Characterization of projective sets by finer topologies

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1 Characterization of Borel and analytic sets by finer topologies

We work in Polish spaces

Definition 1.1. A topological space (X,τ) is called a **Polish space** if (X,τ) is a separable, completely metrizable space, i.e., there exists a complete metric on X and τ is the topology of the metric space.

The following result is classical:

Theorem 1.2. *Let X be a Polish space. $A \subseteq X$ is Borel iff there exists a finer topology t on A which is Polish.*

For a construction of the finer topology see [Kech95, 13.1]. For the converse note that the image of an Borel set under a continuous injection is again Borel ([Kech95, 15.1]), so using the identity mapping $(A,t) \longrightarrow (X,\tau)$ proves the result.

Professor Howard Becker found results for $\mathbf{\Sigma}^1_n$ sets in the spirit of the above theorem in [Be1] and [Be2]. He used for this a characterization of a Polish space by a strong Choquet space. We introduce this notion:

Let X be a nonempty topological space. The **Choquet game** G_X of X is defined as follows: Players I and II take turns in playing nonempty open subsets of X

$$\begin{array}{lccccc} \text{I} & U_0 & & U_1 & & \dots \\ \text{II} & & V_0 & & V_1 & \dots \end{array} \quad ,$$

such that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$
We say II wins this run of the game if $\bigcap_n V_n = \bigcap_n U_n \neq \emptyset$. Otherwise I wins.

The **strong Choquet game** G^s_X is defined similiar, but in addition to the Choquet game player I is required to play a point $x_n \in U_n$ on every turn and then player II must play $V_n \subseteq U_n$ with $x_n \in V_n$. So we have

$$\begin{array}{cccccc} \mathrm{I} & U_0, x_0 & & U_1, x_1 & & \dots \\ \mathrm{II} & & V_0 & & V_1 & \dots \end{array},$$

$U_0 \supseteq V_0 \supseteq \dots, x_n \in U_n, x_n \in V_n$ for $n \in \omega$.
A (winning) strategy for one of the players is defined in the usual way.

Definition 1.3. A nonempty topological space is called a **(strong) Choquet space** if player II has a winning strategy in the (strong) Choquet game G_X (G_X^s).

Now the characterization of the Polish space looks as follows:

Theorem 1.4 (Choquet). *A nonempty, second countable topological space is Polish iff it is T1, regular and strong Choquet.*

It is an easy exercise to prove that a Polish space has this properties. The converse is somewhat more difficult and makes use of the axiom of choice.

This characterization gives now a result for $\mathbf{\Sigma}_1^1$ sets instead of Borel sets as in the above theorem. We only have to drop the regularity property.

Theorem 1.5. *Let X be a Polish space. A subset $A \subseteq X$ is $\mathbf{\Sigma}_1^1$ iff there exists a topology t on A such that*

- $t \supseteq \mathcal{T}|A$
- *t is second countable*
- *t is strong Choquet.*

The topology we construct on A is the Gandy-Harrington topology (or its relativized version). For a definition and the proof that this topology is strong Choquet see [HKL90]. The converse is proved in [Be1].

2 A finer topology on a $\mathbf{\Sigma}_n^1$ set

We are now interested in results similar to that of chapter 1 for higher classes of the projective hierarchy. So we have to consider further ways of weakening the topological conditions in our space. A finer topology of a Hausdorff space will always remain Hausdorff and we already dropped the regularity property. One could ask what happens if we weaken the strong Choquet property to the Choquet property. The next Proposition shows that this leads nowhere.

