To: fom@math.psu.edu
Subject: FOM: A proof of not-CH
From: Soren Moller Riis <smriis@daimi.aau.dk>
Date: Sun, 13 Sep 1998 18:24:49 +0200 (MET DST)
Cc: smriis@daimi.aau.dk
Sender: owner-fom@math.psu.edu

A proof of not-CH:

Joe Shipman writes [Shipman, September 12, 1998] that there was no "proof" of The continuum Hypothesis (or of its negation) using ANY axioms that had any plausibility. I think it is important that Joe Shipman very carefully uses the past tense in his formulation.

On Fom there have been quite a lot of discussions of the status of CH. In my oppinions these discussions have been somewhat irrelevant as there is indeed a proof of not-CH. I have tried this proof on many mathematicians (certainly at least 10) and I never found anyone who did not accept the proof.

My proof is a variant of a related well known argument by Chris Freiling. Freilings argument was published in JSL 1986 (See "Axioms of Symmetry: Throwing Darts at the Real Line", Journal of Symbolic Logic, 51, pages 190-200).

Suppose two players play the following game: First player I choose a real number r. Player I sends this number to the referee. Then player II choose (without knowing r) a countable set B of reals. Player II wins the game if the number r happens to belong to the set B.

Most mathematicians would have no problems in putting their money on player I. Thus they will give a negative answer to the following question:

Question:

> Does player II have a strategy which will guarantee him/her victory almost certainly? More specific: does Player II have a method of picking the countable set B (using any kind of selection mechanism he/she would like) which will win the game with probability 1?

The mathematicians I spoke to either ought-right deny that player II should have such a strategy, or suggest (in rare cases and with some hesitation) that the question might not be well defined.

Now the mathematicians I spoke to certainly accepts naive set-theory (including the axiom of choice) as foundation of mathematics. Thus they are FORCED to accept $\neg\mathsf{CH}$. Indeed they all accepted the validity of this argument, and thus they all accepted $\neg\mathsf{CH}$. First we notice:

Theorem (ZFC/naive set theory):

If CH holds, then player II can choose a measure space and select a set B such that r belongs to B with probability 1.

Thus it does not matter whether you think player I ought to win with probability 1, OR you think the question not is well defined. In both cases you are in effect denying the antecedent in the above theorem. Thus in both cases you are in effect accepting not-CH.

Proof of theorem:

Assume CH is valid. Before the game begin player II fix a well-ordering $<$ of the reals which have the property that each real have only countable many smaller reals (smaller in the well-ordering). This can be done if CH is valid. Player II also select a non-singular probability measure M on the reals. Thus any countable set have measure 0 with respect to this measure.

Now we are ready to play the game: Player I select a real. This real can be selected in any way we wish. For ANY real r chosen by player I, the set $A := \{r' \,;\, r' < r\}$ is countable. Now player II choose s according to the probability distribution. Player II then select the countable set $B := \{s' \,;\, s' < s\}$ as the countable set B. With probability 1, $s \notin A$ (because the set A has measure 0). Thus with probability 1, r belongs to B and player II wins the game. q.e.d.

Why I think my twist of Freilings argument makes a stronger argument:

My argument is (as I already pointed out) a variant of Chris Freilings argument. I think however there is an important difference between Freilings original argument and my modification. In Freilings version people might discard the whole argument as meaningless by appealing to the danger of building arguments on probabilities. After all (one might claim) the whole notion of probability is in a very strong sense in contradiction with the axiom of choice (as part of naive set theory). For this reason Chris Freilings argument might be viewed with some skepsis (though I completely accepts it). Technically Freilings argument uses a version of Fubinis Theorem which might be considered as invalid. It is certainly not provable in ZFC.

The mathematicians who believe that the Question (above) is ill-posed will of course feel no strong reason to accept Freilings argument.

The point by my formulation is that mathematicians who discards the question as meaningless in effect are ACCEPTING $\neg\mathsf{CH}$. This is the major point in my version of Freilings argument.

