Axiomatic Set Theory II

1 The Generalized Continuum Hypothesis (GCH)

Felix Hausdoff made several seminal contributions to cardinal arithmetic. He generalized Cantor's continuum hypothesis from $\aleph_0$ to all cardinals and proved the Hausdorff recursion formula:

Theorem 1. *For all cardinals* $\kappa \in \mathrm{Card}$ *and* $\lambda \in \mathrm{Cd} \setminus \{0\}$ *holds* $(\kappa^+)^\lambda = \kappa^\lambda \cdot \kappa^+$.

Proof. Case 1: $\lambda \geqslant \kappa^+$. Then $(\kappa^+)^\lambda = \kappa^\lambda = \kappa^\lambda \cdot \kappa^+$.

Case 2: $\lambda < \kappa^+$. Then ${}^\lambda(\kappa^+) = \bigcup_{\alpha < \kappa^+} {}^\lambda\alpha$ by the regularity of κ^+. Hence

$$(\kappa^+)^\lambda = \overline{\overline{{}^\lambda(\kappa^+)}} = \overline{\overline{\bigcup_{\alpha < \kappa^+} {}^\lambda\alpha}} \leqslant \sum_{\alpha < \kappa^+} \kappa^\lambda = \kappa^\lambda \cdot \kappa^+.$$

The converse inequality is obvious. □

Definition 2. *The* ***generalized continuum hypothesis*** (GCH) *is the statement: for all* α *holds* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

The GCH determines cardinal exponentiation completely:

Theorem 3. *For* $\kappa \in \mathrm{Card}$ *and* $\lambda < \mathrm{cof}(\kappa)$ *holds* $\kappa^\lambda = \kappa$.

For $\kappa \in \mathrm{Card}$ *and* $\mathrm{cof}(\kappa) \leqslant \lambda \leqslant \kappa$ *holds* $\kappa^\lambda = \kappa^+$.

For $\kappa \in \mathrm{Card}$ *and* $\kappa \leqslant \lambda$ *holds* $\kappa^\lambda = \lambda^+$.

Proof. Exercise. □

This theorem motivates the special attention towards the *continuum function* $\kappa \mapsto 2^\kappa$. We shall see by the methods of forcing that the continuum function is hardly controlled by the axiom of ZFC. Odd configurations like $2^{\aleph_0} = \aleph_{1511}$, $2^{\aleph_1} = \aleph_{\aleph_3}$ and $2^{\aleph_2} = \aleph_{\aleph_{2001}}$can be realized in generic extensions. Somewhat more control exists for singular cardinals κ: if $\mathrm{cof}(\kappa) > \omega$ and $\forall \lambda < \kappa\ 2^\lambda = \lambda^+$ then $2^\kappa = \kappa^+$. Indeed, it suffices that the continuum hypothesis is true for a "large" set of $\lambda < \kappa$. In the next chapter we give an important characterization of "largeness" within set theory.

2 Filters

Definition 4. *For* $X \in V$ *and* $F \subseteq \mathfrak{P}(X)$ *we say that* F *is a* ***filter on*** X *if*

a) $X \in F$, $\emptyset \notin F$,

b) *for* $Y \in F$ *and* $Y \subseteq Y' \subseteq X$ *holds* $Y' \in F$,

c) *for* $Y, Z \in F$ *holds* $Y \cap Z \in F$.

Definition 5. *A filter* F *on* X *is an* ***ultrafilter*** *on* X *if for* $Y \subseteq X$ *holds* $Y \in F$ *or* $X \setminus Y \in F$.

Using bijections we are mainly interested in filters on cardinals. On uncountable cardinals one can define filters which have no analogue in the countable realm:

Definition 6. *Let* γ *be a limit ordinal and* $C \subseteq \gamma$. *We have defined the notion of unboundedness before.*

The set C *is called* ***closed*** *in* γ *if for all* $\beta < \gamma$, $C \cap \beta$ *unbounded in* β *holds* $\beta \in C$. *The set* C *is called* ***closed unbounded*** *or* ***club*** *in* γ *if it is both closed and unbounded in* γ.

Define $\mathcal{C}_\gamma = \{D \subseteq \gamma | \exists C \subseteq D\, C \text{ is club in } \gamma\}$.

Theorem 7. *For γ a limit ordinal, $\mathrm{cof}(\gamma) > \omega$ holds that $\mathcal{C}_\gamma$ is a filter on γ.*

Proof. We only check c) of Definition 4. Consider $Y, Z \in \mathcal{C}_\gamma$. Choose $C \subseteq Y$, $D \subseteq Z$, C, D club in γ. It suffices to show that $C \cap D$ is club in γ.
"closed": straightforward.
"unbounded": Consider $\alpha < \gamma$. Define sequences $(\gamma_i | i < \omega)$ and $(\delta_i | i < \omega)$ by simultaneous recursion: γ_i is the minimal element of C such that $\gamma_i > \alpha$, for all $j < i$ holds $\gamma_i > \gamma_j$ and $\gamma_i > \delta_j$; δ_i is the minimal element of D such that $\delta_i > \gamma_i$, for all $j < i$ holds $\delta_i > \delta_j$ and $\delta_i > \gamma_j$. Set $\eta = \bigcup_{i<\omega} \gamma_i = \bigcup_{i<\omega} \delta_i$. $\eta < \gamma$ since $\mathrm{cof}(\gamma) > \omega$. By construction, $C \cap \eta$ and $D \cap \eta$ are unbounded in η. $\eta \in C$ and $\eta \in D$ since C und D are closed in γ. $\eta > \alpha$ and $\eta \in C \cap D$. □

The filter $\mathcal{C}_\gamma$ has some strong combinatorial properties:

Definition 8. *Let κ be a cardinal and F a filter on κ. Define*

*a) F is $<\kappa$-**closed** if for all $\eta < \kappa$, $\{X_\xi | \xi < \eta\} \subseteq F$ holds $\bigcap_{\xi<\eta} X_\xi \in F$.*

*b) For a sequence $(A_\xi | \xi < \kappa)$ define the **diagonal intersection** $\Delta_{\xi<\kappa} A_\xi = \{\alpha < \kappa | \forall \xi < \alpha\ \alpha \in A_\xi\}$.*

*c) F is **diagonally closed** or **normal** if for all $\{X_\xi | \xi < \kappa\} \subseteq F$ holds $\Delta_{\xi<\kappa} A_\xi \in F$.*

Theorem 9. *Let κ be a regular uncountable cardinal. Then*

a) $\mathcal{C}_\kappa$ is $<\kappa$-closed.

b) $\mathcal{C}_\kappa$ is normal.