Proposition 2.1. *Let $(X, \mathcal{T})$ be a Polish space and A an arbitrary subset of X. Then there exists a topology t on A such that t is finer than $\mathcal{T}$ and t is regular, second countable and Choquet.*

Proof: Let $\mathcal{B}$ be a basis for $(A, \mathcal{T}|A)$. Let $\mathcal{C}$ be the closure of $\mathcal{B}$ under complements and finite intersections. Pick a point x_C in each nonempty $C \in \mathcal{C}$.
Now let $\mathcal{D} = \mathcal{C} \cup \{\{x_C\} \mid C \in \mathcal{C}\}$ be the countable basis for the topology t. Since the basis consists of clopen sets t is regular. The isolated points are dense in t, so player II wins the Choquet game in his first move by playing one of the x_C's. $\square$

By this proposition the only topological condition that remains to be considered is the second countable condition. The key fact for the proof of the strong Choquet property in the Gandy-Harrington topology was that $\mathbf{\Sigma}^1_1$ sets are ω-Suslin. If we would have a Suslin representation for $\mathbf{\Sigma}^1_n$ sets we could imitate this proof. We work now under ZF + DC + projective determinacy where we have indeed such an result.

Theorem 2.2 (PD). *For all $n \geq 0$ the following holds:*
(i) Every $\mathbf{\Sigma}^1_{2n+2}$-set is $\boldsymbol{\delta}^1_{2n+1}$-Suslin.
(ii) Every $\mathbf{\Sigma}^1_{2n+2}$-set A is $\kappa_{2n+1}(A)$-Suslin for a cardinal $\kappa_{2n+1}(A) < \boldsymbol{\delta}^1_{2n+1}$.

Proof: First notice that if a set A is κ-Suslin then $p[A]$ is κ-Suslin. To see this let T be a tree on $\omega^{k+1} \times \kappa$ such that $A \subseteq \mathcal{N}^k$ is given by $A = p[T]$. Fix a bijection

$$f : \omega \times \kappa \longrightarrow \kappa$$

This leads to a bijection

$$f^* : (\omega \times \kappa)^{<\omega} \longrightarrow k^{<\omega}$$

We define a tree T' on $\omega^k \times \kappa$ by

$$(s_1, \dots , s_k, \eta) \in T' :\Leftrightarrow (s_1, \dots , s_k, f^{*-1}(\eta)) \in T$$

(1) $p[T'] = p[A]$
Proof:

$$\begin{aligned}
(x_1, \dots , x_k) \in p[T'] &\Leftrightarrow \exists u \in \kappa^\omega\, (x_1, \dots , x_k, u) \in [T'] \\
&\Leftrightarrow \exists u \in \kappa^\omega\, \forall n(x_1|n, , \dots , x_k|n, u|n) \in T' \\
&\Leftrightarrow \exists u \in \kappa^\omega\, \forall n(x_1|n, \dots , x_y|n, f^{*-1}(u|n)) \in T \\
&\Leftrightarrow \exists u \in \kappa^\omega \exists x_{k+1} \in \mathcal{N}\, \forall n(x_1|n, \dots , x_{k+1}|n, u|n) \in T \\
&\Leftrightarrow \exists u \in \kappa^\omega \exists x_{k+1} \in \mathcal{N}\, (x_1, \dots , x_{k+1}, u) \in T \\
&\Leftrightarrow \exists x_{k+1} \in \mathcal{N}(x_1, \dots , x_{k+1}) \in p[T] = A \\
&\Leftrightarrow (x_1, \dots , x_k) \in p[A]
\end{aligned}$$

q.e.d.(1)

So for (i) it is enough to prove that each $\mathbf{\Pi}^1_{2n+1}$-set is $\boldsymbol{\delta}^1_{2n+1}$-Suslin. But by Second Periodicity Theorem from Moschovakis we know that each $\mathbf{\Pi}^1_{2n+1}$ set has a $\mathbf{\Pi}^1_{2n+1}$-scale. Since a $\mathbf{\Pi}^1_{2n+1}$-scale on a $\mathbf{\Pi}^1_{2n+1}$-set is a $\boldsymbol{\delta}^1_{2n+1}$- scale we are done since we know that if a set admits a λ-scale this set is λ-Suslin
For (ii) Let A be a $\mathbf{\Sigma}^1_{2n+1}$ set and let $B \in \mathbf{\Pi}^1_{2n}$ such that $A = p[B]$. Since $B \in \mathbf{\Delta}^1_{2n+1}$ there exists a $\mathbf{\Pi}^1_{2n+}$-scale $(\varphi_i)_{i\in\omega}$ on B. Each φ_i is a $\mathbf{\Delta}^1_{2n+1}$-norm on B, so $|\varphi_i| < \boldsymbol{\delta}^1_{2n+1}$. The length of the scale is $\sup_{i<\omega} |\varphi_i|$ and since $\mathrm{cf}(\boldsymbol{\delta}^1_{2n+1}) > \omega$ the sequence $|\varphi_i|$ is bounded below $\boldsymbol{\delta}^1_{2n+1}$. Hence there is a cardinal $\kappa_{2n+1}(A) < \boldsymbol{\delta}^1_{2n+1}$ such that $|\varphi_i| \leq \kappa_{2n+1}(A)$ for all $i \in \omega$. Thus $(\varphi_i)_i$ is a $\kappa_{2n+1}(A)$-scale and the associated tree tells us that B is $\kappa_{2n+1}(A)$-Suslin. By the above remark we are done. □

