Varia of ideals and ERs

Vladimir Kanovei

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Abstract

This a survey of some basic results on ideals and ERs, first of all those which admit rather short proofs.

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1 Technical introduction

1.a Notation

- $\mathbb{N} = \{0, 1, 2, ...\}$: natural numbers. $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$.
- $\mathbb{N}^\mathbb{N}$ is *the Baire space*. If $s \in \mathbb{N}^{<\omega}$ (a finite sequence of natural numbers) then $\mathcal{O}_s(\mathbb{N}^\mathbb{N}) = \{x \in \mathbb{N}^\mathbb{N} : s \subset x\}$, a basic clopen nbhd in $\mathbb{N}^\mathbb{N}$.
- $X \subseteq^* Y$ means that the difference $X \setminus Y$ is finite.
- If a basic set A is fixed then $\complement X = X^\complement = A \setminus X$ for any $X \subseteq A$.
- If $X \subseteq A \times B$ and $a \in A$ then $(X)_a = \{b : \langle a, b\rangle \in X\}$, a *cross-section.*
- $\#X = \#(X)$ is the number of elements of a finite set X.
- $f"X = \{f(x) : x \in X \cap \mathtt{dom} f\}$, the f-*image* of X.
- Δ is the symmetric difference.
- $\exists^\infty x$ ... means: "there exist infinitely many x such that ...",
  $\forall^\infty x$ ... means: "for all but finitely many x, ... holds".
- *An ideal on a set* A is, as usual, any set $\emptyset \neq \mathcal{I} \subseteq \mathcal{P}(A)$, closed under $\cup$ and satisfying $x \in \mathcal{I} \Longrightarrow y \in \mathcal{I}$ whenever $y \subseteq x \subseteq A$. Thus, any ideal contains $\emptyset$. We'll usually consider only *nontrivial* ideals, *i.e.*, those which contain all singletons $\{a\} \subseteq A$ and do not contain A, *i.e.*, $\mathcal{P}_{\mathtt{fin}}(A) \subseteq \mathcal{I} \subsetneqq \mathcal{P}(A)$.
- If $\mathcal{I}$ is an ideal on a set A then let $\mathsf{E}_\mathcal{I}$ be an equivalence relation (ER, for brevity) on $\mathcal{P}(A)$, defined as follows: $X \mathrel{\mathsf{E}_\mathcal{I}} Y$ iff $X \,\Delta\, Y \in \mathcal{I}$.
- If E is an ER on a set X then $[y]_\mathsf{E} = \{x \in X : y \mathrel{\mathsf{E}} x\}$ for any $y \in X$ (the E-*class* of x) and $[Y]_\mathsf{E} = \bigcup_{y\in Y} [y]_\mathsf{E}$ (the E-*saturation* of Y) for $Y \subseteq X$. A set $Y \subseteq X$ is E-*invariant* if $[Y]_\mathsf{E} = Y$.
- If E is an ER on a set X then a set $Y \subseteq X$ is *pairwise* E-*equivalent,* resp., *pairwise* E-*inequivalent*, if $x \mathrel{\mathsf{E}} y$, resp., $x \not\mathrel{\mathsf{E}} y$ holds for all $x, y \in Y$.
- If X, Y are sets and E any binary relation then $X \mathrel{\mathsf{E}} Y$ means that we have both $\forall x \in X\, \exists y \in Y\, (x \mathrel{\mathsf{E}} y)$ and $\forall y \in Y\, \exists x \in X\, (x \mathrel{\mathsf{E}} y)$.

1.b Descriptive set theory

A basic knowledge of Borel and projective hierarchy, both classical and effective, in *the Baire space* $\mathbb{N}^\mathbb{N}$ and other (recursively presented, in the effective case) Polish spaces, is assumed.

A map f (between Borel sets in Polish spaces) is *Borel* iff its graph is a Borel set iff all f-preimages of open sets are Borel. A map f is *Baire measurable* (*BM*, for brevity) iff all f-preimages of open sets are Baire measurable sets.

1.c Trivia of "effective" descriptive set theory

Apart of the very common knowledge, the whole instrumentarium of "effective" descriptive set theory employed in the study of reducibility of ideals and ERs, can be summarized in, surprisingly, a very few key facts.

Reduction and Separation: If X and Y are Π^1_1 sets of a recursively presented Polish space then there disjoint Π^1_1 sets $X' \subseteq X$ and $Y' \subseteq Y$ with $X' \cup Y' = X \cup Y$. If X and Y are disjoint Σ^1_1 sets of a recursively presented Polish space then there is a Δ^1_1 set Z with $X \subseteq Z$ and $Y \cap Z = \emptyset$.

Countable-to-1 Projection: If P is a Δ^1_1 subset of the product $\mathbb{X} \times \mathbb{Y}$ of two recursively presented Polish spaces and for any $x \in \mathbb{X}$ the cross-section $P_x = \{y : P(x,y)\}$ is at most countable then $\operatorname{dom} P$ is a Δ^1_1 set in $\mathbb{X}$.

It follows that images of Δ^1_1 sets via countable-to-1, in particular, 1-to-1 Δ^1_1 maps are Δ^1_1 sets, while images via arbitrary Δ^1_1 maps are, generally, Σ^1_1.