Proof. a) Consider $\eta < \kappa$, $\{X_\xi | \xi < \eta\} \subseteq \mathcal{C}_\kappa$. Choose $(C_\xi | \xi < \eta)$ such that for all $\xi < \eta$ holds $C_\xi \subseteq X_\xi$ and C_ξ is club in κ. It suffices to see that $\bigcap_{\xi<\eta} C_\xi$ is club in κ.
"closed": straightforward as in Theorem 7.
"unbounded": Consider $\alpha < \kappa$. For $k < \eta$ define sequences $(\gamma_{i,k} | i < \omega)$ by simultaneous recursion: $\gamma_{i,k}$ is the minimal element of C_k such that $\gamma_{i,k} > \alpha$, for all $j < i$ and $l < \eta$ holds $\gamma_{i,k} > \gamma_{j,l}$ and for all $l < k$ holds $\gamma_{i,k} > \gamma_{i,l}$. The construction is possible by the regularity of κ. Set $\delta = \bigcup_{i<\omega} \gamma_{i,0}$. For $k < \eta$ holds $\delta = \bigcup_{i<\omega} \gamma_{i,k}$. $\delta < \kappa$ since $\mathrm{cof}(\kappa) > \omega$. For $k < \eta$ we have that $C_k \cap \delta$ is unbounded in η. $\delta \in C_k$ since C_k is closed in κ. $\delta > \alpha$ and $\delta \in \bigcap_{\xi<\eta} C_\xi$.

b) Consider $\{X_\xi | \xi < \kappa\} \subseteq \mathcal{C}_\kappa$. Choose $(C_\xi | \xi < \kappa)$ such that for all $\xi < \kappa$ holds $C_\xi \subseteq X_\xi$ and C_ξ is club in κ. It suffices to see that $\Delta_{\xi<\kappa} C_\xi$ is club in κ.
"closed": straightforward.
"unbounded": Consider $\alpha < \kappa$. Define a sequence $(\delta_i | i < \omega)$ by recursion: $\delta_0 = \alpha$; δ_{i+1} is the minimal element of $\bigcap_{\xi<\delta_i} C_\xi$ (see a)). Then $\eta = \bigcup_{i<\omega} \delta_i > \alpha$ and $\eta \in \Delta_{\xi<\kappa} C_\xi$. □

3 Stationary sets

To a filter is associated a dual ideal and the complement of that ideal, i.e., those object which are positive with respect to the ideal.

Definition 10. *Let κ be an uncountable cardinal. A set $S \subseteq \kappa$ is **stationary** in κ if for all $C \in \mathcal{C}_\kappa$ holds $S \cap C \neq \emptyset$.*

Theorem 11. *Let κ be regular and uncountable.*

i. If C is club in κ then C is stationary in κ.

ii. Define $\mathrm{Cof}_\gamma = \{\alpha < \kappa | \mathrm{cof}(\alpha) = \gamma\}$. Then for regular $\gamma < \kappa$ holds $\mathrm{Cof}_\gamma \cap \kappa$ is stationary in κ.

Proof. Exercise. □

Stationary sets satisfy a remarkable canonisation property for regressive functions:

Definition 12. *A function f is* ***regressive*** *if for all $x \in \mathrm{dom}(f)$, $x \neq \varnothing$ holds $f(x) \in x$.*

Theorem 13. *(*Fodor*) Let κ be an uncountable regular cardinal, S stationary in κ and $f: S \to \kappa$ regressive. Then there is $T \subseteq S$ stationary in κ such that f is constant on T, i.e., there is ζ such that for all $\xi \in T$ holds $f(\xi) = \zeta$.*

Proof. Assume for a contradiction that this fails. Then for all $\zeta < \kappa$ holds that $f^{-1}[\zeta]$ is not stationary in κ. Choose $(C_\zeta | \zeta < \kappa)$ such that for $\zeta < \kappa$ holds C_ζ is club in κ and $C_\zeta \cap f^{-1}[\zeta] = \varnothing$. By Theorem 9, $\Delta_{\xi<\kappa} C_\xi$ is club in κ. By the stationarity of S, choose $\zeta_0 \in (\Delta_{\xi<\kappa} C_\xi) \cap S$. Let $\xi_0 = f(\zeta_0) < \zeta_0$. By definition of the diagonal intersection $\zeta_0 \in C_{\xi_0}$. $\zeta_0 \in f^{-1}[\xi_0]$ and so $C_\zeta \cap f^{-1}[\zeta] \neq \varnothing$. Contradiction. □

We give a combinatorial application which will be used in later forcing constructions.

Definition 14. *A family $\mathcal{T}$ of sets is called a* ***Δ-system*** *if there is a set a such that for $x, y \in \mathcal{T}$, $x \neq y$ holds $x \cap y = a$. The set a is called the* ***root*** *of the Δ-system.*

Under certain conditions, families of sets can be thinned out to large Δ-systems:

Theorem 15. *Let κ, λ be regular cardinals, $\kappa > \lambda$ regular, such that $\forall \mu < \kappa \forall \nu < \lambda\ \mu^\nu < \kappa$. Let $\mathcal{S}$ be a family of sets such that $\mathrm{card}(\mathcal{S}) = \kappa$ and for all $s \in \mathcal{S}$ holds $\mathrm{card}(s) < \lambda$. Then there is a Δ-system $\mathcal{T} \subseteq \mathcal{S}$ such that $\mathrm{card}(\mathcal{T}) = \kappa$.*

Proof. We may assume without loss of generality that the given family $\mathcal{S}$ is living on κ, i.e., $\forall x \in \mathcal{R}\, x \subseteq \kappa$. Let $(s_\alpha | \alpha < \kappa)$ be an enumeration of $\mathcal{S}$ without repetition. The set $S = \{\alpha < \kappa | \mathrm{cof}(\alpha) = \lambda\}$ is stationary in κ. The function $f: S \to \kappa$, $\alpha \mapsto \sup\, (s_\alpha \cap \alpha)$ is regressive. By Fodor's theorem choose a stationary $T_0 \subseteq S$ and an ordinal ζ such that for all $\alpha \in T_0$ holds $\sup\, (s_\alpha \cap \alpha) = \zeta$. For $\alpha \in T_0$ holds $s_\alpha \cap \alpha \in [\zeta]^{<\lambda}$. $\mathrm{card}([\zeta]^{<\lambda}) \leqslant \sum_{\gamma<\lambda} \mathrm{card}(\zeta)^{\mathrm{card}(\gamma)} < \kappa$; observe that by the cardinal arithmetic assumption $\mathrm{card}(\zeta)^{\mathrm{card}(\gamma)} < \kappa$; then the sum is $< \kappa$ since $\lambda < \mathrm{cof}(\kappa) = \kappa$. Again applying Fodor's theorem we can choose a stationary $T_1 \subseteq T_0$ such that the function $\alpha \mapsto s_\alpha \cap \alpha$ is constant on T_1. Choose $a \in [\zeta]^{<\lambda}$ such that for all $\alpha \in T_1$ holds $s_\alpha \cap \alpha = a$. Finally by the regularity of κ we can choose a stationary $T \subseteq T_1$ such that for all $\alpha, \beta \in T$, $\alpha < \beta$ holds $s_\alpha \subseteq \beta$. For $\alpha, \beta \in T$, $\alpha < \beta$ holds $s_\alpha \cap s_\beta = s_\alpha \cap s_\beta \cap \beta = s_\alpha \cap a = a$. Thus $\mathcal{T} = \{s_\alpha | \alpha \in T\}$ is a Δ-system with the desired properties. □

Remark 16. 1. The cardinal theoretic assumption is satisfied for $\lambda = \aleph_0$ and $\kappa = \aleph_1$.
2. If we assume GCH, the cardinal arithmetic property is satisfied if κ is a regular limit cardinal and $\kappa > \lambda$: for $\mu < \kappa$ and $v < \lambda$ holds $\mu^\nu < \max\, (\mu, \nu)^{\max(\mu,\nu)} = \max\, (\mu, \nu)^+ < \kappa$.
3. Also if we assume GCH, the property is satisfied if κ is a successor cardinal, $\kappa = \rho^+$ and $\mathrm{cof}(\rho) \geqslant \lambda$: for $\mu < \kappa$ and $v < \lambda$ holds $\mu^\nu < \rho^\nu = \rho < \kappa$.