Now we can give the main result of this section:

Theorem 2.3 (PD). *Let $(X,\mathcal{T})$ be a Polish space. For $n \geq 1$ let $A \in \mathbf{\Sigma}^1_n(X)$. Then there exists a finer topology t on A such that t is strong Choquet and has a basis of length a cardinal less than $\boldsymbol{\delta}^1_n$ (less as an ordinal, not necessarily less in cardinality).*

Proof: Let $n \geq 1$. WLOG we can assume that A is a subset of $\mathcal{N}$. Let κ be a cardinal less than $\boldsymbol{\delta}^1_n$. Let T be a tree on $\omega \times \kappa$ such that $A = p[T]$. (Note that using a bijection between an ordinal λ and its cardinality κ each λ-Suslin set is κ-Suslin.) For $s \in T$ the subtree T_s consists of all sequences compatible with s. Let $\mathcal{A} = \{p[T_s] \mid s \in T\}$. Then $\mathcal{A}$ is of cardinality κ. Let $\mathcal{B}$ be the closure of $\mathcal{A}$ and all the $\mathcal{T}$-basic open sets of A under finite intersections. The cardinality of $\mathcal{B}$ is κ. $\mathcal{B}$ is the basis for the topology t.
It remains to show that this topology t is strong Choquet. We do this by describing a winning strategy for II.
Assume player I starts by playing (x_0, U_0). Then choose a basic open set of the form $p[T_{r^0_0}] \cap p[T_{r^1_0}] \cap \ldots \cap p[T_{r^{m_0}_0}] \cap N_{u_0}, u_0 \in \omega^{<\omega}$, such that this set is a subset of U_0 and contains the point x_0. We want to make sure our set is not just a $\mathcal{T}$-basic open set, so intersect the basic set with $p[T]$ if necessary. Since $x_0 \in p[T_{r^i_0}], 0 \leq i \leq m_0$, there exists an $\eta^i_0 \in \kappa^\omega$ such that $(x_0, \eta^i_0) \in T_{r^i_0}$. Set $s_0 = x_0|1, t^{0,i}_0 = \eta^i_0|1$. Then $(x_0, \eta^i_0) \in (T_{r^i_0})_{(s_0, t^{0,i}_0)}$ (Of course this operation really only applies here if r^i_0 is the empty sequence). Let II play

$$V_0 = p[(T_{r^0_0})_{(s_0,t^{0,0}_0)}] \cap \ldots \cap p[(T_{r^{m_0}_0})_{(s_0,t^{0,m_0}_0)}] \cap N_{u_0}$$

