup_{\alpha<\lambda} L_\alpha, \textit{ for } \lambda \textit{ a limit ordinal.} \end{aligned}$$

The ***constructible universe*** L *is the union of that hierarchy:*

$$L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha .$$

The hierarchy satisfies natural hierarchical laws.

Theorem 8.2. *a)* $\alpha \leqslant \beta$ *implies* $L_\alpha \subseteq L_\beta$

b) L_β *is transitive*

c) $L_\beta \subseteq V_\beta$

d) $\alpha < \beta$ *implies* $L_\alpha \in L_\beta$

e) $L_\beta \cap \mathrm{Ord} = \beta$

f) $\beta \leqslant \omega$ *implies* $L_\beta = V_\beta$

g) $\beta \geqslant \omega$ *implies* $\mathrm{card}(L_\beta) = \mathrm{card}(\beta)$

Proof. By induction on $\beta \in \mathrm{Ord}$. The cases $\beta = 0$ and β a limit ordinal are easy and do not depend on the specific definition of the L_β-hierarchy.

Let $\beta = \gamma + 1$ where the claims hold for γ.
a) It suffices to show that $L_\gamma \subseteq L_\beta$. Let $x \in L_\gamma$. By b), L_γ is transitive and $x \subseteq L_\gamma$. Hence

$$x = \{v \in L_\gamma \mid v \in x\} = \{v \in L_\gamma \mid (L_\gamma, \in) \vDash (v \in w)\frac{x}{w}\} = I(L_\gamma, v \in w, x) \in L_{\gamma+1} = L_\beta .$$

b) Let $x \in L_\beta$. Let $x = I(L_\gamma, \varphi, \vec{p})$. Then by a) $x \subseteq L_\gamma \subseteq L_\beta$.
c) By induction hypothesis,

$$L_\beta = \mathrm{Def}(L_\gamma) \subseteq \mathcal{P}(L_\gamma) \subseteq \mathcal{P}(V_\gamma) = V_{\gamma+1} = V_\beta .$$

d) It suffices to show that $L_\gamma \in L_\beta$.

$$L_\gamma = \{v \in L_\gamma | v = v\} = \{v \in L_\gamma | (L_\gamma, \in) \vDash v = v\} = I(L_\gamma, v = v, \emptyset) \in L_{\gamma+1} = L_\beta.$$

e) $L_\beta \cap \mathrm{Ord} \subseteq V_\beta \cap \mathrm{Ord} = \beta$. For the converse, let $\delta < \beta$. If $\delta < \gamma$ the inductive hypothesis yields that $\delta \in L_\gamma \cap \mathrm{Ord} \subseteq L_\beta \cap \mathrm{Ord}$. Consider the case $\delta = \gamma$. We have to show that $\gamma \in L_\beta$. There is a formula $\varphi(v)$ which is Σ_0 and formalizes being an ordinal. This means that all quantifiers in φ are bounded and if z is transitive then

$$\forall v \in z\, (v \in \mathrm{Ord} \leftrightarrow (z, \in) \vDash \varphi(v)).$$

By induction hypothesis

$$\begin{aligned} \gamma &= \{v \in L_\gamma | v \in \mathrm{Ord}\} \\ &= \{v \in L_\gamma \,|\, (L_\gamma, \in) \vDash \varphi(v)\} \\ &= I(L_\gamma, \varphi, \emptyset) \\ &\in L_{\gamma+1} = L_\beta. \end{aligned}$$

f) Let $\beta < \omega$. By c) it suffices to see that $V_\beta \subseteq L_\beta$. Let $x \in V_\beta$. By induction hypothesis, $L_\gamma = V_\gamma$. $x \subseteq V_\gamma = L_\gamma$. Let $x = \{x_0, \ldots, x_{n-1}\}$. Then

$$\begin{aligned} x &= \{v \in L_\gamma \,|\, v = x_0 \vee v = x_1 \vee \ldots \vee v = x_{n-1}\} \\ &= \{v \in L_\gamma \,|\, (L_\gamma, \in) \vDash (v = v_0 \vee v = v_1 \vee \ldots \vee v = v_{n-1}) \frac{x_0\, x_1 \ldots x_{n-1}}{v_0\, v_1 \ldots v_{n-1}}\} \\ &= I(L_\gamma, (v = v_0 \vee v = v_1 \vee \ldots \vee v = v_{n-1}), x_0, x_1, \ldots, x_{n-1}) \\ &\in L_{\gamma+1} = L_\beta. \end{aligned}$$

g) Let $\beta > \omega$. By induction hypothesis $\mathrm{card}(L_\gamma) = \mathrm{card}(\gamma)$. Then

$$\begin{aligned} \mathrm{card}(\beta) &\leqslant \mathrm{card}(L_\beta) \\ &\leqslant \mathrm{card}(\{I(L_\gamma, \varphi, \vec{p}) \,|\, \varphi \in \mathrm{Fml}, \vec{p} \in L_\gamma\}) \\ &\leqslant \mathrm{card}(\mathrm{Fml}) \cdot \mathrm{card}({}^{<\omega}L_\gamma) \\ &\leqslant \mathrm{card}(\mathrm{Fml}) \cdot \mathrm{card}(L_\gamma)^{<\omega} \\ &= \aleph_0 \cdot \mathrm{card}(\gamma)^{<\omega} \\ &= \aleph_0 \cdot \mathrm{card}(\gamma), \text{ since } \gamma \text{ is infinite,} \\ &= \mathrm{card}(\gamma) \\ &= \mathrm{card}(\beta). \end{aligned}$$

□

The properties of the constructible hierarchy immediately imply the following for the constructible universe.

Theorem 8.3. *a) L is transitive.*

b) $\mathrm{Ord} \subseteq L$.

Theorem 8.4. *$(L, \in)$ is a model of* ZF.

Proof. By a previous theorem it suffices that L is transitive, *almost universal* and *closed under definitions*.
(1) L is almost universal, i.e., $\forall x \subseteq L \exists y \in L\, x \subseteq y$.
Proof. Let $x \subseteq L$. For each $u \in L$ let $\mathrm{rk}(u) = \min\{\alpha \,|\, u \in L_\alpha\}$ be its *constructible rank*. By replacement in V let $\beta = \bigcup \{\mathrm{rk}(u) \,|\, u \in x\} \in \mathrm{Ord}$. Then

$$x \subseteq L_\beta \in L.$$

(2) L is closed under definition, i.e., for every $\in$-formula $\varphi(x, \vec{y}\,)$ holds

$$\forall a, \vec{y} \in L\, \{x \in a \,|\, \varphi^L(x, \vec{y}\,)\} \in L.$$

Proof. Let $\varphi(x, \vec{y}\,)$ be an $\in$-formula and $a, \vec{y} \in L$. Let $a, \vec{y} \in L_{\theta_0}$. By the LEVY reflection theorem there is some $\theta \geqslant \theta_0$ such that φ is L_θ-L-absolute, i.e.,

$$\forall u, \vec{v} \in L_\theta\, (\varphi^{L_\theta}(u, \vec{v}\,) \leftrightarrow \varphi^L(u, \vec{v}\,)).$$

Then

$$\begin{aligned}
\{x \in a \,|\, \varphi^L(x, \vec{y}\,)\} &= \{x \in L_\theta \,|\, x \in a \wedge \varphi^L(x, \vec{y}\,)\} \\
&= \{x \in L_\theta \,|\, x \in a \wedge \varphi^{L_\theta}(x, \vec{y}\,)\} \\
&= \{x \in L_\theta \,|\, (x \in a \wedge \varphi(x, \vec{y}\,))^{L_\theta}\} \\
&= I(L_\theta, (x \in z \wedge \varphi(x, \vec{v}\,)), \frac{a\,\vec{y}}{z\,\vec{v}}) \in L_{\theta+1} \subseteq L.
\end{aligned}$$

$\square$

The recursive and definite definition of the L_α-hierarchy implies immediately:

Theorem 8.5. *The term L_α is definite.*

8.1 Wellordering *L*

We shall now prove an *external* choice principle and also an *external* continuum hypothesis for the constructible sets. These will later be internalized through the *axiom of constructibility*. Every constructible set x is of the form

$$x = I(L_\alpha, \varphi, \vec{p}\,);$$

$(L_\alpha, \varphi, \vec{p}\,)$ is a *name* for x.

Definition 8.6. *Define the class of (**constructible**) **names** or **locations** as*

$$\tilde{L} = \{(L_\alpha, \varphi, \vec{p}\,) | \alpha \in \mathrm{Ord}, \varphi(v, \vec{v}\,) \in \mathrm{Fml}, \vec{p} \in L_\alpha, \mathrm{length}(\vec{p}\,) = \mathrm{length}(\vec{v}\,)\}.$$

This class has a natural stratification

$$\tilde{L}_\alpha = \{(L_\beta, \varphi, \vec{p}\,) \in \tilde{L} | \beta < \alpha\} \text{ for } \alpha \in \mathrm{Ord}.$$

*A location of the form $(L_\alpha, \varphi, \vec{p}\,)$ is called an α-**location**.*

Definition 8.7. *Define wellorders* $<_\alpha$ *of* L_α *and* $\tilde{<}_\alpha$ *of* $\tilde{L}_\alpha$ *by recursion on* α.

- $<_0 = \tilde{<}_0 = \emptyset$ *is the vacuous ordering on* $L_0 = \tilde{L}_0 = \emptyset$;
- *if* $<_\alpha$ *is a wellordering of* L_α *then define* $\tilde{<}_{\alpha+1}$ *on* $\tilde{L}_{\alpha+1}$ *by:*
  $(L_\beta, \varphi, \vec{x}) \tilde{<}_{\alpha+1} (L_\gamma, \psi, \vec{y})$ *iff*
  $(\beta < \gamma)$ *or* $(\beta = \gamma \wedge \varphi < \psi)$ *or*
  $(\beta = \gamma \wedge \varphi = \psi \wedge \vec{x}$ *is lexicographically less than* $\vec{y}$ *with respect to* $<_\alpha$ $)$;
- *if* $\tilde{<}_{\alpha+1}$ *is a wellordering on* $\tilde{L}_{\alpha+1}$ *then define* $<_{\alpha+1}$ *on* $L_{\alpha+1}$ *by:*
  $y <_{\alpha+1} z$ *iff there is a name for* y *which is* $\tilde{<}_{\alpha+1}$*-smaller then every name for* z.
- *for limit* λ, *let* $<_\lambda = \bigcup_{\alpha<\lambda} <_\alpha$ *and* $\tilde{<}_\lambda = \bigcup_{\alpha<\lambda} \tilde{<}_\alpha$.