Thus the ONLY position which is consistent with naive set theory and with CH, is to take it as a fact that the answer to the question is positive and that Player II indeed has a method of picking a countable set B such that any fixed but unknown r belongs to B with probability 1!

Gödel thought it was possible to decide CH on the basis of some obvious principles. I think he was right and that a suitable version of Freilings argument does the job. I leave it to future historians to explain why Freilings argument (or perhaps even more the version I suggest) have not been generally accepted as a valid proof of $\neg\mathsf{CH}$.

Question: Could some of you help me collecting a list of mathematical statements (like CH) which are independent of set theory. I think it could be interesting to try to see if some of these statements could be settled by thought experiments involving some kind of idealised dialog/discussion/game involving two players and a referee.

Søren Riis

*

```
To:  fom@math.psu.edu
Subject:  FOM: Elaboration on not-CH proof:
From:  Soren Moller Riis <smriis@daimi.aau.dk>
Date:  Mon, 14 Sep 1998 20:01:21 +0200 (MET DST)
Cc:  smriis@daimi.aau.dk
Sender:  owner-fom@math.psu.edu
```

Elaboration on not-CH proof:

Yesterday I presented a variant of a proof of $\neg\mathsf{CH}$ [Riis, September 13, 1998]. The argument was a variant of a wellknown argument by Chris Freiling. Off the list I have been asked if not my argument was dubious. The following question was put forward:

> *Suppose that Player I and Player II use the same method to pick their reals r and s. Then the situation is symmetric. You argue, in effect, that $r < s$ with probability 1. But the same argument also shows that $s < r$ with*

> *probability 1. So I guess the notion of picking a real according to a probability distribution does not make sense. Are you then concluding that this not making sense implies* $\neg\mathsf{CH}$*?*

Let me clarify this point:

I was actually very careful the way I defined the game. It is no coincidence that I let player I makes the first choice!! Suppose that player II make the first choice (by choosing a countable set B). Then it follows (even without using CH) that player I by selecting r randomly (according to any non-singular probability distribution) will win with probability 1.

I defined the game such that player I must choose first. It seems to be plain that this should not change the odds in the game. However one might suggest that the game is not well-defined (like the game where two players (independently of each other) have to try to select the largest integer).

The point I am making is that CH implies that NOT only is the game well-defined [when we carefully demand that player I choose first], but player II actually has a strategy which guarantee victory with probability 1.

In short: If we accepts CH, we have to accept that

(1) The game [as I defined it] is well-defined
(2) The related game [where player II choose first] is also well defined, but the expected outcome is totally different.

The underlying principle is the following:

Suppose we are given a mathematical proposition A, as well as two experts (players) which are arguing about the validity of A.

One expert (player I) supports A, while the other expert (player II) supports $\neg A$. The experts communicates via a referee. If one expert always can persuade the referee she is right (i.e., are capable to win the game) with some frequency p (say 80frequency p will appear irrespectively of the order by which the referee received the independent information.

Hope this clarifies my argument,

Søren Riis

$*$

```
To:  fom@math.psu.edu
Subject:  FOM: Re:  Elaboration on not-CH proof:
From:  Martin Schlottmann <martin_schlottmann@math.ualberta.ca>
Date:  Tue, 15 Sep 1998 14:40:46 -0600
Organization:  Department of Mathematical Sciences, University
of Alberta
References:  <199809141801.UAA03255@humulus.daimi.aau.dk>
Sender:  owner-fom@math.psu.edu
```

To my understanding the analysis of the game is not correct unless we also stipulate a strategy for player I. Given the other rules of the game, it is completely irrelevant that I has the first pick.