Let player I's answer be (x_1, U_1) with $x_1 \in U_1 \subseteq V_0$.
Since $x_1 \in V_0$ there exists for $0 \leq i \leq m_0$ an $\overline{\eta}^i_0 \in \kappa^\omega$ such that $(x_1, \overline{\eta}^i_0) \in (T_{r^i_0})_{(s_0,t^{0,i}_0)}$. Set $s_1 = x_1|2, t^{0,i}_1 = \overline{\eta}^i_0|2$. Then $s_0 \subseteq s_1, t^{0,i}_0 \subseteq t^{0,i}_1$ and $x_1 \in p[(T_{r^i_0})_{(s_1,t^{0,i}_0)}]$.
Choose now again a basic set $p[T_{r^0_1}] \cap p[T_{r^1_1}] \cap \ldots \cap p[T_{r^{m_1}_1}] \cap N_{u_1}$ such that this is a subset of U_1 and contains x_1. Let η^i_1 be in κ^ω for $0 \leq i \leq m_1$ such that $(x_1, \eta^i_1) \in T_{r^i_1}$. Set $t^{1,i}_0 = \eta^i_1|1$. Then $x_1 \in p[(T_{r^i_1})_{(s_0,t^{1,m_1}_0)}]$. In particular $x_1 \in p[(T_{r^0_1})_{(s_0,t^{1,0}_0)}] \cap \ldots \cap p[(T_{r^{m_1}_1})_{(s_0,t^{1,m_1}_0)}] \cap N_{u_1}$. Now

$$\begin{aligned} V_1 &= p[(T_{r^0_0})_{(s_1,t^{0,1}_1)}] \cap \ldots \cap p[(T_{r^{m_0}_0})_{(s_0,t^{0,m_0}_0)}] \cap N_{u_0} \cap p[(T_{r^0_1})_{(s_0,t^{1,0}_0)}] \cap \ldots \\ &\quad \cap p[(T_{r^{m_1}_1})_{(s_0,t^{1,m_1}_0)}] \cap N_{u_1} \end{aligned}$$

is a legal move for player II.
Proceeding this way, when I plays $(x_0, U_0), (x_1, U_1), \ldots$ II produces $V_0, V_1, \ldots$ with $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \ldots, x_n \in V_n$ and moreover one defines for each n basic sets A_n with $x_n \in A_n = p[T_{r^0_n}] \cap \ldots \cap p[T_{r^{m_n}_n}] \cap N_{u_n} \subseteq U_n$ and sequences $s_0 \subseteq s_1 \subseteq s_2 \ldots, t^{n,0}_o \subseteq t^{n,0}_1 \subseteq t^{n,0}_2 \ldots, \ldots, t^{n,m_n}_0 \subseteq t^{n,m_n}_1 \subseteq t^{n,m_n}_2 \subseteq \ldots$ with $(s_k, t^{n,i}_k) \in T_{r^i_n}, 0 \leq i \leq m_n$, such that for each k the finite sequences $s_k, t^{n,i}_k$

have length $k+1$ and

$$\begin{aligned} V_k \;=\;& p[(T_{r_0^0})_{(s_k,t_k^{0,0})}] \cap \ldots \cap p[(T_{r_0^{m_0}})] \cap N_{u_0} \\ & \cap p[(T_{r_1^0})_{(s_{k-1},t_{k-1}^{1,0})}] \cap \ldots \cap p[(T_{r_1^{m_1}})_{(s_{k-1},t_{k-1}^{1,m_1})}] \cap N_{u_1} \\ & \cap \ldots \\ & \cap p[(T_{r_k^0})_{(s_0,t_0^{k,0})}] \cap \ldots \cap p[(T_{r_k^{m_k}})_{(s_0,t_0^{k,m_k})}] \cap N_{u_k} \end{aligned}$$

Let $x = \bigcup_k s_k$.
Claim: $x \in \bigcap_n A_n = \bigcap_n V_n$
Proof: Consider $A_n = p[T_{r_n^0}] \cap \ldots \cap p[T_{r_n^{m_n}}] \cap N_{u_n}$. Let $0 \leq i \leq m_n$. Let $\eta_n^i = \bigcup_k t_k^{n,i}$. We have $(s_k, t_k^{n,i}) \in T_{r_n^i}$ for all k. Therefore $(x, \eta_n^i) \in [T_{r_n^i}]$ and thus $x \in p[T_{r_n^i}]$. It remains to show that $x \in N_{u_n}$. For this let S be the full tree on ω^ω. For a sequence $s \in \omega^{<\omega}$ we have $[S_s] = N_s$ our basic set in the Baire space. It suffices to show now that $s_k \in S_{u_n}$ for every k. Note that $s_n = x_n|n+1$. Since $x_n \in N_{u_n}$ we have s_n and u_n compatible, therefore $s_k \in S_{u_n}$ for $k \leq n$. Assume towards a contradiction that for $k > n$ the sequence s_k is not in S_{u_n}. This implies $x_k \notin N_{u_n}$. In particular, $x_k \notin V_n$, but $x_k \in V_k \subseteq V_n$, a contradiction. □