This defines two hierarchies of wellorderings linked by the interpretation function I.

Theorem 8.8. *a)* $<_\alpha$ *and* $\tilde{<}_\alpha$ *are well-defined*

b) $\tilde{<}_\alpha$ *is a wellordering of* $\tilde{L}_\alpha$

c) $<_\alpha$ *is a wellordering of* L_α

d) $\beta < \alpha$ *implies that* $\tilde{<}_\beta$ *is an initial segment of* $\tilde{<}_\alpha$

e) $\beta < \alpha$ *implies that* $<_\beta$ *is an initial segment of* $<_\alpha$

Proof. By induction on $\alpha \in \mathrm{Ord}$. $\square$

We can thus define wellorders $<_L$ and $\tilde{<}$ of L and $\tilde{L}$ respectively:

$$<_L = \bigcup_{\alpha\in\mathrm{Ord}} <_\alpha \quad \text{and} \quad \tilde{<} = \bigcup_{\alpha\in\mathrm{Ord}} \tilde{<}_\alpha$$

Theorem 8.9. $<_L$ *is a wellordering of* L.

The above recursions are definite and yield:

Theorem 8.10. *The terms* $<_\alpha$ *and* $\tilde{<}_\alpha$ *are definite.*

8.2 An external continuum hypothesis

Theorem 8.11. $\mathcal{P}(\omega) \cap L \subseteq L_{\aleph_1}$.

"**Proof**". Let $m \in \mathcal{P}(\omega) \cap L$. By the downward LÖWENHEIM SKOLEM theorem let $K \prec L$ be a "sufficiently elementary" substructure such that

$$m \in K \text{ and } \mathrm{card}(K) = \aleph_0.$$

Let $\pi\colon (K, \in) \cong (K', \in)$ be the MOSTOWSKI transitivisation of K defined by

$$\pi(u) = \{\pi(v) | v \in u \wedge v \in K\}.$$

$\pi \upharpoonright \omega = \mathrm{id} \upharpoonright \omega$ and

$$\pi(m) = \{\pi(i) | i \in m \wedge i \in X\} = \{\pi(i) | i \in m\} = \{i | i \in m\} = m.$$

A *condensation argument* will show that there is $\eta \in \mathrm{Ord}$ with
$K' = L_\eta$. $\mathrm{card}(\eta) \leqslant \mathrm{card}(L_\eta) = \mathrm{card}(K) = \aleph_0$ and $\eta < \aleph_1$. Hence

$$m \in K' = L_\eta \subseteq L_{\aleph_1}.$$

Chapter 9
The Axiom of Constructibility

If $V = L$ holds then every set is constructible, and the above external arguments become internal. We shall show that $(V = L)^L$.

Definition 9.1. *The* ***axiom of constructibility*** *is the property* $V = L$*.*

Theorem 9.2. (ZF^-) *The axiom of constructibility holds in* L*. This can be also written as* $(V = L)^L$ *or* $L = L^L$*.*

Proof. By Theorem 8.5, the term L_α is definite. Thus the formula $x \in L_\alpha$ is absolute for the transitive ZF^--model L. Since $L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$ we have

$\forall x \in L\, \exists \alpha \in \mathrm{Ord}\, x \in L_\alpha$
$\forall x \in L\, \exists \alpha \in L\, (\alpha \in \mathrm{Ord} \wedge x \in L_\alpha)$
$\forall x \in L\, \exists \alpha \in L\, ((\alpha \in \mathrm{Ord})^L \wedge (x \in L_\alpha)^L)$
$\forall x \in L\, \exists \alpha \in L\, ((\alpha \in \mathrm{Ord})^L \wedge (x \in L_\alpha)^L)$
$(\forall x \exists \alpha\, x \in L_\alpha)^L$
$(\forall x\, x \in L)^L$
$(V = L)^L$. □

Theorem 9.3. (ZF^-) *The axiom of choice holds in* L*:* AC^L*.*

Theorem 9.4. *If the theory* ZF *is consistent then the theory* $\mathrm{ZFC} = \mathrm{ZF} + \mathrm{AC}$ *is also consistent.*

L ist minimal.

Chapter 10

Constructible Operations and condensation

There are various ways of ensuring the condensation property for the structure K as used in the above argument for the continuum hypothesis. We shall only require closure under some basic operations of constructibility theory, in particular the interpretation operator I. An early predecessor for this approach to condensation and to hyperfine structure theory can be found in Gödel's 1939 paper [2]:

> Proof: Define a set K of constructible sets, a set O of ordinals and a set F of Skolem functions by the following postulates I-VII:
>
> I. $M_{\omega_\mu} \subseteq K$ and $m \in K$.
>
> II. If $x \in K$, the order of x belongs to O.
>
> III. If $x \in K$, all constants occuring in the definition of x belong to K.
>
> IV. If $\alpha \in O$ and $\phi_\alpha(x)$ is a propositional function over M_α all of whose constants belong to K, then:
>
> 1. The subset of M_α defined by ϕ_α belongs to K.
> 2. For any $y \in K \cdot M_\alpha$ the designated Skolem functions for ϕ_α and y or $\sim \phi_\alpha$ and y (according as $\phi_\alpha(y)$ or $\sim \phi_\alpha(y)$) belong to F.
>
> V. If $f \in F$, $x_1, \ldots, x_n \in K$ and $(x_1, \ldots, x_n)$ belongs to the domain of definition of f, then $f(x_1, \ldots, x_n) \in K$.
>
> VI. If $x, y \in K$ and $x - y \neq \Lambda$ the first element of $x - y$ belongs to K.
>
> VII. No proper subsets of K, O, F satisfy I--VI.
>
> ....
>
> ....

Theorem 5. There exists a one-to-one mapping x' of K on M_η such that $x \in y \equiv x' \in y'$ for $x, y \in K$ and $x' = x$ for $x \in M_{\omega_\mu}$.

Proof: The mapping x' (....) is defined by transfinite induction on the order, ...

10.1 Constructible operations

A substructure of the kind considered by Gödel may be obtained as a closure with respect to certain *constructible operations*.

Definition 10.1. *Define the constructible operations I, N, S by:*

a) ***Interpretation**: for a name $(L_\alpha, \varphi, \vec{x})$ let*
$I(L_\alpha, \varphi, \vec{x}) = \{y \in L_\alpha \mid (L_\alpha, \in) \vDash \varphi(y, \vec{x})\}$;

b) ***Naming**: for $y \in L$ let*
$N(y) =$ the $\tilde{<}$-least name $(L_\alpha, \varphi, \vec{x})$ such that $I(L_\alpha, \varphi, \vec{x}) = y$.

c) ***Skolem function**: for a name $(L_\alpha, \varphi, \vec{x})$ let*
$S(L_\alpha, \varphi, \vec{x}) =$ the $<_L$ - least $y \in L_\alpha$ such that $L_\alpha \vDash \varphi(y, \vec{x})$ if such a y exists; set $S(L_\alpha, \varphi, \vec{x}) = 0$ if such a y does not exist.

As we do not assume that α is a limit ordinal and therefore do not have pairing, we make the following convention.

For $X \subseteq L$, $(L_\alpha, \varphi, \vec{x})$ a name we write $(L_\alpha, \varphi, \vec{x}) \in X$ to mean that L_α and each component of $\vec{x}$ is an element of X.

Definition 10.2. *$X \subseteq L$ is **constructibly closed**, $X \lhd L$, iff X is closed under I, N, S:*

$$\begin{aligned} (L_\alpha, \varphi, \vec{x}) \in X &\longrightarrow I(L_\alpha, \varphi, \vec{x}) \in X \text{ and } S(L_\alpha, \varphi, \vec{x}) \in X, \\ y \in X &\longrightarrow N(y) \in X. \end{aligned}$$

*For $X \subseteq L$, $L\{X\} =$ the $\subseteq$-smallest $Y \supseteq X$ such that $Y \lhd L$ is called the **constructible hull** of X.*

The constructible hull $L\{X\}$ of X can be obtained by closing X under the functions I, N, S in the obvious way. Hulls of this kind satisfy certain "algebraic" laws which will be stated later in the context of *fine* hulls. Clearly each L_α is constructibly closed.

Theorem 10.3. (Condensation Theorem) *Let X be constructibly closed and let π: $X \cong M$ be the Mostowski collapse of X onto the transitive set M. Then there is an ordinal α such that $M = L_\alpha$, and π preserves I, N, S and $<_L$:*

$$\pi: (X, \in, <_L, I, N, S) \cong (L_\alpha, \in, <_L, I, N, S).$$

Proof. We first show the legitimacy of performing a Mostowski collapse.
(1) $(X, \in)$ is extensional.
Proof. Let $x, y \in X$, $x \neq y$. Let $N(x) = (L_\alpha, \varphi, \vec{p}) \in X$ and $N(y) = (L_\beta, \psi, \vec{q}) \in X$.
Case 1. $\alpha < \beta$. Then $x \in L_\beta$ and $(L_\beta, \in) \vDash \exists v(v \in x \nleftrightarrow \psi(v, \vec{q}))$. Let

$$z = S(L_\beta, (v \in u \nleftrightarrow \psi(v, \vec{w})), \frac{x\,\vec{q}}{u\,\vec{w}}) \in X$$

Then $z \in x \nleftrightarrow z \in y$. *qed*(1)

We prove the theorem for $X \subseteq L_\gamma$, by induction on γ. There is nothing to show in case $\gamma = 0$. For γ a limit ordinal observe that

$$\pi = \bigcup_{\alpha < \gamma} \pi \restriction (X \cap L_\gamma)$$

where each $\pi \restriction (X \cap L_\gamma)$ is the Mostowski collapse of the constructibly closed set $X \cap L_\gamma$ which by induction already satisfies the theorem.