We can now apply the Δ-system property to a specific partial order, i.e., a set of forcing conditions:

Definition 17. *Let $\mathrm{Fn}(X, Y, \lambda)$ denote the set of partial functions $\{p | p: \mathrm{dom}(p) \to Y, \mathrm{dom}(p) \subseteq X, \mathrm{card}(p) < \lambda\}$, partially ordered by reverse inclusion ($\supseteq$).*

Theorem 18. *Let κ, λ be regular cardinals, $\kappa > \lambda$ regular, such that $\forall \mu < \kappa \forall \nu < \lambda\ \mu^\nu < \kappa$. Let $\mathrm{card}(Y) < \kappa$. Then $\mathrm{Fn}(X, Y, \lambda)$ has the κ chain condition, i.e., every antichain in $\mathrm{Fn}(X, Y, \lambda)$ has cardinality $< \kappa$.*

Proof. Assume instead, that A is an antichain in $\mathrm{Fn}(X, Y, \lambda)$ of size κ. Let $\mathcal{S} = \{\mathrm{dom}(p) | p \in A\}$ be the associated family of domains. In case $\mathrm{card}(\mathcal{S}) < \kappa$ we can choose a subset $B \subseteq A$, $\mathrm{card}(B) = \kappa$ and an $a \subseteq X$ such that for all $p \in B$ holds $\mathrm{dom}(p) = a$. In case $\mathrm{card}(\mathcal{S}) = \kappa$ we can use the Δ-system theorem to choose a subset $B \subseteq A$, $\mathrm{card}(B) = \kappa$ and an $a \subseteq X$ such that $\{\mathrm{dom}(p) | p \in B\}$ forms a Δ-system with root a. So in either case we may assume that there is an antichain A of size κ and a set a such that $\{\mathrm{dom}(p) | p \in A\}$ is a Δ-system with root a. For $p, q \in A$, $p \neq q$ holds p is incompatible with q and hence $p \upharpoonright a \neq q \upharpoonright a$. $p \upharpoonright a: a \to Y$ and so $\mathrm{card}({}^aY) \geqslant \kappa$. But by the cardinal arithmetic assumption, $\mathrm{card}({}^aY) \leqslant \mathrm{card}(Y)^{\mathrm{card}(a)} < \kappa$, contradiction. □

An immediate corollary is the following theorem which was proved differently in the forcing construction for $\neg$CH.

Theorem 19. *Let* $\mathrm{card}(Y) < \aleph_1$. *Then* $\mathrm{Fn}(X, Y, \aleph_0)$ *has the countable chain condition, i.e., the* $\aleph_1$*-c.c.*

Theorem 20. *Assume* GCH *and let* λ *be a regular cardinal. Then* $\mathrm{Fn}(X, 2, \lambda)$ *has the* λ^+*-c.c.*

Proof. We have to check the cardinal arithmetic hypothesis of the theorem. Consider $\mu < \lambda^+$ and $\nu < \lambda$. Then $\mu^\nu \leqslant \lambda^\nu \leqslant \lambda < \lambda^+$ by previous results on the GCH. □

4 Destroying the Continuum Hypothesis at Higher Cardinals

We intend to generalise COHEN's forcing construction for $\neg$CH to cardinals $\kappa > \aleph_0$. Let us proceed similarly to the $\neg$CH-case.

Fix a *ground model* M in which GCH holds. Let κ be a regular cardinal in M and let λ be a cardinal in M such that $M \vDash \mathrm{cof}(\lambda) > \kappa$. Define a forcing partial order within M:

$$P = \mathrm{Fn}(\lambda \times \kappa, 2, \lambda)^M = \{p \in M | p\colon \mathrm{dom}(p) \to 2 \wedge \mathrm{dom}(p) \subseteq \lambda \times \kappa \wedge \mathrm{card}^M(p) < \kappa\}.$$

The partial order on P is defined by reverse inclusion: $p \leqslant q$ iff $p \supseteq q$, $1_P = \emptyset$.

Let G be a P-generic filter over M with correponding generic extension $M[G]$. We know that $M[G] \vDash$ ZFC. We want to show that $M[G] \vDash 2^\kappa = \lambda$ and that cardinals and cofinalities are absolute between M and $M[G]$.

Let $f = \bigcup G$.

(1) $f\colon \lambda \times \kappa \to 2$.

Proof. This is proved as in the $\neg$CH case. □

For $\alpha < \lambda$ define characteristic functions $c_\alpha = (f(\alpha, i) | i < \kappa)$. These correspond to the COHEN reals generated in the $\neg$CH forcing.

(2) For $\alpha, \beta < \lambda$, $\alpha \neq \beta$ holds $c_\alpha \neq c_\beta$.

Proof. Define a dense set

$$D = \{p \in P | \exists i < \kappa\ (\alpha, i) \in \mathrm{dom}(p) \wedge (\beta, i) \in \mathrm{dom}(p) \wedge p(\alpha, i) \neq p(\beta, i)\}.$$

By absoluteness, $D \in M$. Choose $p \in G \cap D$. Choose $i < \kappa$ such that $(\alpha, i) \in \mathrm{dom}(p) \wedge (\beta, i) \in \mathrm{dom}(p) \wedge p(\alpha, i) \neq p(\beta, i)$. Then

$$c_\alpha(i) = f(\alpha, i) = p(\alpha, i) \neq p(\beta, i) = f(\beta, i) = c_\beta(i).$$

□

Thus $M[G]$ satisfies:

$$2^\kappa = \mathrm{card}({}^\kappa 2) \geqslant \mathrm{card}(\{c_\alpha | \alpha < \lambda\}) = \mathrm{card}(\lambda).$$

The intended absolutenesses between M and $M[G]$ rest on the following combinatorial properties of P. In the next chapter we shall see that the following combination of combinatorial properties of P do indeed imply the absoluteness properties.

Definition 21. *Let* κ *be a regular cardinal. A forcing partial order* $(P, \leqslant, 1_P)$ *is called* κ-***closed*** *if* $\forall \nu < \kappa \forall \vec{p} \in {}^\nu P \exists q \in P (\forall i, j < \nu (i \leqslant j \to \vec{p}(i) \geqslant \vec{p}(j)) \to \forall i < \nu\ \vec{p}(i) \geqslant q)$.

Obviously, the ground model M models that $\mathrm{Fn}(\lambda \times \kappa, 2, \lambda)^M$ is κ-closed.

Definition 22. *Let* μ *be a regular cardinal. A forcing partial order* $(P, \leqslant, 1_P)$ *has the* μ ***chain condition*** *(*μ*-c.c.) if* $\forall A (A$ *is an antichain in* $P \to \mathrm{card}(A) < \mu)$.

By the GCH assumption, the ground model M models that $\mathrm{Fn}(\lambda\times\kappa,2,\lambda)^M$ has the $(k^+)^M$-c.c.

5 Preservation Properties

Theorem 23. *Let M be a ground model, $M\vDash\kappa$ is a regular cardinal and $P=(P,\leqslant,1_P)$ is a κ-closed forcing partial order. Let G be P-generic over M. Then for $\nu<\kappa$, $h\in M[G]$, $h:\nu\to M$ holds $h\in M$.*

Proof. Consider $\nu<\kappa$, $h\in M[G]$, $h:\nu\to M$. Since $M[G]$ satisfies the replacement scheme we can choose $b\in M$ such that $h:\nu\to b$. Choose $\dot{h}\in M$, $p\in G$ such that $h=\dot{h}^G$ and $p\Vdash\dot{h}:\check{\nu}\to\check{b}$. It suffices to see that the set

$$D=\{q\,|\,\exists f\in M\, q\Vdash\dot{h}=\check{f}\}$$

is dense in P below p.