3 Characterization of projective sets by finer topologies

In this chapter we want to give a sketch of the proof for the converse of Theorem 2.3. Unfortunately we are not able to prove this under the same assumptions as Theorem 2.3. Instead of PD we need the much stronger axiom $\mathrm{AD}_{\mathbb{R}}$. We put down the theorem:

Theorem 3.1 (ZF+DC+AD$_{\mathbb{R}}$). *Let $(X, \mathcal{T})$ be a Polish space, $A \subseteq X$ and $n \in \omega$. If there exists a topology t on A such that*

- *t is finer that $\mathcal{T}$*
- *t is strong Choquet*
- *t has a basis of length less than $\boldsymbol{\delta}_n^1$*

then A is a $\boldsymbol{\Sigma}_n^1$ subset of $(X, \mathcal{T})$.

3.1 Sketch of the proof

Fix now an $n \in \omega$. Note first that under AD the projective ordinals are successor cardinals, see for this [Kech78]. So in the spirit of our theorem we are interested in the predecessors of the $\boldsymbol{\delta}_n^1$. Let κ be the predecessor of $\boldsymbol{\delta}_n^1$. Second we claim that there exists a $\boldsymbol{\Delta}_n^1$-scale $(\varphi_i)_{i\in\omega}$ on a $\boldsymbol{\Delta}_n^1$ subset $W \subseteq \mathcal{N}$ such that φ_0 is a surjection on κ and the length of all other norms of the scale is less or equal κ. We code elements from κ by φ_0.
Let now $X, \mathcal{T}, A, t$ be given. Let $\mathcal{B} = \{B_i \mid i \in \omega\}$ be a basis for $(X, \mathcal{T})$.

Let d be a complete metric compatible with $\mathcal{T}$.
Let $\mathcal{C} = \{C_\xi \mid \xi < \kappa\}$ be a basis for (A, t).
Let σ be a winning strategy for player II in the strong Choquet game $G^s_{(A,t)}$ that plays elements of $\mathcal{C}$ with diameter less than $\frac{1}{i}$ in the i-th move.
Let $W \subseteq \mathcal{N}$ be a $\mathbf{\Delta}^1_n$ set, $\varphi_i : W \longrightarrow \kappa$ a $\mathbf{\Delta}^1_n$ scale on W such that $im(\varphi_0) = \kappa$.

Definition 3.1.1. A countable subset $S \subseteq \kappa$ is called **ξ-honest**, if $\xi \in S$ and there exists an $w \in W$ such that $\varphi_0(w) = \xi$ and $\varphi_i(w) \in S$ for all $i \geq 1$.
S is called **honest**, if it is ξ-honest for all $\xi \in S$.

We need now the following Lemma. A proof can be found in [KW80].

Lemma 3.1.2. *There exists a Lipschitz function $F : \kappa^\omega \longrightarrow \omega^\omega$ such that*

(a) $\forall f \in \kappa^\omega \, \forall i \, (F(f))_i \in W$

(b) $\forall f \in \kappa^\omega : \ \{f(0), f(1), \dots\}$ *is honest* $\ \Rightarrow \ \forall i \in \omega \, \varphi_0((F(f))_i) = f(i)$

Fix such an F.

A game G

We define now a game G in the following way:

$$\begin{array}{lccccc} \text{I} & \alpha_0, \xi_0, x_0 & & \alpha_1, \xi_1, x_1 & & \dots \\ \text{II} & & \beta_0, \eta_0 & & \beta_1, \eta_1 & \dots \end{array}$$

where $\alpha_i, \xi_i, \beta_i, \eta_i \in \kappa$ and $x_i \in A$ for $i \in \omega$.
The players must obey the following **rule** $\mathcal{R}$:
The players must play so that the finite initial segments of $(*)$