So let $\gamma = \beta + 1$, $X \subseteq L_{\beta+1}$, $X \nsubseteq L_\beta$, and the theorem holds for β. Let

$$\pi \colon (X, \in) \cong (\bar{X}, \in)$$

be the Mostowski collapse of X. $X \cap L_\beta$ is an $\in$-initial segment of X, hence $\pi \restriction X \cap L_\beta$ is the Mostowski collapse of $X \cap L_\beta$. $X \cap L_\beta$ is constructibly closed and so by the inductive assumption there is some ordinal $\bar{\beta}$ such that

$$\pi \restriction X \cap L_\beta \colon (X \cap L_\beta, \in, <_L, I, N, S) \cong (L_{\bar{\beta}}, \in, <_L, I, N, S).$$

Note that the inverse map $\pi^{-1}\colon L_{\bar{\beta}} \to L_\beta$ is elementary since $X \cap L_\beta$ is closed under Skolem functions for L_β.
(2) $L_\beta \in X$.
Proof. Take $x \in X \setminus L_\beta$. Let $N(x) = (L_\gamma, \varphi, \vec{p})$. Then $L_\gamma \in X$ and $L_\gamma = L_\beta$ since $x \notin L_\beta$. *qed*(2)
(3) $\pi(L_\beta) = L_{\bar{\beta}}$.
Proof. $\pi(L_\beta) = \{\pi(x) | x \in L_\beta \wedge x \in X\} = \{\pi(x) | x \in X \cap L_\beta\} = L_{\bar{\beta}}$.
(4) $X = \{I(L_\beta, \varphi, \vec{p}) | \vec{p} \in X \cap L_\beta\}$.
Proof. $\supseteq$ is clear. For the converse let $x \in X$.
Case 1. $x \in L_\beta$. Then $x = I(L_\beta, v \in v_1, \frac{x}{v_1})$ is of the required form.
Case 2. $x \in L \setminus L_\beta$. Let $N(x) = (L_\beta, \varphi, \vec{p})$, noting that the first component cannot be smaller than L_β. $\vec{p} \in X$ and $x = I(N(x)) = I(L_\beta, \varphi, \vec{p})$ is of the required form. *qed*(4)
(5) Let $\vec{x} \in X$. Then $\pi(I(L_\beta, \varphi, \vec{x})) = I(L_{\bar{\beta}}, \varphi, \pi(\vec{x}))$.
Proof.

$$\begin{aligned}
\pi(I(L_\beta, \varphi, \vec{x})) &= \{\pi(y) | y \in \pi(I(L_\beta, \varphi, \vec{x})) \wedge y \in X\} \\
&= \{\pi(y) | (L_\beta, \in) \vDash \varphi(y, \vec{x}) \wedge y \in X\} \\
&= \{\pi(y) | (L_{\bar{\beta}}, \in) \vDash \varphi(\pi(y), \pi(\vec{x})) \wedge y \in X\} \\
&= \{z \in L_{\bar{\beta}} | (L_{\bar{\beta}}, \in) \vDash \varphi(z, \pi(\vec{x}))\} \\
&= I(L_{\bar{\beta}}, \varphi, \pi(\vec{x})).
\end{aligned}$$

qed(5)
(6) $\bar{X} = L_{\bar{\beta}+1}$.
Proof. By (4,5),

$$\begin{aligned} L_{\bar{\beta}+1} &= \{I(L_{\bar{\beta}}, \varphi, \vec{x}) \,|\, \vec{x} \in L_{\bar{\beta}}\} \\ &= \{I(L_{\bar{\beta}}, \varphi, \pi(\vec{p})) \,|\, \vec{p} \in X \cap L_\beta\}, \text{ since } \pi \upharpoonright X \cap L_\beta : X \cap L_\beta \cong L_{\bar{\beta}}\ , \\ &= \{\pi(I(L_\beta, \varphi, \vec{p})) \,|\, \vec{p} \in X \cap L_\beta\} \\ &= \pi''\{I(L_\beta, \varphi, \vec{p}) \,|\, \vec{p} \in X \cap L_\beta\} \\ &= \pi''X = \bar{X}. \end{aligned}$$

qed(6)
(7) Let $y \in X$. Then $\pi(N(y)) = N(\pi(y))$. This means: if $N(y) = (L_\delta, \varphi, \vec{x})$ then $N(\pi(y)) = (\pi(L_\delta), \varphi, \pi(\vec{x})) = (L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$.
Proof. Let $N(y) = (L_\delta, \varphi, \vec{x})$. Then $y = I(L_\delta, \varphi, \vec{x})$ and by (5) we have $\pi(y) = I(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$. Assume for a contradiction that $(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) \neq N(\pi(y))$. Let $N(\pi(y)) = (L_\eta, \psi, \vec{y})$. By the minimality of names we have $(L_\eta, \psi, \vec{y}) \tilde{<} (L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$. Then $(L_{\pi^{-1}(\eta)}, \psi, \pi^{-1}(\vec{y})) \tilde{<} (L_\delta, \varphi, \vec{x})$. By the minimality of $(L_\delta, \varphi, \vec{x}) = N(y)$, $I(L_{\pi^{-1}(\eta)}, \psi, \pi^{-1}(\vec{y})) \neq I(L_\delta, \varphi, \vec{x}) = y$. Since π is injective and by (5),

$$\begin{aligned} \pi(y) &\neq \pi(I(L_{\pi^{-1}(\eta)}, \psi, \pi^{-1}(\vec{y}))) \\ &= I(L_\eta, \psi, \vec{y}) \\ &= I(N(y)) = y. \end{aligned}$$

Contradiction. *qed*(7)
(8) Let $x, y \in X$. Then $x <_L y$ iff $\pi(x) <_L \pi(y)$.
Proof. $x <_L y$ iff $N(x) \tilde{<} N(y)$ iff $\pi(N(x)) \tilde{<} \pi(N(y))$ (since inductively π preserves $<_L$ on $X \cap L_\beta$ and $\tilde{<}$ is canonically defined from $<_L$) iff $N(\pi(x)) \tilde{<} N(\pi(y))$ iff $\pi(x) <_L \pi(y)$. *qed*(8)
(9) Let $(L_\delta, \varphi, \vec{x}) \in X$. Then $\pi(S(L_\delta, \varphi, \vec{x})) = S(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$.
Proof. We distinguish cases according to the definition of $S(L_\delta, \varphi, \vec{x})$.
Case 1. There is no $v \in I(L_\delta, \varphi, \vec{x})$, i.e., $I(L_\delta, \varphi, \vec{x}) = \emptyset$ and $S(L_\delta, \varphi, \vec{x}) = \emptyset$. Then by (5),

$$I(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) = \pi(I(L_\delta, \varphi, \vec{x})) = \pi(\emptyset) = \emptyset$$

and $S(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) = \emptyset$. So the claim holds in this case.
Case 2. There is $v \in I(L_\delta, \varphi, \vec{x})$, and then $S(L_\delta, \varphi, \vec{x})$ is the $<_L$-smallest element of $I(L_\delta, \varphi, \vec{x})$. Let $y = S(L_\delta, \varphi, \vec{x})$. By (5),

$$\pi(y) \in \pi(I(L_\delta, \varphi, \vec{x})) = I(L_{\pi(\delta)}, \varphi, \pi(\vec{x})).$$

So $S(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$ is well-defined as the $<_L$ -minimal element of $I(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$. Assume for a contradiction that $S(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) \neq \pi(y)$. Let $z = S(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) \in I(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$. By the minimality of Skolem values, $z <_L \pi(y)$. By (8), $\pi^{-1}(z) <_L y$. Since π is $\in$-preserving, $\pi^{-1}(z) \in I(L_\delta, \varphi, \vec{x})$. But this contradicts the $<_L$-minimality of $y = S(L_\delta, \varphi, \vec{x})$ □

Chapter 11
GCH in L

Theorem 11.1. $(L, \in) \vDash \mathrm{GCH}$.

Proof. $(L, \in) \vDash V = L$. It suffices to show that

$$\mathrm{ZFC} + V = L \vdash \mathrm{GCH}.$$

Let $\omega_\mu \geqslant \aleph_0$ be an infinite cardinal.
(1) $\mathcal{P}(\omega_\mu) \subseteq L_{\omega_\mu^+}$.
Proof. Let $m \in \mathcal{P}(\omega_\mu)$. Let $K = L\{L_{\omega_\mu} \cup \{m\}\}$ be the constructible hull of $L_{\omega_\mu} \cup \{m\}$. By the Condensation Theorem take an ordinal η and and the Mostowski isomorphism

$$\pi\colon (K, \in) \cong (L_\eta, \in).$$

Since $L_{\omega_\mu} \subseteq K$ we have $\pi(m) = m$.

$$\eta < \mathrm{card}(\eta)^+ = \mathrm{card}(L_\eta)^+ = \mathrm{card}(K)^+ = \mathrm{card}(L_{\omega_\mu})^+ = \omega_\mu^+.$$

Hence $m \in L_\eta \subseteq L_{\omega_\mu^+}$. *qed*(1)

Thus $\omega_\mu^+ \leqslant \mathrm{card}(\mathcal{P}(\omega_\mu)) \leqslant \mathrm{card}(L_{\omega_\mu^+}) = \omega_\mu^+$. □

Chapter 12
Trees

Throughout these lectures we shall prove combinatorial principles in L and apply them to construct specific structures that cannot be proved to exist in ZFC alone. We concentrate on the construction of infinite *trees* since they are purely combinatorial objects which are still quite close to ordinals and cardinals. One could extend these considerations and also construct unusual topological spaces or uncountable groups.