Consider $r\leqslant p$. Work in M and define a function $f:\nu\to b$ and a sequence $(r_i|i<\nu)$ of conditions in P by simultaneous recursion on $i<\nu$ such that

a) $\forall i,j<\nu\,(i\leqslant j\to r_i\geqslant r_j)$;

b) $\forall i<\nu\; r_i\Vdash\dot{h}(i)=f(i)$.

Consider $j<\nu$ and assume that $f\upharpoonright j$ and $(r_i|i<j)$ are defined according to a) and b). By the κ-closure of P choose $s\in P$ such that $\forall i<j\; r_i\geqslant s$. $s\Vdash\dot{h}:\check{\nu}\to\check{b}$. Choose $r_j\leqslant s$ and $f(j)\in b$ such that $r_j\Vdash\dot{h}(j)=f(j)$. This concludes the simultaneous recursion.

Again by the κ-closure of P choose $q\in P$ such that $\forall i<\nu\; r_i\geqslant q$. Obviously, $q\Vdash\dot{h}=\check{f}$ and so $q\in D$ as required. □

Theorem 24. *Let M be a ground model, $M\vDash$ "κ is a regular cardinal and $P=(P,\leqslant,1_P)$ is a κ-closed forcing partial order". Let G be P-generic over M. Then all cardinals and cofinalities $\leqslant\kappa$ are absolute between M and $M[G]$, i.e.,*

i. $\forall\mu\leqslant\kappa\, M\vDash\mu$ *is a cardinal iff* $M[G]\vDash\mu$ *is a cardinal.*

ii. $\forall\nu<\kappa\forall\zeta\;\mathrm{cof}^M(\zeta)=\nu$ *iff* $\mathrm{cof}^{M[G]}(\zeta)=\nu$.

Proof. By the previous theorem, M and $M[G]$ agree with respect to their ν-sequences of ordinals. The notions in i. and ii. only depend on the class of ν-sequences of ordinals. □

Theorem 25. *Let M be a ground model, $M\vDash$ "κ is a regular cardinal and $P=(P,\leqslant,1_P)$ has the κ-chain condition". Let G be P-generic over M. Then for every $\nu\in\mathrm{On}$, $h\in M[G]$, $h:\nu\to M$ there is $f\in M$, $f:\nu\to M$ such that $\forall i<\nu\;(h(i)\in f(i)\wedge\mathrm{card}^M(f(i))<\kappa)$.*

Proof. Consider $\nu\in\mathrm{On}$, $h\in M[G]$, $h:\nu\to M$. Since $M[G]$ satisfies the replacement scheme we can choose $b\in M$ such that $h:\nu\to b$. Choose $\dot{h}\in M$, $p\in G$ such that $h=\dot{h}^G$ and $p\Vdash\dot{h}:\check{\nu}\to\check{b}$. In M define $f:\nu\to V$, $f(i)=\{c\in b|\exists q\leqslant p\, q\Vdash\dot{h}(\check{i})=\check{c}\}$. Consider $i<\nu$. Let $c=h(i)$. Choose $r\in G$ such that $r\Vdash\check{c}=\dot{h}(\check{i})$. Choose $q\leqslant p$, $q\leqslant r$. Then $q\Vdash\dot{h}(\check{i})=\check{c}$ and so $h(i)=c\in f(i)$. For the cardinality estimate choose a sequence $(q_c|c\in f(i))$ such that for $c\in f(i)$ holds $q_c\leqslant p$ and $q_c\Vdash\dot{h}(\check{i})=\check{c}$. For $c,d\in f(i)$, $c\neq d$ holds $q_c\perp q_d$ since if there were some $r\leqslant q_c,q_d$ then $r\Vdash\dot{h}(\check{i})=\check{c}$, $r\Vdash\dot{h}(\check{i})=\check{d}$, $r\Vdash\check{c}=\check{d}$, contradiction. Hence $\{q_c|c\in f(i)\}$ is an antichain. By the κ-chain condition, $\mathrm{card}(\{q_c|c\in f(i)\})<\kappa$. Thus $\mathrm{card}(f(i))<\kappa$ as required. □

The previous theorem is a kind of covering theorem between M and $M[G]$: the function $h\in M[G]$ can be "covered" by the function $f\in M$.

Theorem 26. *Let M be a ground model, $M\vDash$ "κ is a regular cardinal and $P=(P,\leqslant,1_P)$ has the κ-chain condition". Let G be P-generic over M. Then all cardinals and cofinalities $\geqslant\kappa$ are absolute between M and $M[G]$, i.e.,*

i. $\forall\mu\geqslant\kappa\, M\vDash\mu$ *is a cardinal iff* $M[G]\vDash\mu$ *is a cardinal.*

ii. $\forall\mu\geqslant\kappa\forall\zeta\,(\mathrm{cof}^M(\zeta)=\mu\longrightarrow\mathrm{cof}^M(\zeta)=\mathrm{cof}^{M[G]}(\zeta)$.

Proof. i. Consider $\mu\geqslant\kappa$. The implication from right to left is obvious. Now assume that $M[G]\vDash\mu$ is not a cardinal. Choose $\nu<\mu$, $h\in M[G]$, $h:\nu\to\mu$ surjective. By the previous theorem choose $f\in M$, $f:\nu\to M$ such that $\forall i<\nu\ (h(i)\in f(i)\wedge\mathrm{card}^M(f(i))<\kappa)$. Then $\mu=\mathrm{range}(h)\subseteq\bigcup_{i<\nu}f(i)$. We can give the following cardinality estimates in M:
Case 1: $\mu=\kappa$. Then $\mathrm{card}^M(\mu)\leqslant\sum_{i<\nu}\mathrm{card}^M(f(i))<\kappa$, since κ is regular in M. But then $\mu=\kappa$ is not a cardinal in M, contradiction.
Case 2: $\mu>\kappa$. Then $\mathrm{card}^M(\mu)\leqslant\sum_{i<\nu}\mathrm{card}^M(f(i))\leqslant\mathrm{card}^M(\nu)\cdot\kappa=\max(\mathrm{card}^M(\nu),\kappa)\leqslant\max(\nu,\kappa)<\mu$. But then μ is not a cardinal in M.

ii. Consider $\mu\geqslant\kappa$ and $\zeta\geqslant\kappa$. Assume that $\mathrm{cof}^M(\zeta)=\mu$. Obviously, $\mathrm{cof}^{M[G]}(\zeta)\leqslant\mathrm{cof}^M(\zeta)$. Assume for a contradiction that $\mathrm{cof}^{M[G]}(\zeta)<\mathrm{cof}^M(\zeta)$. Choose $\nu<\mu$, $h\in M[G]$, $h:\nu\to\zeta$ cofinal. By the previous theorem choose $f\in M$, $f:\nu\to M$ such that $\forall i<\nu\ (h(i)\in f(i)\wedge\mathrm{card}^M(f(i))<\kappa)$. We may assume that $\forall i<\nu\, f(i)\subseteq\zeta$. Then $\mathrm{range}(h)\subseteq\bigcup_{i<\nu}f(i)$, hence $\bigcup_{i<\nu}f(i)$ is cofinal in ζ. We can make the following cofinality estimates in M:
Case 1: $\zeta=\kappa$. Then $\mathrm{cof}^M(\zeta)\leqslant\sum_{i<\nu}\mathrm{card}^M(f(i))<\kappa$, since κ is regular in M. But then $\zeta=\kappa$ is not regular in M, contradiction.
Case 2: $\zeta>\kappa$. Then $\mathrm{cof}^M(\zeta)\leqslant\sum_{i<\nu}\mathrm{card}^M(f(i))\leqslant\mathrm{card}^M(\nu)\cdot\kappa=\max\,(\mathrm{card}^M(\nu),\kappa)\leqslant\max\,(\nu,\kappa)<\mu$. But this contradicts the assumption $\mathrm{cof}^M(\zeta)=\mu$. □

We now give an upper bound for sizes of powersets in forcing extensions.