$$\begin{array}{lccccc} \text{I} & (C_{\xi_0}, x_0) & & (C_{\xi_1}, x_1) & & \dots \\ \text{II} & & C_{\eta_0} & & C_{\eta_1} & \dots \end{array}$$

are legal moves in the strong Choquet game for (A, t) with $\mathrm{diam}(C_{\eta_i}) < \frac{1}{i}$.
The first player to fail loses.
The **payoff set** is the following:
Let us assume no player violates the rule.
Let $f = (\alpha_0, \xi_0, \beta_0, \eta_0, \alpha_1, \xi_1, \beta_1, \eta_1, \dots) \in \kappa^\omega$.
Set $\hat{\eta}_i(f) = \varphi_0((F(f))_{4i+3})$. Let $P = \{f \in \kappa^\omega \mid \exists x \in A \forall m \in \omega x \in C_{\hat{\eta}_m}(f)\}$.
II wins the round of the game $\Leftrightarrow f \in P$.
Remark: The definition of F implies that if f is honest, then

$$\hat{\eta}_i(f) = \varphi_0((F(f))_{4i+3} = f(4i+3) = \eta_i$$

Hence if f is honest, then $f \in P \Leftrightarrow$ II wins the round of the strong Choquet game $(*)$

Lemma 3.1.3. *II has a winning quasi-strategy τ for G with the following property:*
Suppose $s = ((\alpha_0, \xi_0, x_0), (\beta_0, \eta_0), \dots, \alpha_{m-1}, \xi_{m-1}, x_{m-1}))$
and $s' = ((\alpha_0, \xi_0, x'_0), (\beta_0, \eta_0), \dots, \alpha_{m-1}, \xi_{m-1}, x'_{m-1}))$
are two positions in G which are legal and consistent with τ. Let $(\beta_{m-1}, \eta_{m-1}) \in \kappa \times \kappa$. If $x_{m-1}, x'_{m-1} \in C_{\eta_{m-1}}$ and $s^\wedge(\beta_{m-1}, \eta_{m-1})$ is consistent with τ, then $s'^\wedge(\beta_{m-1}, \eta_{m-1})$ is also consistent with τ.

The tree T_0
Let τ be a winning strategy for II as in the above Lemma 3.1.3. τ is essentially a tree on $(\kappa \times \kappa \times A \times \kappa \times \kappa)$. Let us call this tree T_0 and we assume all positions in T_0 legal, that is if I loses by violating the rule $(*)$ we remove this finite branch from the tree.

The tree T_1
Define a tree T_1 on κ^4 by

$$\begin{aligned} ((\alpha_0, \xi_0, \beta_0, \eta_0), \dots, (\alpha_{m-1}, \xi_{m-1}, \beta_{m-1}, \eta_{m-1})) \in T_1 \\ \Leftrightarrow \exists x_0, \dots x_{m-1} \in A \text{ such that} \\ ((\alpha_0, \xi_0, x_0, \beta_0, \eta_0), \dots, (\alpha_{m-1}, \xi_{m-1}, x_{m-1}, \beta_{m-1}, \eta_{m-1})) \in T_0 \end{aligned}$$

Lemma 3.1.4. *Let $f = (\alpha_0, \xi_0, \beta_0, \eta_0, \alpha_1, \xi_1, \beta_1, \eta_1, \dots)$ be an infinite branch in T_1. If f is honest, then $\bigcap_i C_{\eta_i}$ contains a point $x_f \in A$.*

The tree T
Let finally T be the following tree on $\omega \times \kappa^4$:
$((i_0, \alpha_0, \xi_0, \beta_0, \eta_0), \dots, (i_{m-1}, \alpha_{m-1}, \xi_{m-1}, \beta_{m-1}, \eta_{m-1})) \in T \Leftrightarrow$

(i) For all k $\mathrm{diam}(B_{i_k}) < \frac{1}{k}$

(ii) For all k $\overline{B_{i_{k+1}}} \subseteq B_{i_k}$

(iii) $((\alpha_0, \xi_0, \beta_0, \eta_0), \dots, (\alpha_{m-1}, \xi_{m-1}, \beta_{m-1}, \eta_{m-1})) \in T_1$