Definition 12.1. *A **tree** is a strict partial order $T = (T, <_T)$, such that $\forall t \in T\,\{s \in T \mid s <_T t\}$ is well-ordered by $<_T$. For $t \in T$ let $\mathrm{ht}_T(t) = \mathrm{otp}(\{s \in T \mid s <_T t\})$ be the **height** of t in T. For $X \subseteq \mathrm{Ord}$ let T_X be the set of points in the tree whose heights lie in X:*

$$T_X = \{t \in T \mid \mathrm{ht}_T(t) \in X\}.$$

*In particular, $T_{\{\alpha\}}$ is the α-th **level** of the tree and T_α is the initial segment of T below α. We let*

$$\mathrm{ht}(T) = \min\{\alpha \mid T = T_\alpha\}$$

*be the **height** of the tree T.*

*A **chain** in T is a linearly ordered subset of T. An $\subseteq$-maximal chain is called a **branch**.*

Definition 12.2. *A tree T of cardinality λ all of whose levels and branches are of cardinality $< \lambda$ is called a λ-**Aronszajn tree**. If $\lambda = \omega_1$, T is called an **Aronszajn tree**.*

Theorem 12.3. *Let κ be regular and $\forall \lambda < \kappa\; 2^\lambda \leqslant \kappa$. Then there is a κ^+-Aronszajn tree.*

Hence in ZFC one can show the existence of an (ω_1-)Aronszajn tree. The generalized continuum hypothesis implies the assumption $\forall \lambda < \kappa\; 2^\lambda \leqslant \kappa$, so in L there are κ^+-Aronszajn trees for every regular κ.

Theorem 12.4. *Let κ be an infinite cardinal. Then there is a linear order $(Q, \prec)$ such that $\mathrm{card}(Q) = \kappa$ and every $\alpha < \kappa^+$ can be order-embedded into every proper interval of Q.*

Proof. Let $Q = \{a \in {}^\omega\kappa \mid \exists m \in \omega \forall n \in \omega\, (n > m \rightarrow a(n) = 0\}$ be the set of ω-sequences from κ which are eventually zero and define the lexicographic linear order $\prec$ on Q by:

$$a \prec b \leftrightarrow \exists n \in \omega (a \upharpoonright n = b \upharpoonright n \wedge a(n) < b(n)).$$

We first prove the embedding property for $\alpha = \kappa$:
(1) If $a \prec b$ then there is an order-preserving embedding

$$f\colon (\kappa, <) \to ((a,b), \prec)$$

into the interval $(a,b) = \{c \mid a \prec c \prec b\}$.
Proof. Take $n \in \omega$ such that

$$a \upharpoonright n = b \upharpoonright n \wedge a(n) < b(n).$$

Then define $f\colon (\kappa, <) \to ((a,b), \prec)$ by

$$f(i) = (a \upharpoonright n+1) \cup \{(n+1, a(n+1)+1+i)\} \cup (0 \mid n+2 \leqslant l < \omega).$$

qed(1)

We prove the full theorem by induction on $\alpha < \kappa^+$. Let $\alpha < \kappa^+$ and assume that the theorem holds for all $\beta < \alpha$. Let (a,b) be a proper interval of Q, $a \prec b$.
Case 1: $\alpha = \beta + 1$ is a successor ordinal. By (1) take $b' \in (a, b)$. By the inductive assumption take an order-preserving map $f'\colon (\beta, <) \to (a, b')$. Extend f' to an order-preserving map $f\colon (\alpha, <) \to (a,b)$ by setting $f(\beta) = b'$.
Case 2: α is a limit ordinal. Since $\alpha < \kappa^+$ let $\alpha = \bigcup_{i<\kappa} \alpha_i$ such that $\forall i < \kappa\, \alpha_i < \alpha$. By (1) let $f\colon (\kappa, <) \to ((a,b), \prec)$ order-preservingly. By the inductive assumption choose a sequence $(g_i \mid i < \kappa)$ of order-preserving embeddings

$$g_i\colon (\alpha_i, <) \to ((f(i), f(i+1)), \prec).$$

Then define an order-preserving embedding

$$h\colon (\alpha, <) \to ((a,b), \prec)$$

by $h(\beta) = g_i(\beta)$, where $i < \kappa$ is minimal such that $\beta \in \alpha_i$. □

Proof of Theorem 12.3. Let $(Q, \prec)$ be a linear order as in Theorem 12.4. We define a tree

$$T \subseteq \{t \mid \exists \alpha < \kappa^+\ t\colon (\alpha, <) \to (Q, \prec) \text{ is order-preserving}\}$$

with strict inclusion $\subset$ as the tree order such that:

a) T is closed under initial segments, i.e., $\forall t \in T\, \forall \xi \in \mathrm{Ord}\ t \upharpoonright \xi \in T$;

b) for all $\alpha < \kappa^+$, $T_{\{\alpha\}} = \{t \in T \mid \mathrm{dom}(t) = \alpha\}$ has cardinality $\leqslant \kappa$;

c) for all limit ordinals $\alpha < \kappa^+$ with $\mathrm{cof}(\alpha) < \kappa$

$$T_{\{\alpha\}} = \{t \mid t\colon \alpha \to Q \wedge \forall \beta < \alpha\ \ t \upharpoonright \beta \in T_{\{\beta\}}\}.$$

d) for all $\alpha < \beta < \kappa^+$, $t \in T_{\{\alpha\}}$, $a \prec b \in Q$ such that $\forall \xi \in \alpha\ t(\xi) \prec a$ there exists $t' \in T_{\{\beta\}}$ such that $t \subset t'$ and $\forall \xi \in \beta\, t'(\xi) \prec b$.

We define the levels $T_{\{\alpha\}}$ by recursion on $\alpha < \kappa^+$.
Let $T_{\{0\}} = \{\emptyset\}$.
Let $\alpha = \beta + 1$ and assume that $T_{\{\beta\}}$ is defined according to a) - d). For any $t \in T_{\{b\}}$ and $a \prec b \in Q$ such that $\forall \xi \in \beta\, t(\xi) \prec a$ choose an extension $t'_{a,b}$ such that

– $t'_{a,b}\colon (\alpha, <) \to (Q, \prec)$ is order-preserving;

- $t'_{a,b} \supset t$;
- $\forall \xi \in \alpha\; t'_{a,b}(\xi) \prec b$.

One could for example set $t'_{a,b}(\beta) = a$. Then set

$$T_{\{\alpha\}} = \{t'_{a,b} \,|\, t \in T_{\{\beta\}}, a \prec b, \forall \xi \in \beta\, t(\xi) \prec a\}.$$

Obviously, conditions a) - d) are satisfied.
Let $\alpha < \kappa^+$ be a limit ordinal so that for all $\beta < \alpha$ $T_{\{\beta\}}$ is defined according to a) - d). These are the levels of the tree T_α.
Case 1: $\mathrm{cof}(\alpha) < \kappa$. Let the sequence $(\alpha_i \,|\, i < \mathrm{cof}(\alpha))$ be continuous and cofinal in α with $\mathrm{cof}(\alpha) < \alpha_0$. By c) we must set

$$T_{\{\alpha\}} = \{t \,|\, t\colon \alpha \to Q \wedge \forall \beta < \alpha\;\; t \upharpoonright \beta \in T_{\{\beta\}}\}.$$

Let us check that properties a) - d) hold for this definition. a) is immediate. For b), note that every $t \in T_{\{\alpha\}}$ is determined by $(t \upharpoonright \beta \,|\, \beta \in C)$:

$$\begin{aligned}
\mathrm{card}(T_{\{\alpha\}}) &\leqslant \mathrm{card}({}^{\mathrm{cof}(\alpha)}(T_\alpha)) \\
&\leqslant \mathrm{card}({}^{\mathrm{cof}(\alpha)} \bigcup_{\beta<\alpha} T_{\{\beta\}}) \\
&\leqslant \mathrm{card}({}^{\mathrm{cof}(\alpha)}\kappa \cdot \kappa) \\
&= \kappa^{\mathrm{cof}(\alpha)} \\
&\leqslant \sum_{\nu<\kappa} \nu^{\mathrm{cof}(\alpha)} \\
&\leqslant \sum_{\nu<\kappa} 2^{\nu \cdot \mathrm{cof}(\alpha)} \\
&\leqslant \sum_{\nu<\kappa} \kappa \text{ , by the assumption } \forall \lambda < \kappa\, 2^\lambda \leqslant \kappa, \\
&= \kappa.
\end{aligned}$$

For d), let $t \in T_\alpha$ and $a \prec b \in Q$ such that $\forall \xi \in \mathrm{dom}(s)\; t(\xi) \prec a$. By Theorem 12.4 there is an order-preserving embedding $f\colon (\mathrm{cof}(\alpha), <\,) \to ((a, b), \prec\,)$. We may assume that $\mathrm{ht}(t) < \alpha_0$. We may recursively choose sequences $t_i \in T_{\{\alpha_i\}}$ such that

- $\forall i < j < \mathrm{cof}(\alpha)\, t \subset t_i \subset t_j$;
- $\forall i < \mathrm{cof}(\alpha) \forall \xi \in \alpha_i\; t_i(\xi) \prec f(i)$.

For non-limit ordinals $i < \mathrm{cof}(\alpha)$ use the extension property d). For limit ordinals $i < \mathrm{cof}(\alpha)$ note that α_i is the limit of $(\alpha_j \,|\, j < i)$ and is thus singular with $\mathrm{cof}(\alpha_i) \leqslant i < \mathrm{cof}(\alpha) < \kappa$. We can then take $t_i = \bigcup_{j<i} t_j$ which is an element of $T_{\{\alpha_i\}}$ by c).

Then take $t' = \bigcup_{i<\mathrm{cof}(\alpha)} t_i$. $t' \in T_{\{\alpha\}}$ by the definition of $T_{\{\alpha\}}$. t' is an extension of t and $\forall \xi \in \alpha\, t'(\xi) \prec b$ as required.
Case 2: $\mathrm{cof}(\alpha) = \kappa$. Let the sequence $(\alpha_i \,|\, i < \kappa)$ be continuous and cofinal in α. For each $t \in T_\alpha$ and $a \prec b \in Q$ with $\forall \xi \in \mathrm{dom}(t)\, t(\xi) \prec a$ we shall construct an extension $t'_{a,b}$ in T appropriate for the extension property d): By Theorem 12.4 there is an order-preserving embedding $f\colon (\kappa, <\,) \to ((a, b), \prec\,)$. We may assume that $\mathrm{ht}(t) < \alpha_0$. Recursively choose sequences $t_i \in T_{\{\alpha_i\}}$ such that

- $\forall i < j < \kappa\;\; t \subset t_i \subset t_j$;

– $\forall i<\kappa \forall \xi \in \alpha_i\; t_i(\xi) \prec f(i)$.