Theorem 27. *Let M be a ground model, $M\vDash$ "κ is a regular cardinal and $P=(P,\leqslant,1_P)$ has the κ-chain condition". Let G be P-generic over M. Then for every cardinal μ holds* $\mathrm{card}(\wp(\mu))^{M[G]}\leqslant((\mathrm{card}(P)^{<\kappa})^\mu)^M$.

Proof. We define the following set of *canonical names* for subsets of μ in the ground model M:

$$Z=\{z\subseteq\{\check{\alpha}\,|\alpha<\mu\}\times P|\forall\alpha<\mu\,\{p|(\check{\alpha},p)\in z\}\text{ is an antichain in }P\}.$$

We claim that $\wp(\mu)^{M[G]}=\{z^G|z\in Z\}$. The inclusion $\supseteq$ is obvious. For the converse let $x\in\wp(\mu)\cap M[G]$. Choose a name $\dot{x}\in M$ such that $x=\dot{x}^G$. For $\alpha<\mu$ let A_α be a maximal antichain in the set $\{p\in P|p\Vdash\check{\alpha}\in\dot{x}\vee p\Vdash\check{\alpha}\notin\dot{x}\}$. Let $B_\alpha=\{p\in A_\alpha|p\Vdash\check{\alpha}\in\dot{x}\}$ and set

$$z=\{(\check{\alpha},p)|\alpha<\mu,p\in B_\alpha\}.$$

We claim that $x=z^G$. $z^G=\{\alpha<\mu|\exists p\in G\,(\check{\alpha},p)\in z\}=\{\alpha<\mu|\exists p\in G\,p\in B_\alpha\}=\{\alpha<\mu|\exists p\in G\ p\Vdash\check{\alpha}\in\dot{x}\}=\{\alpha<\mu|\alpha\in\dot{x}^G\}=x$. To justify the third equality, assume that $p\in G$ and $p\Vdash\check{\alpha}\in\dot{x}$. By the genericity of G choose $q\in G\cap A_\alpha$. Since q is compatible with p, we cannot have $q\Vdash\check{\alpha}\notin\dot{x}$. Hence $q\Vdash\check{\alpha}\in\dot{x}$ and $q\in B_\alpha$.

Then an upper bound for the cardinality of $\wp(\mu)\cap M[G]$ in $M[G]$ is given by the M-cardinality of Z. Every element of Z can be viewed as a function from μ taking antichains in P as its values. By the κ-chain condition every antichain in P is an element of $[P]^{<\kappa}$. Hence

$$M\vDash\mathrm{card}(Z)\leqslant(\mathrm{card}(P)^{<\kappa})^\mu.$$

□

Let us now apply our calculations to the forcing construction considered above. We fixed a ground model M of GCH, and for M-cardinals κ, λ, $\mathrm{cof}^M(\lambda)>\kappa$ we defined: $P=\mathrm{Fn}(\lambda\times\kappa,2,\lambda)^M=\{p\in M|p\colon\mathrm{dom}(p)\to2\wedge\mathrm{dom}(p)\subseteq\lambda\times\kappa\wedge\mathrm{card}^M(p)<\kappa\}$. This forcing is κ-closed and has the κ^+-chain condition. Thus all cardinals are absolute between M and $M[G]$. We already checked that $M[G]\vDash2^\kappa\geqslant\lambda$. The converse inequality follows from the previous theorem: work in M: by the GCH, $\mathrm{card}(P)=(\lambda\cdot\kappa)^{<\kappa}=\lambda^{<\kappa}=\lambda$ and $(\mathrm{card}(P)^{<\kappa^+})^\kappa=(\lambda^{<\kappa^+})^\kappa=\lambda^\kappa=\lambda$. Hence $M[G]\vDash2^\kappa\leqslant\mathrm{card}(\wp(\kappa))\leqslant\lambda$. By choosing appropriate scenarios one can now deduce a host of relative consistency results of which we list some typical cases.

Theorem 28. *Assume the theory* ZFC + GCH *is consistent. Then the following are consistent:*

i. $\mathrm{ZFC}+2^{\aleph_0}=\aleph_2$;

ii. $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_3$;

iii. $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_{\omega+1}$;

iv. $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_{\omega_1}$;

v. $\mathrm{ZFC} + 2^{\aleph_1} = \aleph_3$;

vi. $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_3$.

Note that we have some constraints by earlier results: e.g. we cannot have $2^{\aleph_0} = \aleph_\omega$. One can elaborate the above techniques by determining the value of 2^μ in $M[G]$ for cardinals $\mu \neq \kappa$. In subsequent chapters we shall now approach the problem of changing the continuum function simultaneously at several places.

6 Silver's Theorem

We shall see by forcing constructions that the value of 2^κ for regular cardinals κ is hardly determined by the value of 2^λ at other cardinals. The situation at singular cardinals is more subtle, and the first result in this area is the following theorem by J. SILVER:

Theorem 29. *(Silver) Let* $\omega < \lambda = \mathrm{cof}(\kappa) < \kappa \in \mathrm{Card}$. *Let* $2^\mu = \mu^+$ *for all* $\omega \leq \mu \in \kappa \cap \mathrm{Card}$. *Then* $2^\kappa = \kappa^+$.

Proof. We define $f, g \colon \lambda \to V$ to be **almost disjoint** iff there exists $\alpha < \lambda$ such that $f(\beta) \neq g(\beta)$ for all $\alpha < \beta < \lambda$. $\mathcal{F} \subseteq \{f \mid f \colon \lambda \to v\}$ is called **almost disjoint** if all $f \neq g \in \mathcal{F}$ are almost disjoint.
Let $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ be normal and cofinal in κ such that $\omega \leq \kappa_\alpha \in \kappa \cap \mathrm{Card}$.
(1) Let $\mathcal{F} \subseteq \prod_{\alpha<\lambda} A_\alpha$, $\mathrm{card}(A_\alpha) \leq \kappa_\alpha$, be almost disjoint. Then $\mathrm{card}(\mathcal{F}) \leq \kappa$.
Proof. Assume w.l.o.g. $A_\alpha \subseteq \kappa_\alpha$. For $f \in \mathcal{F}$ define

$$h(\alpha) := \text{ the least } \beta \text{ such that } f(\alpha) \in \kappa_\beta.$$

Then $h \upharpoonright \mathrm{Lim}$ is regressive. So, by Fodor's lemma, there is a stationary $S_f \subseteq \mathrm{Lim} \cap \lambda$ such that h is constant on S_f. Hence f is on S_f bounded in κ. If $f \upharpoonright S_f = g \upharpoonright S_g$, then $f = g$, since $\mathcal{F}$ is almost disjoint. So $f \mapsto f \upharpoonright S_f$ is one-to-one. For a fixed S, the set of functions on S that are bounded in κ has the cardinality $\sup \{\kappa_\alpha^\lambda \mid \alpha \in \lambda\} = \kappa$, since GCH holds below κ. But again by GCH below κ, we have $\mathrm{card}(\wp(\lambda)) = 2^\lambda < \kappa$. Hence $\mathrm{card}(\mathcal{F}) \leq \kappa$. qed(1)
(2) Let $\mathcal{F} \subseteq \prod_{\alpha<\lambda} A_\alpha$ be almost disjoint, $\mathrm{card}(A_\alpha) \leq \kappa_\alpha^+$. Then $\mathrm{card}(\mathcal{F}) \leq \kappa^+$.
Proof. Assume w.l.o.g. that $A_\alpha \subseteq \kappa_\alpha^+$. Let $S \subseteq \lambda$ be stationary and $f \in \mathcal{F}$. Let

$$\mathcal{F}_{f,S} = \{g \in \mathcal{F} \mid (\forall \alpha \in S)(g(\alpha) \leq f(\alpha))\}.$$