(iv) For all k $B_{i_k} \cap C_{\eta_k} \neq \emptyset$

The set A'
Define $A' \subseteq X$ by

$$\begin{aligned} x \in A' \quad \Leftrightarrow \quad & \exists y \in \omega^\omega \exists \overline{\alpha}, \overline{\xi}, \overline{\beta}, \overline{\eta} \in \kappa^\omega \\ & [(y, \overline{\alpha}, \overline{\xi}, \overline{\beta}, \overline{\eta}) \in T \text{ and } x \in \bigcap_m B_{y(m)} \\ & \text{and } \{\overline{\alpha}(m), \overline{\xi}(m), \overline{\beta}(m), \overline{\eta}(m) \mid m \in \omega\} \text{ is honest }] \end{aligned}$$

We claim that this set A' is $\mathbf{\Sigma}^1_n$. To see this we use the powerful Coding Lemma from Moschovakis that assures that T is $\mathbf{\Delta}^1_n$-in-the-codes.

Now we finish the proof by showing that $A = A'$.
$\boldsymbol{A' \subseteq A}$
Let $x \in A'$. Let $y, \overline{\alpha}, \overline{\xi}, \overline{\beta}, \overline{\eta}$ witness that $x \in A'$.

Let $f = (\alpha_0, \xi_0, \beta_0, \eta_0, \alpha_1, \xi_1, \beta_0, \eta_0, \dots)$. Then $(y, f) \in [T]$ and f is honest. By definition of T we have $f \in [T_1]$. So by Lemma 3.1.4 there exists an $x_f \in A$ such that $x_f \in \bigcap_m C_{\eta_m}$. Since x is the only point in $\bigcap_m B_{y(m)}$ (by (i) of the definition of T) it suffices to show that $x_f \in \bigcap_m B_{y(m)}$ because then $x = x_f \in A$.
Claim: $x_f \in \bigcap_m B_{y(m)}$
Proof: Assume not. So there is an $m \in \omega$ with $x_f \notin B_{y(m)}$. Therefore $d(B_{y(m)}, x_f) > 0$, let us say $d(B_{y(m)}, x_f) = \varepsilon > 0$. But now there exists an $k > m$ with $\mathrm{diam}(C_{\eta_k}) < \frac{\varepsilon}{4}$ (because of the rule $\mathcal{R}$ in the definition of G) and $x_f \in C_{\eta_k}$. Also $\mathrm{diam}(B_{y(k)}) < \frac{\varepsilon}{4}$ by (i) of the definition of T and $B_{y(k)} \subseteq B_{y(m)}$. By (iv) of the definition there is an $z \in B_{y(k)} \cap C_{\eta_k}$. Since $z, x_f \in C_{\eta_k}$ we have $d(z, x_f) < \frac{\varepsilon}{4}$. But also $z \in B_{y(k)} \subseteq B_{y(m)}$ and hence $d(B_{y(m)}, x_f) = \inf\{d(z', x_f) \mid z' \in B_{y(m)}\} \leq d(z, x_f) < \frac{\varepsilon}{4}$. This contradicts $d(B_{y(m)}, x_f) > \varepsilon$.

$A \subseteq A'$

Let $x \in A$. Let $h : \omega^\omega \twoheadrightarrow A$ be a coding of A by the reals.
Let Z be a countable subset of ω^ω such that

1. there is an $\overline{x} \in Z$ with $h(\overline{x}) = x$
2. the set of ordinals less than κ with codes in Z is honest
3. $\exists \nu < \kappa$ such that $x \in C_\nu$ and ν has a code in Z
4. every position in G consistent with τ with all moves from elements coded from Z has an extension consistent with τ and with all moves from elements coded from Z