For non-limit ordinals $i<\mathrm{cof}(\alpha)$ use the extension property d). For limit ordinals $i<\mathrm{cof}(\alpha)$ note that α_i is the limit of $(\alpha_j \mid j<i)$ and is thus singular with $\mathrm{cof}(\alpha_i) \leqslant i<\kappa$. We can then take $t_i=\bigcup_{j<i} t_j$ which is an element of $T_{\{\alpha_i\}}$ by c).

Then set $t'_{a,b}=\bigcup_{i<\kappa} t_i$. $t'_{a,b}$ is an extension of t and $\forall \xi \in \alpha\, t'_{a,b}(\xi) \prec b$ as required in c).

Now define

$$T_{\{\alpha\}}=\{t'_{a,b} \mid t\in T_\alpha, a\prec b, \forall \xi \in \mathrm{dom}(t)\; t(\xi)\prec a\}.$$

The properties a) - d) are easily checked. a) follows by construction. For b) note that

$$\begin{aligned} \mathrm{card}(T_{\{\alpha\}}) &\leqslant \mathrm{card}(T_\alpha)\cdot \mathrm{card}(Q)\cdot \mathrm{card}(Q) \\ &\leqslant (\mathrm{card}(\alpha)\cdot\kappa)\cdot\kappa\cdot\kappa \\ &\leqslant \kappa\cdot\kappa\cdot\kappa\cdot\kappa \leqslant \kappa. \end{aligned}$$

c) does not apply for $T_{\{\alpha\}}$ and d) holds by construction.

This defines the tree $T=\bigcup_{\alpha<\kappa^+} T_{\{\alpha\}}$. We show that T is a κ^+-Aronszajn tree.

(1) $\mathrm{ht}(T)=\kappa^+$.

Proof. Property d) ensures that $\forall \alpha<\kappa^+\; T_{\{\alpha\}}\neq\emptyset$. By construction, $T_{\{\kappa^+\}}=\emptyset$, hence $\mathrm{ht}(T)=\kappa^+$. *qed*(1)

(2) $\mathrm{card}(T)=\kappa^+$, since by property b) $\kappa^+=\mathrm{ht}(T)\leqslant \mathrm{card}(T)\leqslant\kappa^+\cdot\kappa=\kappa^+$.

(3) $\forall\alpha<\mathrm{ht}(T)\,\mathrm{card}(T_{\{\alpha\}})\leqslant\kappa$, by property b).

(4) Every branch of T has cardinality $\leqslant\kappa$.

Proof. Let $B\subseteq T$ be a branch of T. Then $\bigcup B\colon(\theta,<)\to(Q,\prec)$ is an order-preserving embedding for some $\theta\in\mathrm{Ord}$. Since $\bigcup B$ is an injection from θ into Q, $\mathrm{card}(\theta)\leqslant\kappa$. Then $\mathrm{card}(B)\leqslant\theta\leqslant\kappa$. □

Chapter 13

The principle $\diamondsuit$

We shall study a principle which was introduced by Ronald Jensen and may be seen as a strong form of a continuum hypothesis. We shall use the principle to construct Aronszajn trees with stronger properties. The principle $\diamondsuit$ involves notions for "large" subsets of a regular uncountable cardinal: *closed unbounded* and *stationary* sets.

Definition 13.1. *Let κ be a regular uncountable cardinal.*

*a) $C \subseteq \kappa$ is **closed unbounded** in κ if C is cofinal in κ and*

$$\forall \alpha < \kappa (C \cap \alpha \text{ is cofinal in } \alpha \rightarrow \alpha \in C).$$

*b) $\mathcal{C}_\kappa = \{X \subseteq \kappa \mid \exists C \subseteq X\, C \text{ is closed unbounded in } \kappa\}$ is the **closed unbounded filter** on κ.*

*c) $S \subseteq \kappa$ is **stationary** in κ if $\forall C \in \mathcal{C}_\kappa\, S \cap C \neq \emptyset$.*

Theorem 13.2. *Let $\kappa > \omega$ be a regular cardinal. Then $\mathcal{C}_\kappa$ is a non-trivial filter on κ which is $<\kappa$-complete, i.e.,*

$$\forall \beta < \kappa \forall \{X_\xi \mid \xi < \beta\} \subseteq \mathcal{C}_\kappa \bigcap_{\xi < \beta} X_\xi \in \mathcal{C}_\kappa .$$

Proof. Exercise. □

Definition 13.3. *Let κ be a regular uncountable cardinal. Then $\diamondsuit_\kappa$ is the principle: there is a sequence $(S_\alpha \mid \alpha < \kappa)$ such that*

$$\forall S \subseteq \kappa \; \{\alpha < \kappa \mid S \cap \alpha = S_\alpha\} \text{ is stationary in } \kappa.$$

Theorem 13.4. *Assume $\diamondsuit_{\kappa^+}$. Then $2^\kappa = \kappa^+$.*

Proof. Let $(S_\alpha \mid \alpha < \kappa)$ be a sequence satisfying $\diamondsuit_{\kappa^+}$. Consider $x \subseteq \kappa$. By the $\diamondsuit_{\kappa^+}$-property there is $\alpha \in (\kappa, \kappa^+)$ such that $x = x \cap \alpha = S_\alpha$. Hence

$$\mathcal{P}(\kappa) \subseteq \{S_\alpha \mid \alpha < \kappa^+\}$$

and

$$2^\kappa = \mathrm{card}(\mathcal{P}(\kappa)) \leqslant \kappa^+. \qquad \square$$

Theorem 13.5. *Assume $V = L$. Then $\diamondsuit_\kappa$ holds for all regular uncountable cardinals κ.*

Proof. Define $(S_\alpha \,|\, \alpha < \kappa)$ by recursion on α. Consider $\beta < \kappa$ and let $(S_\alpha \,|\, \alpha < \beta)$ be appropriately defined. If β is not a limit ordinal, set $S_\beta = \emptyset$. If β is a limit ordinal, let (S_β, C_β) be the $<_L$-minimal pair such that C_β is closed unbounded in β, $S_\beta \subseteq \beta$ and $\forall \alpha \in C_\beta\ S_\beta \cap \alpha \neq S_\alpha$, if this exists; otherwise let $S_\beta = \emptyset$.

We show that $(S_\alpha \,|\, \alpha < \kappa)$ satisfies $\diamondsuit_\kappa$. Assume not. Then there is a set $S \subseteq \kappa$ such that $\{\alpha < \kappa \,|\, S \cap \alpha = S_\alpha\}$ is not stationary in κ. Hence there is a closed unbounded set $C \subseteq \kappa$ such that

$$\{\alpha < \kappa \,|\, S \cap \alpha = S_\alpha\} \cap C = \emptyset,$$

i.e.,

$$\forall \alpha \in C\ S \cap \alpha \neq S_\alpha.$$

We may assume that (S, C) is the $<_L$-minimal pair such that C is closed unbounded in κ and $\forall \alpha \in C\ S \cap \alpha \neq S_\alpha$.

Take a level L_θ such that $(\mathrm{ZF}^-)^{L_\theta}$ and $\kappa, (S_\alpha \,|\, \alpha < \kappa), S, C \in L_\theta$.

(1) There is $X \lhd L$ such that L_θ, κ, $(S_\alpha \,|\, \alpha < \kappa)$, S, $C \in X$, and $\beta = X \cap \kappa$ is a limit ordinal $< \kappa$.

Proof. Define sequence $X_0 \subseteq X_1 \subseteq \ldots$ and $\beta_0 < \beta_1 < \ldots < \kappa$ by recursion so that $\mathrm{card}(X_n) < \kappa$ and $X_n \cap \kappa \subseteq \beta_n$. Let

$$X_0 = L\{\{L_\theta, \kappa, (S_\alpha \,|\, \alpha < \kappa), S, C\}\} \lhd L.$$

X_0 is countable and so $\mathrm{card}(X_0) < \kappa$.

Let X_n be defined such that $\mathrm{card}(X_n) < \kappa$. Since κ is a regular cardinal, $X_n \cap \kappa$ is bounded below κ. Take $\beta_n < \kappa$ such that $X_n \cap \kappa \subseteq \beta_n$. Then let

$$X_{n+1} = L\{X_n \cup (\beta_n + 1)\}.$$

$$\mathrm{card}(X_{n+1}) \leqslant \mathrm{card}(X_n) + \mathrm{card}(\beta_n) + \aleph_0 < \kappa.$$

Let $X = \bigcup_{n<\omega} X_n$ and $\beta = \bigcup_{n<\omega} \beta_n$. Since κ is regular uncountable, β is a limit ordinal and $\beta < \kappa$. By construction,

$$X = \bigcup_{n<\omega} X_n = \bigcup_{n<\omega} L\{X_n \cup (\beta_n + 1)\} = L\{\bigcup_{n<\omega} (X_n \cup (\beta_n + 1))\} \lhd L.$$

$$\beta = \bigcup_{n<\omega} \beta_n \subseteq (\bigcup_{n<\omega} X_{n+1}) \cap \kappa \subseteq X \cap \kappa \subseteq \bigcup_{n<\omega} X_n \cap \kappa \subseteq \bigcup_{n<\omega} \beta_n = \beta.$$

qed(1)

By the condensation theorem let

$$\pi\colon (X, \in, <_L, I, N, S) \cong (L_\delta, \in, <_L, I, N, S)$$

for some $\delta \in \mathrm{Ord}$. We compute the images of various sets.
(2) $\pi \upharpoonright \beta = \mathrm{id} \upharpoonright \beta$, since $\beta = X \cap \kappa \subseteq X$ is transitive.
(3) $\pi(\kappa) = \beta$, since $\pi(\kappa) = \{\pi(\xi) | \xi \in \kappa \wedge \xi \in X\} = \{\pi(\xi) | \xi \in \beta\} = \{\xi | \xi \in \beta\} = \beta$.
(4) $\pi(S) = S \cap \beta$, since

$$\begin{aligned} \pi(S) &= \{\pi(\xi) | \xi \in S \wedge \xi \in X\} \\ &= \{\pi(\xi) | \xi \in S \cap X\} \\ &= \{\pi(\xi) | \xi \in S \cap \beta\} \\ &= \{\xi | \xi \in S \cap \beta\} \\ &= S \cap \beta. \end{aligned}$$

Similarly
(5) $\pi(C) = C \cap \beta$.
(6) $\pi((S_\alpha | \alpha < \kappa)) = (S_\alpha | \alpha < \beta)$.
Proof.

$$\begin{aligned} \pi((S_\alpha | \alpha < \kappa)) &= \pi(\{(\alpha, S_\alpha) | \alpha \in \kappa\}) \\ &= \{\pi((\alpha, S_\alpha)) | \alpha \in \beta\} \\ &= \{(\pi(\alpha), \pi(S_\alpha)) | \alpha \in \beta\} \\ &= \{(\alpha, S_\alpha) | \alpha \in \beta\} \\ &= (S_\alpha | \alpha < \beta). \end{aligned}$$

qed(6)
(7) $X \cap L_\theta$ is an elementary substructure of $(L_\theta, \in)$.
Proof. Since $L_\theta \in X$, the initial segment $X \cap L_\theta$ is closed with respect to the Skolem functions $S(L_\theta, \_, \_)$ for L_θ. *qed*(7)

Let $\bar{\theta} = \pi(\theta)$. Then
(8) $\pi^{-1} \upharpoonright L_{\bar{\theta}} : (L_{\bar{\theta}}, \in) \to (L_\theta, \in)$ is an elementary embedding.