Let P_{I} be the probability measure of player I, P_{II} the probability measure of player II. If we call $A(r) := \{r' ; r' < r\}$ and $B(s) := \{s' ; s' <= s\}$, then the event that II wins is:

$$\{ \text{ II wins } \} = \{(r,s) ; r \in B(s)\} = \{(r,s) ; s \notin A(r)\}$$

which occurs with probability

$$(P_{\mathrm{I}} \times P_{\mathrm{II}})(\{(r,s) ; r \in B(s)\}) = (P_{\mathrm{I}} \times P_{\mathrm{II}})(\{(r,s) ; s \notin A(r)\}),$$

but only in the case that $\{$ II wins $\}$ is a measurable set (then we could use Fubini in order to calculate this probability to be 1, given certain properties of P_{I} and P_{II}).

Therefore, the only thing shown is that certain sets in the product space are not measurable, a rather non-surprise.

Martin Schlottmann

$*$

```
To:  fom@math.psu.edu
Subject:  FOM: Re:  Elaboration on not-CH proof:
From:  Soren Moller Riis <smriis@daimi.aau.dk>
Date:  Wed, 16 Sep 1998 00:13:17 +0200 (MET DST)
Cc:  smriis@daimi.aau.dk
Sender:  owner-fom@math.psu.edu
```

It seems you are addressing Chris Freiling original argument rather than my version of his argument.

Assuming CH, Player II has a winning strategy in the game in the following sense:

A: **Consequence of CH**

For ANY $r \in \mathbb{R}$, $r \in B(s) := \{s' \,;\, s' \leq s\}$ with probability 1, provided s is chosen randomly according to a non-singular probability distribution.

This is a significantly stronger statement than the one you consider and which I think is closer to Freilings argument where the question really is about swapping the order of integration of a function with a non-measurable graph.

You (and Freiling) seems to consider a somewhat weaker consequence of CH:

A': **Weaker consequence of** CH:

If $r \in \mathbb{R}$ is chosen randomly (first) and then player II randomly choose $s \in \mathbb{R}$, then $r \in B(s) := \{s' \,;\, s' \leq s\}$ with probability 1.

You (and Freiling) in effect compare this statement with the statement:

C': **Letting player II move first:**

If $s \in \mathbb{R}$ is chosen randomly (first) (by player II) and player I then randomly choose $r \in \mathbb{R}$, then $r \in B(s) := \{s' \,;\, s' \leq s\}$ with probability 0.

However in my version of the argument I contrast A with:

C: **Letting player II move first:**

For ANY countable set B, we have $r \in B$ with probability 0, provided r is chosen randomly according to a non-singular probability distribution.

You (and essentially also Freiling as he put his argument) seems to contrast A' and C'. The skeptics then argue that there is no contradiction here. Put in your words: "Therefore, the only thing shown is that certain sets in the product space are not measurable, a rather non-surprise".

What you do not seems to recognise is that I am contrasting A and C (rather than A' and C'). If we do this and combine this with the fact that (USING YOUR OWN WORDS): "Given the other rules of the game, it is completely irrelevant that I has the first pick" we must accept $\neg\mathsf{CH}$.

Assuming CH, we have demonstrated that it indeed matter who moves first (compare A and C). Yet, you acknowledge (correctly of course) that it is completely irrelevant who moves first (as I have defined the game).

Thus you fell into the trap!!

Your attempt to show the game is not well defined (because the order shouldn't matter) backfired. You statement that the order does not matter is in direct contradiction with provable facts (compare A, and C).

Well, I should certainly stipulate: provable under the assumption of CH. The contradiction (you seems to acknowledge) demonstrates ¬CH (using an argument which has not (yet?!) been incorporated in naive set theory).

Søren Riis

*

```
To:  fom@math.psu.edu
Subject:  FOM: Further elaboration on not-CH:
From:  Soren Moller Riis <smriis@daimi.aau.dk>
Date:  Wed, 16 Sep 1998 11:57:08 +0200 (MET DST)
Cc:  smriis@daimi.aau.dk
Sender:  owner-fom@math.psu.edu
```

Further elaboration on ¬CH:

In my latest elaboration [Riis, Wed, 16 Sep 1998] I got carried away when I wrote:

> *using an argument which has not (yet?!) been incorporated in naive set theory.*

This is of course debatable and I forgot to add a smiley.