To prove the existence of such a set we define by recursion countable sets Z_i for $i \in \omega$ (using AC_ω in every other step of the construction) and the take Z to be the union of all Z_i. To make an easy thing not look to complicated (by jumping back and forth between the "coded game" and G) note that if there is a countable set of ordinals less than κ one can get by AC_ω a countable subset of W coding these ordinals. Simultaneously one can get for a countable subset of A a countable set of reals coding the elements of the subset throug h.
Let $\overline{x} \in \omega^\omega$ such that $h(\overline{x}) = x \in A$ and let $\overline{y} \in W$ such that $\varphi_0(\overline{y}) = \nu$ and $x \in C_\nu$. Set $Z_0 = \{\overline{x}, \overline{y}\}$.
Let now Z_i countable be given for an $i \in \omega$. We want to define Z_{i+1}. Let $T_0 \restriction Z_i$ be the tree on $\kappa \times \kappa \times A \times \kappa \times \kappa$ restricted to elements coded by Z_i. Consider in $T_0 \restriction Z_i$ the countable set of all finite sequences $s \in T_0 \restriction Z_i$ which have no proper extension. Let s be such a finite sequence and let $s^\wedge = \{(\alpha, \xi, x, \beta, \eta) \in \kappa \times \kappa \times A \times \kappa \times \kappa \mid s^\wedge(\alpha, \xi, x, \beta, \eta) \in T_0\}$. By AC_ω we find a countable set $\tilde{Z}_{i+1}$ such that for all such s there is a proper extension of s from $s^\wedge$ in $\tilde{Z}_{i+1}$. Again by AC_ω and the above remark there is a countable set Z'_{i+1} of reals coding these elements. Choose further codes for the ordinals $\varphi_k(\overline{z})$ for $\overline{z} \in Z_i \cap W, k \in \omega$ and let M_i be the set of these codes. Then let $Z_{i+1} = Z_i \cup M_i \cup Z'_{i+1}$. Z_{i+1} is countable. Set $Z = \bigcup_{i \in \omega} Z_i$.
By definition of Z_0 1. and 3. are satisfied. If $\overline{w} \in Z_i$ for some $i \in \omega$ then for $\varphi_k(\overline{w})$ there is a code in $M_i \subseteq Z_{i+1}$ for all $k \in \omega$. Hence the set of ordinals less

than κ with codes in Z is honest. If s is a position in G consistent with τ and all moves are in Z then there is an $i \in \omega$ such that $s \in T_0 \restriction Z_i$ and there is an extension so $s \in T_0 \restriction Z_{i+1} \subseteq T_0 \restriction Z$. So this extension is also consistent with τ. Using such an Z there is a round of the game G such that

(i) all moves are in Z (that is, coded by Z)

(ii) the round is consistent with II's winning quasi-strategy τ for G

(iii) $x_m = x$ for all m (so player I always plays the same element $x \in A$)

(iv) $\xi_0 = \nu, \xi_{m+1} = \eta_{m+1}$

(v) $\alpha_0, \alpha_1, \dots$ is an enumeration of the ordinals less than κ with codes in Z

Since such a round $g = (\alpha_0, \xi_0, x, \beta_0, \eta_0, \alpha_1, \xi_1, x, \beta_1, \eta_1, \dots)$ of G is consistent with τ, we know that g is an infinite branch in T_0. By definition of T_1 the sequence $f = (\alpha_0, \xi_0, \beta_0, \eta_0, \alpha_1, \xi_1, \beta_1, \eta_1, \dots)$ is an infinite branch through T_1. Using this f we want to get an infinite branch in T. Property (iii) in the definition of T is allready satisfied. Now let $i_0, i_1, \dots$ be such that $x \in \bigcap_m B_{i_m}$ and (i) and (ii) of the definition of T holds. Since II wins the round g of G all C_{η_m} are legal moves of II and therefore are moves in the strong Choquet game. So $x \in C_{\eta_m}$ for all m. This implies $B_{i_m} \cap C_{\eta_m} \neq \emptyset$ for all m and therefore (iv) in the definition of T holds. So $(i_0, \alpha_0, \xi_0, \beta_0, \eta_0, i_1, \alpha_1, \xi_1, \beta_1, \eta_1, \dots) \in [T]$. To show now that $x \in A'$ it remains to show $\{\alpha_m, \xi_m, \beta_m, \eta_m \mid m \in \omega\}$ is honest. But all this elements were chosen in Z and the α_m are all ordinals less than κ coded by Z and this set is honest by the construction of Z.

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