Now we use elementarity and absoluteness to derive a contradiction.
(9) $C \cap \beta$ is closed unbounded in β, $S \cap \beta \subseteq \beta$ and $\forall \alpha \in C \cap \beta \; S \cap \alpha \neq S_\alpha$.
Proof. C is closed unbounded in κ. Since this is a definite property, $(L_\theta, \in) \vDash C$ is closed unbounded in κ. By elementarity, $(L_{\bar{\theta}}, \in) \vDash C \cap \beta$ is closed unbounded in β. By the absoluteness of being closed unbounded, $C \cap \beta$ is closed unbounded in β.

The other properties follow by the assumptions on C and S. *qed*(9)
(10) $(S \cap \beta, C \cap \beta) = (S_\beta, C_\beta)$.
Proof. Assume not. By the minimality of (S_β, C_β) and (9), we get

$$(S_\beta, C_\beta) <_L (S \cap \beta, C \cap \beta).$$

Since $L_{\bar{\theta}}$ is an initial segment of $<_L$ we have $(S_\beta, C_\beta) \in L_{\bar{\theta}}$. The defining properties for (S_β, C_β) are absolute for $(L_{\bar{\theta}}, \in)$:

$$(L_{\bar{\theta}}, \in) \vDash C_\beta \text{ is closed unbounded in } \beta,\ S_\beta \subseteq \beta \text{ and } \forall \alpha \in C_\beta \; S_\beta \cap \alpha \neq S_\alpha.$$

By the elementarity of $\pi^{-1} \upharpoonright L_{\bar{\theta}}$:

$$(L_\theta, \in) \vDash \pi^{-1}(C_\beta) \text{ is cl. unb. in } \kappa,\ \pi^{-1}(S_\beta) \subseteq \kappa,\ \forall \alpha \in \pi^{-1}(C_\beta) \; \pi^{-1}(S_\beta) \cap \alpha \neq S_\alpha.$$

By the absoluteness of these properties for transitive ZF$^-$-models,

$$\pi^{-1}(C_\beta) \text{ is cl. unb. in } \kappa,\ \pi^{-1}(S_\beta) \subseteq \kappa,\ \forall \alpha \in \pi^{-1}(C_\beta)\ \pi^{-1}(S_\beta) \cap \alpha \neq S_\alpha\,,$$

i.e., the pair $(\pi^{-1}(S_\beta),\ \pi^{-1}(C_\beta))$ satisfies the defining property for $(S,\ C)$. Since π^{-1} preserves $<_L$,

$$(\pi^{-1}(S_\beta), \pi^{-1}(C_\beta)) <_L (\pi^{-1}(S \cap \beta), \pi^{-1}(C \cap \beta)) = (S, C).$$

This contradicts the $<_L$-minimal cloice of (S, C). $qed(10)$

By (9), β is a limit point of C and hence $\beta \in C$. By (10), $S \cap \beta = S_\beta$. This contradicts the choice of the pair $(S,\ C)$, i.e., there is no counterexample against the $\diamondsuit_\kappa$-property of the sequence $(S_\alpha \,|\, \alpha < \kappa)$. $\square$

Chapter 14

Combinatorial principles and Suslin trees

Definition 14.1. *Let $T = (T, <_T)$ be a tree.*

a) A set $A \subseteq T$ is an ***antichain*** *in T if $\forall s, t \in A\, (s \neq t \rightarrow (s \not<_T t \wedge t \not<_T s))$.*

*b) Let κ be a cardinal. T is called a κ-**Suslin tree** if $\operatorname{card}(T) = \kappa$ and every chain and antichain in T has cardinality $< \kappa$.*

Obviously every level of a tree is an antichain. Hence a κ-Suslin tree is also a κ-Aronszajn tree.

Theorem 14.2. *Let κ be an infinite cardinal. Let $T = (T, <_T)$ be a tree with $\operatorname{card}(T) = \kappa$ such that every antichain in T has cardinality $< \kappa$ and T is* ***branching****, i.e.*

$$\forall s \in T\, \exists t, t' \in T\, (s <_T t \wedge s <_T t' \wedge \mathrm{ht}_T(t) = \mathrm{ht}_T(t') = \mathrm{ht}_T(s) + 1 \wedge t \neq t').$$

Then T is a κ-Suslin tree.

Proof. It suffices to see that every chain in T has cardinality $< \kappa$. Let $C \subseteq T$ be a chain. For every $s \in C$ choose $t, t' \in T$ such that

$$s <_T t \wedge s <_T t' \wedge \mathrm{ht}_T(t) = \mathrm{ht}_T(t') = \mathrm{ht}_T(s) + 1 \wedge t \neq t'$$

Then at least one of t, t' is not an element of C. So for each $s \in C$ we can choose $s^* >_T s$ such that $s^* \notin C$ and $\mathrm{ht}_T(s^*) = \mathrm{ht}_T(s) + 1$.
(1) If $s, t \in C$ and $s \neq t$ then $s^* \not<_T t^* \wedge t^* \not<_T s^*$.
Proof. Assume not. Without loss of generality assume $s^* <_T t^*$. Since t is the immediate $<_T$-predecessor of t^* we have $s^* \leqslant_T t$ and $s^* \in C$. Contradiction. *qed*(1)

Hence $\{s^* \mid s \in C\}$ is an antichain in T. By assumption $\operatorname{card}(\{s^* \mid s \in C\}) < \kappa$. Since the assignment $s \mapsto s^*$ is injective, we have $\operatorname{card}(C) < \kappa$. $\square$

Theorem 14.3. *Assume $\diamondsuit_{\omega_1}$. Then there exists an ω_1-Suslin tree.*

Proof. Let $(S_\alpha \mid \alpha < \omega_1)$ be a $\diamondsuit_{\omega_1}$-sequence. We construct a tree $T = (T, <_T)$ of the form $T = \bigcup_{\alpha < \omega_1} T_{\{\alpha\}}$ such that every level T_α is countable. We can arrange that

$$T_{\{0\}} = \{0\} \text{ and } \forall \alpha \in [1, \omega_1)\, T_{\{\alpha\}} = \omega \cdot (\alpha + 1) \setminus \omega \cdot \alpha.$$

By recursion on $\alpha<\omega_1$ we shall determine the $<_T$-predecessors of $x\in T_{\{\alpha\}}$. We shall also ensure the following recursive condition which guarantees that the tree can always be continued:
(1) for all $\xi<\zeta\leqslant\alpha$ and $s\in T_{\{\xi\}}$ there exists $t\in T_{\{\zeta\}}$ such that $s<_T t$.

For $\alpha=0$ there is nothing to determine.

For $\alpha=1$, let every element of $T_{\{1\}}$ be a $<_T$-successor of $0\in T_{\{0\}}$.

Let $\alpha=\beta+1>1$ and let $<_T\restriction T_\alpha$ be determined so that (1) is satisfied. We let every $s\in T_{\{\beta\}}$ have two immediate successors in $T_{\{\alpha\}}$: if $s=\omega\cdot\beta+m\in T_{\{\beta\}}$ and $t=\omega\cdot\alpha+n\in T_{\{\alpha\}}$ then set

$$s<_T t \text{ iff } n=2\cdot m \text{ or } n=2\cdot m+1.$$

Since $<_T$ has to be a transitive partial order, this determines all $<_T$-predecessors of $x\in T_{\{\alpha\}}$. Also (1) holds for $<_T\restriction T_{\alpha+1}$.

Let α be a limit ordinal and let $<_T\restriction T_\alpha$ be determined so that (1) is satisfied.
(2) For every $s_0\in T_\alpha$ there is a branch B of the tree $\mathcal{T}_\alpha=(T_\alpha,<_T\restriction T_\alpha)$ such that $s_0\in B$ and $\operatorname{otp}(B)=\alpha$.
Proof. Choose an ω-sequence

$$\operatorname{ht}_T(s)=\alpha_0<\alpha_1<\ldots<\alpha_n<\ldots<\alpha$$

which is cofinal in α. Using (1) choose a sequence

$$s_0<_T s_1<_T\ldots<_T s_n<_T\ldots$$

such that $\forall n<\omega\ \operatorname{ht}_T(s_n)=\alpha_n$. Then

$$B=\{t\in T_\alpha\mid\exists n<\omega\ t<_T s_n\}$$

satisfies the claim. *qed*(2)