I certainly accepts that the proof of ¬CH. So do all the mathematicians I have spoken to.

The point I am making is that I do not think Chris Freiling argument necessarily needs to depend on swapping the order of integration (in a case where the function involved is non-measurable).

The probabilities involved in the variant I have suggested ARE well defined.

Let me give some examples which might clarify my point:

Example 1:

Fix some predicate $A(x, y, z, w)$ and consider the game in which player I choose x, player II choose y, Player I then choose z, and Player II chose w. Let us say player II wins if $A(x, y, z, w)$ holds. Otherwise player I wins.

It is plain that player II has a winning strategy in this game iff and only if $\forall x \exists y \forall z \exists w A(x, y, z, w)$ is valid.

Thus if we play the game between two experts (player I and player II) either player I or player II has a winning strategy. This is "tertium non datur"

The game I just defined (in the example) is well defined though we did not specify any strategy for the losing player. All we did was to present the winning player with a strategy which works against ANY defence.

Now not all games between experts are well defined in the sense that one has a successful winning strategy. Consider for example:

Example 2:

Player I selects a natural number.
Player II selects a natural number.
The one who choose the highest number win the game.

None of the players has a winning strategy and it clearly makes no sense to assign a probability (say $\frac{1}{2}$) that player I (player II) wins.

We can easily prove (in ZFC) that none of the players has a strategy which guarantee victory with any probability $p > 0$. More specifically there is no probability space U, and map $f : U \to \mathbb{N}$, such that for ANY $n \in \mathbb{N}$, the probability $f(u) > n$ is at least p.

Thus we can show (in ZFC) that none of the players have any strategy which guarantee victory with some non-zero probability.

Now consider the game:

Example 3:

Player I selects $r \in \mathbb{R}$.
Player II selects $B \subseteq \mathbb{R}$, B countable.
Player II wins if and only if $r \in B$. Otherwise player I wins.

This game is well defined if we require player I "moves" first. In this case (assuming CH) player II has a winning strategy in the sense that there exists a probability space U and a map f from U into the collection of countable subsets of $\mathbb{R}$, such that for ANY $r \in \mathbb{R}$, the probability $r \in f(u)$ is 1.

Thus player II has a winning strategy and this winning strategy only involves well defined measurable functions.

Now consider the game:

Example 4:

Player II selects $B \subseteq \mathbb{R}$, B countable.

Player I selects $r \in \mathbb{R}$.
Player II wins if and only if $r \in B$. Otherwise player I wins.

This game is also well defined. In this case player I has a winning strategy in the sense that there exists a probability space U and a map g from U into $\mathbb{R}$ such that for ANY countable $B \subseteq \mathbb{R}$ the probability $g(u) \notin B$ is 1.

The games (presented in example 3 and example 4) can be played in the same sense as we could play the game in Example 1. And the winning strategies are completely well defined (nothing involving non-measurable functions). Yet, the outcome depends (if we assume CH) on whether player I or player II moves first. If the players moves simultaneously the game is not even well defined (again provided we assume CH).

A traditional contradiction arise when we can deduce that each of two experts in a game like the one in example 1, has a winning strategy.

The contradiction we arrive at is of course somewhat different, but again the essential point is that in some sense both experts has a winning strategy. This is indeed a contradiction (though not a traditional one).

Thus we have to (and the mathematicians I discussed this with indeed did) accept $\neg\mathsf{CH}$.

Should we consider the above argument as part of naive set theory?

I do NOT think it should. Rather it seems to belong naturally to some meta-theory of set-theory. Since Gödel we know that it sometimes might be necessary to step outside a given system and have a look at things from a higher perspective. I think this is what is happening in the variant of Chris Freiling proof which I have presented.

Søren Riis