Then $\mathcal{F}_{f,S} \subseteq \prod_{\alpha<\lambda} B_\alpha$ where $B_\alpha = f(\alpha) + 1$. But $\mathrm{card}(B_\alpha) = \mathrm{card}(f(\alpha) + 1) \leq \kappa_\alpha$. So $\mathrm{card}(\mathcal{F}_{f,S}) \leq \kappa$ by (1).
Define

$$\mathcal{F}_f = \bigcup \{\mathcal{F}_{f,S} \mid S \subseteq \lambda \text{ is stationary}\}.$$

Then $\mathrm{card}(\mathcal{F}_f) \leq \kappa$, too.
Now, construct a sequence $\langle f_\xi \mid \xi < \delta \rangle$ of functions in $\mathcal{F}$ such that $\delta \leq \kappa^+$ and $\mathcal{F} = \bigcup \{\mathcal{F}_{f_\xi} \mid \xi < \delta\}$. Do this by induction. Take any $f \in \mathcal{F}$ to be f_0. If $\langle f_v \mid v < \xi \rangle$ is already defined, take any $f \notin \bigcup \{\mathcal{F}_{f_\nu} \mid v < \xi\}$ to be f_ξ. If there is no such f, take $\xi = \delta$ and we are finished. By definition, $\{\alpha \mid f_\xi(\alpha) \leq f_v(\alpha)\}$ is non-stationary for all $v < \xi$. So $f_v \in \mathcal{F}_{f_\xi}$ for all $v < \xi$. But $\mathrm{card}(\mathcal{F}_{f_\xi}) \leq \kappa$. Therefore, $\delta \leq \kappa^+$.
Altogether, $\mathrm{card}(\mathcal{F}) = \mathrm{card}(\bigcup \{\mathcal{F}_{f_\xi} \mid \xi < \delta\}) \leq \kappa^+$. qed(2)
(3) $2^\kappa = \kappa^+$
Proof. For $X \subseteq \kappa$, let $f_X = \langle X \cap \kappa_\alpha \mid \alpha < \lambda \rangle$. Then $\mathcal{F} := \{f_X \mid X \leq \kappa\}$ is almost disjoint and $\mathcal{F} \subseteq \prod_{\alpha<\lambda} \wp(\kappa_\alpha)$. But $\mathrm{card}(\wp(\kappa_\alpha)) = \kappa_\alpha^+$, since GCH holds below κ. So (3) follows from (2). $\square$

7 Product Forcing

We have employed the forcing method to manipulate the size of powersets of specific cardinals like $\aleph_0$, $\aleph_1$, $\aleph_{\omega+1}$. In order to obtain models with global cardinal arithmetic properties we want to apply these techniques simultaneously at several places. This is possible through the method of product forcing where several forcings are performed in parallel and somewhat independently of each other. The product forcing partial orders are obtained by forming simple products of partial orders.

Definition 30. *Let* $(P_0, \leqslant_0, 1_0)$ *and* $(P_1, \leqslant_1, 1_1)$ *be forcing partial orders. Then define their* ***product*** $P_0 \times P_1 = (P_0 \times P_1, \leqslant, 1)$ *by:* $(p_0, p_1) \leqslant (q_0, q_1) \Leftrightarrow p_0 \leqslant_0 q_0$ and $p_1 \leqslant_1 q_1$; $1 = (1_0, 1_1)$.

We relate forcing with $P_0 \times P_1$ to the two component forcings with P_0 and P_1.

Theorem 31. *Let M be a ground model and let $(P_0, \leqslant_0, 1_0) \in M$ and $(P_1, \leqslant_1, 1_1) \in M$ be forcing partial orders.*

a) Let G be $P_0 \times P_1$-generic over M. Define projections $G_0 = \{p_0 \in P_0 | \exists p_1\, (p_0, p_1) \in G\}$ and $G_1 = \{p_1 \in P_1 | \exists p_0\, (p_0, p_1) \in G\}$. Then $G = G_0 \times G_1$, G_0 is P_0-generic over M, and G_1 is P_1-generic over $M[G_0]$. Moreover, by symmetry, G_0 is P_0-generic over $M[G_1]$.

b) Let G_0 be P_0-generic over M and G_1 be P_1-generic over $M[G_0]$. Then $G_0 \times G_1$ is $P_0 \times P_1$-generic over M.

Proof. a) $G \subseteq G_0 \times G_1$ is obvious. For the converse consider $p_0 \in G_0$ and $p_1 \in G_1$. Choose $(p_0, q_1) \in G$ and $(q_0, p_1) \in G$. Choose $(r_0, r_1) \in G$, $(r_0, r_1) \leqslant (p_0, q_1)$, and $(r_0, r_1) \leqslant (q_0, p_1)$. $(r_0, r_1) \leqslant (p_0, p_1)$ and so $(p_0, p_1) \in G$.

To check the first genericity property consider $D_0 \in M$, D_0 dense in P_0. Define $D = D_0 \times P_1$. $D \in M$ is dense in $P_0 \times P_1$. By genericity choose $(p_0, p_1) \in G \cap D$. Then $p_0 \in G_0 \cap D_0$.

To check the other genericity property consider $D_1 \in M[G_0]$, D_1 dense in P_1. Choose a name $\dot{D}_1 \in M$, $\dot{D}_1^{G_0} = D_1$. Choose a condition $p_0 \in G_0$ such that $p_0 \Vdash'' \dot{D}_1$ is dense in $\check{P}_1''$. Choose $p_1 \in P_1$ such that $(p_0, p_1) \in G$. Define a set $D = \{(q_0, q_1) | q_0 \Vdash \check{q}_1 \in \dot{D}_1\}$. We claim that D is dense in $P_0 \times P_1$ below (p_0, p_1): Consider $(q_0, q_1) \leqslant (p_0, p_1)$. $q_0 \Vdash'' \dot{D}_1$ is dense in $\check{P}_1''$, $q_0 \Vdash \check{q}_1 \in \check{P}_1$. By the laws of forcing, choose $r_0 \leqslant q_0$, $r_1 \leqslant q_1$, $r_0 \Vdash \check{r}_1 \leqslant \check{q}_1$, $r_0 \Vdash \check{r}_1 \in \dot{D}_1$. Then $(r_0, r_1) \leqslant (q_0, q_1)$ and $(r_0, r_1) \in D$. By the genericity of G choose $(q_0, q_1) \in G \cap D$. Then $q_0 \in G_0$, $q_0 \Vdash \check{q}_1 \in \dot{D}_1$, $q_1 \in D_1$, $q_1 \in G_1$. Hence $G_1 \cap D_1 \neq \emptyset$, as required.

b) Trivially, $G_0 \times G_1$ is a filter on $P_0 \times P_1$. To check genericity, consider $D \in M$, D dense in $P_0 \times P_1$. Define $D_1 = \{p_1 | \exists p_0 \in G_0\, (p_0, p_1) \in D\}$.

(1) $D_1 \in M[G_0]$ and D_1 is dense in P_1.