Define a set $S'_\alpha\subseteq T_\alpha$ as follows: if S_α is a maximal antichain in the tree $\mathcal{T}_\alpha=(T_\alpha,<_T\restriction T_\alpha)$ then set

$$S'_\alpha=\{r\in T_\alpha\mid\exists s\in S_\alpha\ s\leqslant_T r\};$$

otherwise set $S'_\alpha=T_\alpha$. The set S'_α is countable. Let $S'_\alpha=\{s_i\mid i<\omega\}$ be an enumeration of S'_α. For each $i<\omega$ use (2) to choose a branch B_i of $\mathcal{T}_\alpha$ with $s_i\in B_i$ and $\operatorname{otp}(B_i)=\alpha$. For $x=\omega\cdot\alpha+i\in T_{\{\alpha\}}$ and $s\in T_\alpha$ define

$$s<_T x \text{ iff } s\in B_i.$$

(3) Property (1) holds for $T_{\alpha+1}$.
Proof. Let $s\in T_\alpha$. It suffices to find $t\in T_{\{\alpha\}}$ such that $s<_T t$.
Case 1: $S'_\alpha=T_\alpha$. Then $s=s_i$ for some $i<\omega$, $s_i\in B_i$, and $s_i<_T\omega\cdot\alpha+i\in T_{\{\alpha\}}$.
Case 2: $S'_\alpha=\{r\in T_\alpha\mid\exists s\in S_\alpha\ s\leqslant_T r\}$, where S_α is a maximal antichain in $\mathcal{T}_\alpha=(T_\alpha,<_T\restriction T_\alpha)$. By the maximality of S_α there is $s'\in S_\alpha$ which is comparable with s:

$$s\leqslant_T s' \text{ or } s'\leqslant_T s.$$

Case 2.1: $s\leqslant_T s'$. Then $s'\in S'_\alpha$, say $s'=s_i$, $s\in B_i$, and $s<_T\omega\cdot\alpha+i\in T_{\{\alpha\}}$.
Case 2.2: $s'\leqslant_T s$. Then $s\in S'_\alpha$, say $s=s_i$, $s\in B_i$, and $s<_T\omega\cdot\alpha+i\in T_{\{\alpha\}}$. *qed*(3)

This concludes the recursive definition of the tree $T = (T, <_T)$. It is straightforward to check, that the predetermined sets $T_{\{\alpha\}}$ are indeed the α-th levels of the tree. By the construction at successors, the tree is branching. By the previous theorem it suffices to show that every antichain in T has cardinality $<\omega_1$.

Let $A \subseteq T$ be an antichain in T. Using the lemma of ZORN we may assume that A is maximal with respect to $\subseteq$.
(4) The set $C = \{\alpha < \omega_1 \mid A \cap \alpha \text{ is a maximal antichain in } T_\alpha\}$ is closed unbounded in ω_1.
Proof. Let us first show unboundedness. Let $\alpha_0 < \omega_1$. Construct an ω-sequence

$$\alpha_0 < \alpha_1 < \ldots < \omega_1$$

as follows. Let $\alpha_n < \omega_1$ be defined. By the maximality of A every $s \in T_{\alpha_n}$ is $<_T$-comparable to some $t \in A$. By the regularity of ω_1 one can take $\alpha_{n+1} \in (\alpha_n, \omega_1)$ such that

$$\forall s \in T_{\alpha_n} \exists t \in A \cap \alpha_{n+1} (s \leqslant_T t \vee t \leqslant_T s).$$

Let $\alpha = \bigcup_{n<\omega} \alpha_n < \omega_1$. $A \cap \alpha$ is an antichain in T, since it consists of pairwise incomparable elements. So $A \cap \alpha$ is an antichain in T_α. For the maximality consider $s \in T_\alpha$. Let $s \in T_{\alpha_n}$. By construction there is $t \in A \cap \alpha_{n+1}$ such that $s \leqslant_T t \vee t \leqslant_T s$. So every element of T_α is comparable with some element of $A \cap \alpha$.

For the closure property consider some $\alpha < \omega_1$ such that $C \cap \alpha$ is cofinal in α. To show that $\alpha \in C$ it suffices to show that $A \cap \alpha$ is a *maximal* antichain in T_α. Consider $s \in T_\alpha$. Take $\beta \in C \cap \alpha$ such that $s \in T_\beta$. Then $A \cap \beta$ is a maximal antichain in T_β and there exists $t \in A \cap \beta \subseteq A \cap \alpha$ which is comparable with s. Thus for every $s \in T_a$ there exists $t \in A \cap \alpha$ which is comparable with s. Thus $\alpha \in C$. *qed*(4)

By the $\diamondsuit_{\omega_1}$-property, $\{\alpha < \omega_1 \mid A \cap \alpha = S_\alpha\}$ is stationary in ω_1. Take $\alpha \in C$ such that $A \cap \alpha = S_\alpha$. Then $A \cap \alpha = S_\alpha$ is a maximal antichain in T_α.
(5) $A = A \cap \alpha$.
Proof. Let $t \in A$. We show that every $r \in T$ is comparable with some $s \in A \cap \alpha$. Since $A \cap \alpha$ is a maximal antichain in T_α this is clear for $r \in T_\alpha$ and we may assume that $r \in T \setminus T_\alpha$. Then $\mathrm{ht}_T(r) \geqslant \alpha$ and we can take the unique $\bar{r} \in T_{\{\alpha\}}$ such that $\bar{r} \leqslant_T r$. By construction of $T_{\{\alpha\}}$ there is some $s \in S_\alpha = A \cap \alpha$ such that

$$s <_T \bar{r} \leqslant_T r$$

qed(5)

By (5), $A = A \cap \alpha$ is countable. Since T is a branching tree all whose antichains are countable, T is a Suslin tree. □

We shall now study generalizations from ω_1-Suslin trees to κ^+-Suslin trees for $\kappa > \omega$. We first consider the case when κ is regular. There are now different kinds of limit cases α in the construction: $\mathrm{cof}(\alpha) < \kappa$ and $\mathrm{cof}(\alpha) = \kappa$. To ensure the analogue of property (1) of the previous proof, we

- extend all paths through T_α when $\mathrm{cof}(\alpha) < \kappa$;
- use the set S_α of the $\diamondsuit$-sequence as above when $\mathrm{cof}(\alpha) = \kappa$.

In the first case one assumes that $\kappa^{\mathrm{cof}(\alpha)} \leqslant \kappa^{<\kappa} = \kappa$ which is a consequence of GCH. For the second case to yield the desired result a $\diamondsuit$-principle for ordinals of cofinality κ is needed. Note that the set $\mathrm{Cof}_\kappa = \{\alpha < \kappa^+ \mid \mathrm{cof}(\alpha) = \kappa\}$ is stationary in κ^+.

Definition 14.4. *Let κ be a regular uncountable cardinal and let $D \subseteq \kappa$ be stationary in κ. Then $\diamondsuit_\kappa(D)$ is the principle: there is a sequence $(S_\alpha \mid \alpha < \kappa)$ such that*

$$\forall S \subseteq \kappa \, \{\alpha \in D \mid S \cap \alpha = S_\alpha\} \text{ is stationary in } \kappa.$$

Theorem 14.5. *Assume $V = L$. Let κ be a regular uncountable cardinal and $D \subseteq \kappa$ be stationary. Then $\diamondsuit_\kappa(D)$ holds.*

This is very much proved like $\diamondsuit_\kappa = \diamondsuit_\kappa(\kappa)$. We only indicate the necessary changes in the previous proof.

Proof. Let $\beta < \kappa$ and let $(S_\alpha \mid \alpha < \beta)$ be appropriately defined. If β is not a limit ordinal or $\beta \notin D$, set $S_\beta = \emptyset$. If β is a limit ordinal and $\beta \in D$, let (S_β, C_β) be the $<_L$-minimal pair such that C_β is closed unbounded in β, $S_\beta \subseteq \beta$ and $\forall \alpha \in D \cap C_\beta \; S_\beta \cap \alpha \neq S_\alpha$, if this exists; otherwise let $S_\beta = \emptyset$.

Assume that $(S_\alpha \mid \alpha < \kappa)$ does not satisfy $\diamondsuit_\kappa$. Then there is a set $S \subseteq \kappa$ such that $\{\alpha \in D \mid S \cap \alpha = S_\alpha\}$ is not stationary in κ. Let (S, C) be the $<_L$-minimal pair such that C is closed unbounded in κ and $\forall \alpha \in D \cap C \; S \cap \alpha \neq S_\alpha$.

Take a level L_θ such that $(\mathrm{ZF}^-)^{L_\theta}$ and $\kappa, D, (S_\alpha \mid \alpha < \kappa), S, C \in L_\theta$.

(1) There is $X \lhd L$ such that L_θ, κ, D, $(S_\alpha \mid \alpha < \kappa)$, S, $C \in X$, $\beta = X \cap \kappa$ is a limit ordinal, and $\beta \in D$.

Proof. We basically show that the set of $\beta < \kappa$ with the first two properties is closed unbounded in κ. Let

$$A = \{\beta < \kappa \mid \beta = L\{\beta \cup \{L_\theta, \kappa, D, (S_\alpha \mid \alpha < \kappa), S, C\}\} \cap \kappa\}.$$

We first show the unboundedness of A. Let $\beta_0 < \kappa$ and define an ω-sequence $\beta_0 < \beta_1 < \ldots < \kappa$ by recursion: if $\beta_n < \kappa$ is defined, let $\beta_{n+1} < \kappa$ be minimal such that $\beta_{n+1} > \beta_n$ and

$$L\{\beta_n \cup \{L_\theta, \kappa, D, (S_\alpha \mid \alpha < \kappa), S, C\}\} \cap \kappa < \beta_{n+1}.$$

β_{n+1} exists, since

$$\mathrm{card}(L\{\beta_n \cup \{L_\theta, \kappa, D, (S_\alpha \mid \alpha < \kappa), S, C\}\}) \leqslant \mathrm{card}(\beta_n) + \aleph_0 < \kappa$$

and since κ is regular.

Let $\beta = \bigcup_{n<\omega} \beta_n$. Since κ is regular uncountable, β is a limit ordinal and $\beta < \kappa$. By construction,

$$\begin{aligned} \beta &\subseteq L\{\beta \cup \{L_\theta, \kappa, D, (S_\alpha \mid \alpha < \kappa), S, C\}\} \cap \kappa \\ &= \bigcup_{n<\omega} (L\{\beta_n \cup \{L_\theta, \kappa, D, (S_\alpha \mid \alpha < \kappa), S, C\}\} \cap \kappa) \\ &\subseteq \bigcup_{n<\omega} \beta_{n+1} \\ &= \beta, \end{aligned}$$

hence $\beta \in A$.

A similar argument shows that A is closed in κ. Since D is stationary in κ take $\beta \in D \cap A$. Then

$$X = L\{\beta \cup \{L_\theta, \kappa, D, (S_\alpha \mid \alpha < \kappa), S, C\}\}$$

has the required properties. *qed*(1)

By the condensation theorem let

$$\pi: (X, \in, <_L, I, N, S) \cong (L_\delta, \in, <_L, I, N, S)$$

for some $\delta \in \mathrm{Ord}$. The proof then follows the previous proof of $\diamondsuit_\kappa$. $\square$

Theorem 14.6. *Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Assume $\diamondsuit_{\kappa^+}(\{\alpha < \kappa^+ \mid \mathrm{cof}(\alpha) = \kappa\})$. Then there exists a κ^+-Suslin tree.*

Proof. Let $(S_\alpha \mid \alpha < \kappa^+)$ be a $\diamondsuit_{\kappa^+}(\{\alpha < \kappa^+ \mid \mathrm{cof}(\alpha) = \kappa\})$-sequence. We construct a tree $T = (T, <_T)$ of the form $T = \bigcup_{\alpha < \kappa^+} T_{\{\alpha\}}$ such that every level T_α has cardinality $\leqslant \kappa$. We can arrange that

$$T_{\{0\}} = \{0\} \text{ and } \forall \alpha \in [1, \kappa^+)\, T_{\{\alpha\}} = \kappa \cdot (\alpha + 1) \setminus \kappa \cdot \alpha.$$

By recursion on $\alpha < \kappa^+$ we shall determine the $<_T$-predecessors of $x \in T_{\{\alpha\}}$. We shall also ensure the following two recursive conditions which guarantee that the tree can always be continued:
(1) for all $\xi < \zeta \leqslant \alpha$ and $s \in T_{\{\xi\}}$ there exists $t \in T_{\{\zeta\}}$ such that $s <_T t$;
(2) if $\alpha' < \alpha$ is a limit ordinal with $\mathrm{cof}(\alpha') < \kappa$ and B is a branch through $T_{\alpha'}$ with $\mathrm{otp}(B) = \alpha'$ then there is $t \in T_{\{\alpha'\}}$ such that $\forall s \in B\, s <_T t$.

For $\alpha = 0$ there is nothing to determine.

For $\alpha = 1$, let every element of $T_{\{1\}}$ be a $<_T$-successor of $0 \in T_{\{0\}}$.

Let $\alpha = \beta + 1 > 1$ and let $<_T \restriction T_\alpha$ be determined so that (1), (2) are satisfied. We let every $s \in T_{\{\beta\}}$ have two immediate successors in $T_{\{\alpha\}}$: if $s = \kappa \cdot \beta + \mu + m \in T_{\{\beta\}}$ and $t = \omega \cdot \alpha + \nu + n \in T_{\{\alpha\}}$ with limit ordinals $\mu, \nu < \kappa$ and $m, n < \omega$ then set

$$s <_T t \text{ iff } \mu = \nu \text{ and } (n = 2 \cdot m \text{ or } n = 2 \cdot m + 1).$$

Since $<_T$ has to be a transitive partial order, this determines all $<_T$-predecessors of $x \in T_{\{\alpha\}}$. Also (1) and (2) hold for $<_T \restriction T_{\alpha+1}$.

Let α be a limit ordinal and let $<_T \restriction T_\alpha$ be determined so that (1) is satisfied.
(2) For every $s_0 \in T_\alpha$ there is a branch B of the tree $T_\alpha = (T_\alpha, <_T \restriction T_\alpha)$ such that $s_0 \in B$ and $\mathrm{otp}(B) = \alpha$.
Proof. Let $\gamma = \mathrm{cof}(\alpha)$. Take a γ-sequence

$$\mathrm{ht}_T(s) = \alpha_0 < \alpha_1 < \ldots < \alpha_i < \ldots < \alpha, i < \gamma$$

which is cofinal in α and continuous, i.e., if $i < \gamma$ is a limit ordinal then

$$\alpha_i = \bigcup_{j < i} \alpha_j.$$

Recursively choose a γ-sequence

$$s_0 <_T s_1 <_T \ldots <_T s_i <_T \ldots, i < \gamma$$

such that $\forall i < \gamma\, \mathrm{ht}_T(s_i) = \alpha_i$. The recursive construction is possible at successor ordinals $i<\gamma$ by (1). If $i<\gamma$ is a limit ordinal then

$$\mathrm{cof}(\alpha_i) \leqslant i < \gamma = \mathrm{cof}(\alpha) \leqslant \kappa .$$

Let $B_i = \{t \in T_{\alpha_i} \mid \exists j < i\, t <_T s_j\}$ be the branch through T_{α_i} determined so far. Then $s_i \in T_{\{\alpha_i\}}$ can be found by property (2). Then

$$B = \{t \in T_\alpha \mid \exists i < \gamma\, t <_T s_i\}$$

satisfies the claim. *qed*(2)
Case 1: $\mathrm{cof}(\alpha) < \kappa$. Then
(3) $\mathrm{card}(\{B \mid B$ is a branch through T_α of ordertype $\alpha\}) = \kappa$.
Proof. Let $\gamma = \mathrm{cof}(\alpha)$. Take a γ-sequence

$$\mathrm{ht}_T(s) = \alpha_0 < \alpha_1 < \ldots < \alpha_i < \ldots < \alpha, \, i < \gamma$$

which is cofinal in α. A branch B through T_α of ordertype α is determined by the set $\{B \cap T_{\alpha_i} \mid i < \gamma\}$. The letter is basically a function from γ into κ. Hence

$$\kappa \leqslant \mathrm{card}(\{B \mid B \text{ is a branch through } T_\alpha \text{ of ordertype } \alpha\} \leqslant \mathrm{card}({}^\gamma\kappa) \leqslant \kappa^{<\kappa} \leqslant \kappa .$$

qed(3)

Let $(B_i \mid i < \kappa)$ be an injective enumeration of all branches through T_α of ordertype α. For $x = \kappa \cdot \alpha + i \in T_{\{\alpha\}}$, $i < \kappa$ and $s \in T_\alpha$ define

$$s <_T x \text{ iff } s \in B_i .$$

Obviously properties (1) and (2) hold for α.

Case 2: $\mathrm{cof}(\alpha) = \kappa$.

Define a set $S'_\alpha \subseteq T_\alpha$ as follows: if S_α is a maximal antichain in the tree $T_\alpha = (T_\alpha, <_T \restriction T_\alpha)$ then set

$$S'_\alpha = \{r \in T_\alpha \mid \exists s \in S_\alpha\, s \leqslant_T r\};$$

otherwise set $S'_\alpha = T_\alpha$. Obviously $\mathrm{card}(S'_\alpha) = \kappa$. Let $S'_\alpha = \{s_i \mid i < \kappa\}$ be an enumeration of S'_α . For each $i < \kappa$ use (2) to choose a branch B_i of T_α with $s_i \in B_i$ and $\mathrm{otp}(B_i) = \alpha$. For $x = \kappa \cdot \alpha + i \in T_{\{\alpha\}}$ and $s \in T_\alpha$ define

$$s <_T x \text{ iff } s \in B_i .$$

(3) Property (1) holds for $T_{\alpha+1}$.
Proof. Let $s \in T_\alpha$. It suffices to find $t \in T_{\{\alpha\}}$ such that $s <_T t$.
Case 1: $S'_\alpha = T_\alpha$. Then $s = s_i$ for some $i < \kappa$, $s_i \in B_i$, and $s_i <_T \kappa \cdot \alpha + i \in T_{\{\alpha\}}$.
Case 2: $S'_\alpha = \{r \in T_\alpha \mid \exists s \in S_\alpha\ s \leqslant_T r\}$, where S_α is a maximal antichain in $T_\alpha = (T_\alpha, <_T \restriction T_\alpha)$. By the maximality of S_α there is $s' \in S_\alpha$ which is comparable with s:

$$s \leqslant_T s' \text{ or } s' \leqslant_T s .$$

Case 2.1: $s \leqslant_T s'$. Then $s' \in S'_\alpha$, say $s' = s_i$, $s \in B_i$, and $s <_T \kappa \cdot \alpha + i \in T_{\{\alpha\}}$.
Case 2.2: $s' \leqslant_T s$. Then $s \in S'_\alpha$, say $s = s_i$, $s \in B_i$, and $s <_T \kappa \cdot \alpha + i \in T_{\{\alpha\}}$. *qed*(3)

This concludes the recursive definition of the tree $T = (T, <_T)$. It is straightforward to check, that the predetermined sets $T_{\{\alpha\}}$ are indeed the α-th levels of the tree. By the construction at successors, the tree is branching. By the previous theorem it suffices to show that every antichain in T has cardinality $\leqslant \kappa$.

Let $A \subseteq T$ be an antichain in T. Using the lemma of Zorn we may assume that A is maximal with respect to $\subseteq$. As before one can show

(4) The set $C = \{\alpha < \kappa^+ \mid A \cap \alpha \text{ is a maximal antichain in } T_\alpha\}$ is closed unbounded in κ^+.

By the $\diamondsuit_{\kappa^+}$-property, $\{\alpha < \kappa^+ \mid A \cap \alpha = S_\alpha\}$ is stationary in κ^+. Take $\alpha \in C$ such that $A \cap \alpha = S_\alpha$. Then $A \cap \alpha = S_\alpha$ is a maximal antichain in T_α.

(5) $A = A \cap \alpha$.

Proof. Let $t \in A$. We show that every $r \in T$ is comparable with some $s \in A \cap \alpha$. Since $A \cap \alpha$ is a maximal antichain in T_α this is clear for $r \in T_\alpha$ and we may assume that $r \in T \setminus T_\alpha$. Then $\mathrm{ht}_T(r) \geqslant \alpha$ and we can take the unique $\bar{r} \in T_{\{\alpha\}}$ such that $\bar{r} \leqslant_T r$. By construction of $T_{\{\alpha\}}$ there is some $s \in S_\alpha = A \cap \alpha$ such that

$$s <_T \bar{r} \leqslant_T r$$

qed(5)

By (5), $A = A \cap \alpha$ has cardinality $\leqslant \kappa$. Since T is a branching tree all whose antichains have cardinality $\leqslant \kappa$, T is a κ^+-Suslin tree. □

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