Proof. $D_1 \in M[G_0]$ holds by the definition of D_1. Consider $q_1 \in P_1$. Define $D_0 = \{p_0 | \exists p_1 \leqslant q_1\, (p_0, p_1) \in D\}$. We claim that $D_0 \in M$ is dense in P_0. Consider $q_0 \in P_0$. Choose $(p_0, p_1) \leqslant (q_0, q_1)$, $(p_0, p_1) \in D$. Then $p_0 \leqslant q_0$ and $p_0 \in D_0$. By the genericity of G_0 choose $p_0 \in G_0 \cap D_0$. Choose $p_1 \leqslant q_1$, $(p_0, p_1) \in D$. Then $p_1 \in D_1$ and $p_1 \leqslant q_1$. qed(1)

By the genericity of G_1 choose $p_1 \in G_1 \cap D_1$. Choose $p_0 \in G_0$, $(p_0, p_1) \in D$. Then $(p_0, p_1) \in G_0 \times G_1 \cap D$, as required. □

This theorem has many consequences in the analysis of forcing constructions. Let $P = \mathrm{Fn}(\omega, 2, \omega)$ be the partial order for adding one Cohen real over a ground model M. By splitting ω into even and odd numbers we see that $P \simeq P \times P$ where the isomorphism lies in the ground model M. So Cohen forcing can be construed as a product of two Cohen forcings, and by the previous theorem a generic extension $M[G]$ by P can be split into two succesive extension $M[G_1]$ and $M[G] = M[G_1][G_2]$. Since G_1 and G_2 can again be split in this way we obtain an ascending chain of intermediate models between M and $M[G]$ which is densely ordered by $\subseteq$. Actually the structure of the partial order of all intermediate models of a Cohen generic extension is still the object of set theoretic research.

One can generalise the finite product construction to infinite products. We simplify notation by denoting all partial order relations by $\leqslant$ and all maximal elements by 1.

Definition 32. *Let $((P_i, \leqslant, 1) | i \in I)$ be a sequence of forcing partial orders. Then define the* ***product*** *partial order $(\prod_{i \in I} P_i, \leqslant, 1)$ by $(p_i | i \in I) \leqslant (q_i | i \in I) \Leftrightarrow \forall i\, p_i \leqslant q_i$; $1 = (1 | i \in I)$. For $p = (p_i | i \in I) \in \prod_{i \in I} P_i$ let its* ***support*** *be $s(p) = \{i \in I | p_i \neq 1\}$. There are various suborders of $\prod_{i \in I} P_i$ according to the assumed supports: for λ a cardinal let the λ-product of $((P_i, \leqslant, 1) | i \in I)$ be the suborder $\{p \in \prod_{i \in I} P_i \,|\, \mathrm{card}(s(p)) < \lambda\} \subseteq \prod_{i \in I} P_i$.*

In later applications we shall study even more complicated products. As above we get:

Theorem 33. *Let M be a ground model and let $((P_i, \leqslant, 1) | i \in I) \in M$ be a sequence of forcing partial orders. Let $\lambda \in \mathrm{Card}^M$ and let P be the λ-product of $((P_i, \leqslant, 1) | i \in I)$ formed in M. Let G be P-generic over M. For $i \in I$ define a projection $G_i = \{p(i) \,|\, p \in G\}$. Then G_i is P_i-generic over M.*

The converse of this theorem is not true. If for all $i \in I$ we have that G_i is P_i-generic over M the product $P \cap \prod_{i \in I} G_i$ is in general not generic over M; it is for example possible to code so much information into the sequence $(G_i | i \in I)$ such that $M[(G_i | i \in I)]$ is not a model of ZFC.

8 Forcing the GCH

Definition 34. *Define the $\beth$-sequence ("beth") by recursion on $\alpha \in \mathrm{On}$: $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, and for limit ordinals δ, $\beth_\delta = \bigcup_{\alpha < \delta} \beth_\alpha$.*

Obviously the GCH is equivalent to the statement: $\forall \alpha \beth_\alpha = \aleph_\alpha$. We shall aim to force that property. The idea is to eliminate all cardinals between $\beth_\alpha$ and $\beth_{\alpha+1}$ by generically adjoining surjections from $\beth_\alpha^+$ onto $\beth_{\alpha+1}$. An appropriate forcing for this is given by the partial order of partial functions $\mathrm{Fn}(\beth_\alpha^+, \beth_{\alpha+1}, \beth_\alpha^+)$. This forcing can be construed differently. Since $\beth_{\alpha+1} = \mathrm{card}(\wp(\beth_\alpha))$ we can consider instead $\mathrm{Fn}(\beth_\alpha^+, \wp(\beth_\alpha), \beth_\alpha^+)$. Since the conditions can have size $\beth_\alpha$ and take values in $\wp(\beth_\alpha)$ we can identify conditions with characteristic functions on $\beth_\alpha^+ \times \beth_\alpha$. Since $\mathrm{card}(\beth_\alpha^+ \times \beth_\alpha) = \beth_\alpha^+$ we can even take characteristic functions on $\beth_\alpha^+ \setminus \beth_\alpha$. So we can use the forcing partial order $\mathrm{Fn}(\beth_\alpha^+ \setminus \beth_\alpha, 2, \beth_\alpha^+)$. One can now show:

Theorem 35. *Let M be a ground model and let $P_\alpha = \mathrm{Fn}(\beth_\alpha^+ \setminus \beth_\alpha, 2, \beth_\alpha^+)$ be defined in M. Let G be P_α-generic over M. Then*

a) $\forall \kappa \leqslant (\beth_\alpha^+)^M\, \kappa \in \mathrm{Card}^M \longleftrightarrow \kappa \in \mathrm{Card}^{M[G]}$;

b) $\forall (\beth_\alpha^+)^M < \kappa \leqslant \beth_{\alpha+1}^M\, \mathrm{card}^{M[G]}(\kappa) = (\beth_\alpha^+)^M$;

c) $\forall \kappa > \beth_{\alpha+1}^M\, \kappa \in \mathrm{Card}^M \longleftrightarrow \kappa \in \mathrm{Card}^{M[G]}$;

d) $\wp(\beth_\alpha)^M = \wp(\beth_\alpha)^{M[G]}$ *and* $M[G] \vDash 2^{\beth_\alpha} = \beth_\alpha^+$.

Proof. a) holds since the forcing is $\beth_\alpha^+$-closed in M.
b) It suffices to see that G adjoins a surjection $f: (\beth_\alpha^+)^M \to \wp(\beth_\alpha)^M$ by $f(i) = \{j < \kappa | G(\kappa \cdot i + j) = 1\}$. Since the forcing is $\beth_\alpha^+$-closed in M we have for all $i < (\beth_\alpha^+)^M$ that $f(i) \in \wp(\beth_\alpha)^M$. Conversely, if $a \in \wp(\beth_\alpha)^M$ the set $D = \{p \in P_\alpha | \exists i < (\beth_\alpha^+)^M\, \forall j < \kappa\, (j \in a \longleftrightarrow p(\kappa \cdot i + j) = 1)\}$ is dense in D. If $p \in G \cap D$ then $a \in \mathrm{range}(f)$.
c) In M, $\mathrm{card}(P_\alpha) \leqslant \beth_\alpha^+ \cdot 2^{\beth_\alpha} = 2^{\beth_\alpha} = \beth_{\alpha+1}$. So P_α trivially satisfies the $\beth_{\alpha+1}^+$-chain condition in M which implies the claim.
d) follows from the $\beth_\alpha^+$-closure of P_α in M and from b). □

We can form a kind of product of forcings of the type P_α to obtain GCH for an initial segment of cardinals. To obtain the right preservation of the forcing the product has to be changed somewhat at certain cardinals:

Definition 36. *A cardinal κ is called* ***inaccessible*** *if $\kappa = \beth_\kappa$ and κ is regular.*

Inaccessible cardinals belong to the class of *large cardinals*, i.e., those cardinals which cannot be reached from below and which are strongly closed limit points of the cardinal hierarchy. The existence of inaccessible cardinals cannot be proved in ZFC but we cannot exclude them in the following construction.

Theorem 37. *Let M be a ground model. In M, let θ be a limit ordinal and define a partial order* $P = \{p | p: \mathrm{dom}(p) \to 2, \mathrm{dom}(p) \subseteq \beth_\theta, \forall \alpha < \theta\ \mathrm{card}(\mathrm{dom}(p) \cap \beth_\alpha^+) < \beth_\alpha^+, \forall \kappa, \kappa \text{ inaccessible or } \kappa = \omega: \mathrm{card}(\mathrm{dom}(p) \cap \kappa) < \kappa\}$, *ordered by reverse inclusion. Let G be P-generic over M. Then*

a) For $\alpha < \theta$ holds $\aleph_{\alpha+1}^{M[G]} = (\beth_\alpha^+)^M$.

b) For $\beta \leqslant \theta$, $\mathrm{Lim}(\beta)$ holds $\aleph_\beta^{M[G]} = \beth_\beta^M$.

c) $M \vDash \forall \alpha < \beta\ 2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Proof. Since limits of cardinals are cardinals we only have to prove a) and c) for $\alpha < \theta$. We do a *product analysis* of the forcing P by factoring it in ways such that the factors satisfy certain preservation properties.
(1) Let $\kappa = (\beth_{\alpha+1}^+)^M$, $\alpha < \theta$. Then, in M, $P \cong P_0 \times P_1$ where P_0 is κ-closed and $\mathrm{card}(P_1) < \kappa$.
Proof. Define $P_0 = \{p \in P | \mathrm{dom}(p) \subseteq \beth_\theta \setminus \beth_\alpha^+\}$ and $P_1 = \{p \in P | \mathrm{dom}(p) \subseteq \beth_\alpha^+\}$. $p \mapsto (p \upharpoonright (\beth_\theta \setminus \beth_\alpha^+), p \upharpoonright \beth_\alpha^+)$ defines a canonical isomorphism $P \cong P_0 \times P_1$. By the cardinality requirements in the definition of P, P_0 is clearly κ-closed. $\mathrm{card}(P_1) \leqslant \beth_\alpha^+ \cdot 2^{\beth_\alpha} = 2^{\beth_\alpha} = \beth_{\alpha+1} < \kappa$. qed(1)
(2) $\kappa = (\beth_\alpha^+)^M$, $\alpha < \theta$, and $\beth_\alpha$ inaccessible or $\alpha = 0$. Then, in M, $P \cong P_0 \times P_1$ where P_0 is κ-closed and $\mathrm{card}(P_1) < \kappa$.
Proof. Define $P_0 = \{p \in P | \mathrm{dom}(p) \subseteq \beth_\theta \setminus \beth_\alpha\}$ and $P_1 = \{p \in P | \mathrm{dom}(p) \subseteq \beth_\alpha\}$. $p \mapsto (p \upharpoonright (\beth_\theta \setminus \beth_\alpha), p \upharpoonright \beth_\alpha)$ defines a canonical isomorphism $P \cong P_0 \times P_1$. By the cardinality requirements in the definition of P, P_0 is clearly κ-closed. $\mathrm{card}(P_1) \leqslant \beth_\alpha \cdot 2^{<\beth_\alpha} = \beth_\alpha \cdot \beth_\alpha = \beth_\alpha < \kappa$. qed(2)

Consider a cardinal κ as in (1) or (2). Let G_0 be P_0-generic over M and G_1 be P_1-generic over $M[G_0]$ such that $G \cong G_0 \times G_1$ by the described canonical isomorphism. Since P_0 is κ-closed in M, κ is a cardinal in $M[G_0]$. In M, $\mathrm{card}(P_1) < \kappa$ and so in $M[G_0], \mathrm{card}(P_1) < \kappa$. So P_1 satisfies the κ-chain condition in $M[G_0]$ and κ is a cardinal in $M[G_0][G_1] = M[G]$.

Now consider a cardinal $\kappa = (\beth_\alpha^+)^M$, $\alpha < \theta$ which does not fall under (1) and (2). Then $\mathrm{Lim}(\alpha)$ and $(\beth_\alpha)^M$ is singular in M. Assume for a contradiction that κ is not a cardinal in $M[G]$. Then $\mathrm{cof}^{M[G]}(\kappa) < (\beth_\alpha)^M$ since cofinalities are regular. Choose an ordinal $\beta < \alpha$, $\mathrm{cof}^{M[G]}(\kappa) < \beth_{\beta+1}^M$. Let $\lambda = (\beth_{\beta+1}^+)^M$. By (1), choose a factorisation $P \cong P_0 \times P_1$ where P_0 is λ-closed and P_1 satisfies the λ-chain condition. Let G_0 be P_0-generic over M and G_1 be P_1-generic over $M[G_0]$ such that $G \cong G_0 \times G_1$ by the described canonical isomorphism. Since P_1 satisfies the λ-chain condition in $M[G_0]$ cofinalities $\geqslant \lambda$ are preserved between $M[G_0]$ and $M[G_0][G_1]$. Hence $\mathrm{cof}^{M[G_0]}(\kappa) < \lambda$. Since M and $M[G_0]$ possess the same $< \lambda$-sequences, $\mathrm{cof}(\kappa) < \lambda$. But this is a contradiction.

We have thus proved:
(3) Let $\kappa = (\beth_\alpha^+)^M$, $\alpha < \theta$. Then κ is a cardinal in $M[G]$.
Since limits of cardinals are cardinals we also have:
(4) Let $\kappa = (\beth_\alpha)^M$, $\alpha \leqslant \theta$, $\mathrm{Lim}(\alpha)$ or $\alpha = 0$. Then κ is a cardinal in $M[G]$.

For a) and b) it suffices to show that all other cardinals in M below $\beth_\theta^M$ are *not* preserved between M and $M[G]$. These are the cardinals λ with $(\beth_\alpha^+)^M < \lambda \leqslant \beth_{\alpha+1}^M < (\beth_{\alpha+1}^+)^M$ for some $\alpha < \theta$. Now just as in the previous theorem one obtains $\mathrm{card}^{M[G]}(\lambda) = (\beth_\alpha^+)^M$. This proves a) and b).
(5) $M[G] \vDash \forall \alpha < \theta\ 2^{\aleph_{\alpha+1}} = \aleph_{\alpha+2}$.
Proof. By a), $\lambda = \aleph_{\alpha+1}^{M[G]} = (\beth_\alpha^+)^M$. Let $\bar{\kappa} = \beth_{\alpha+1}^M$ and let $\kappa = \aleph_{\alpha+2}^{M[G]} = (\beth_{\alpha+1}^+)^M$ be the next cardinal in $M[G]$. By (1) there is a factorisation $P \cong P_0 \times P_1$ where P_0 is κ-closed and $\mathrm{card}(P_1) < \kappa$ with a corresponding factorisation $G \cong G_0 \times G_1$ of generic sets. By the κ-closure of P_0 we have $\wp(\lambda) \cap M = \wp(\lambda) \cap M[G_0]$. Since $\mathrm{card}^{M[G_0]}(P_1) \leqslant \bar{\kappa}$ we have

$$\begin{aligned}(2^\lambda)^{M[G]} &= (2^\lambda)^{M[G_0][G_1]}\\ &\leqslant (2^{\bar{\kappa}\cdot\lambda})^{M[G_0]}\\ &= (2\end{aligned}$$

□