Countable-to-1 Enumeration: If P, $\mathbb{X}$, $\mathbb{Y}$ are as in Countable-to-1 Projection then there exists a Δ^1_1 map $f : \operatorname{dom} P \times \mathbb{N} \to \mathbb{Y}$ such that $P_x = \{f(x,n) : n \in \mathbb{N}\}$ for all $x \in \operatorname{dom} P$.

In this case, P can be uniformized by a Δ^1_1 set, of course, but there is a much more useful uniformization theorem !

Kreisel Selection: If $\mathbb{X}$ is a recursively presented Polish space, $P \subseteq \mathbb{X} \times \mathbb{N}$ is a Π^1_1 set, and $X \subseteq \operatorname{dom} P$ is a Δ^1_1 set then there is a Δ^1_1 function $f : X \to \mathbb{N}$ such that $\langle x, f(x) \rangle \in P$ for al $x \in X$.

The proof is surprisingly simple. Let $Q \subseteq P$ be a Π^1_1 set which uniformizes P. For any $x \in X$ let $f(x)$ be the only n with $\langle x, n \rangle \in Q$. Immediately, (the graph of) f is Π^1_1, however, as $\operatorname{ran} f \subseteq \mathbb{N}$, we have $f(x) = n \iff \forall m \neq n\, (f(x) \neq m)$ whenever $x \in X$, which demonstrates that f is Σ^1_1 as well.

Δ^1_1 Enumeration: If $\mathbb{X}$ is a recursively presented Polish space then there exist Π^1_1 sets $C \subseteq \mathbb{N}$ and $W \subseteq \mathbb{N} \times \mathbb{X}$ and a Σ^1_1 set $W' \subseteq \mathbb{N} \times \mathbb{X}$ such that $W_e = W'_e$ for any $e \in C$ and a set $X \subseteq \mathbb{X}$ is Δ^1_1 iff there is $e \in C$ such that $X = W_e = W'_e$. (Here $W_e = \{x : W(e,x)\}$ and similarly W'_e.)

There is a generalization useful for relativised classes $\Delta^1_1(y)$.

Relativized Δ^1_1 Enumeration: If $\mathbb{X}, \mathbb{Y}$ are recursively presented Polish spaces then there exist Π^1_1 sets $C \subseteq \mathbb{Y} \times \mathbb{N}$ and $W \subseteq \mathbb{Y} \times \mathbb{N} \times \mathbb{X}$ and a Σ^1_1 set $W' \subseteq \mathbb{Y} \times \mathbb{N} \times \mathbb{X}$ such that $W_{ye} = W'_{ye}$ for any $\langle y, e \rangle \in C$ and, for any $y \in \mathbb{Y}$, a set $X \subseteq \mathbb{X}$ is $\Delta^1_1(y)$ iff there is e such that $\langle y, e \rangle \in C$ and $X = W_{ye} = W'_{ye}$. ($W_{ye} = \{x : W(y,e,x)\}$ and similarly W'_{ye}.)

Universal Sets: Suppose that $\mathbb{X}$ is a recursively presented Polish space. Then there exists a Π^1_1 set $U \subseteq \mathbb{N} \times \mathbb{X}$, called: *a universal Π^1_1 set*, such that for any Π^1_1 set $X \subseteq \mathbb{X}$ there is an index n with $X = U_n = \{x : \langle n, x \rangle \in U\}$. There also exists a Σ^1_1 set $V \subseteq \mathbb{N} \times \mathbb{X}$, called: *a universal Σ^1_1 set*, such that for any Σ^1_1 set $X \subseteq \mathbb{X}$ there is an index n with $X = V_n = \{x : \langle n, x \rangle \in V\}$. (In fact we can take $V = (\mathbb{N} \times \mathbb{X}) \setminus U$.)

If a universal Π^1_1 set U is fixed then a collection $\mathcal{A}$ of Π^1_1 sets $X \subseteq \mathbb{X}$ is Π^1_1 *in the codes* if $\{n : U_n \in \mathcal{A}\}$ is a Π^1_1 set. Similarly, if a universal Σ^1_1 set V is fixed then a collection $\mathcal{A}$ of Σ^1_1 sets $X \subseteq \mathbb{X}$ is Π^1_1 *in the codes* if $\{n : V_n \in \mathcal{A}\}$ is a Π^1_1 set.

The following important result involves the notion of "good" universal Π^1_1 and Σ^1_1 sets, which we cannot comment here because it is connected with some quite special topics of recursion theory, the only point which is essential for applications is that "good" universal Π^1_1 and Σ^1_1 sets do exists, while, for instance, verification that a given family is Π^1_1 in the codes does not depend on the choice of a universal set.

In this sense, the Reflection principle is less elementary than the results cited above, but it is very useful because it allows to "compress" some sophisticated arguments with multiple applications of Separation and Kreisel selection.

Reflection: Assume that $\mathbb{X}$ is a recursively presented Polish space.

Π^1_1 *form*: Let a collection $\mathcal{A}$ of Π^1_1 sets $X \subseteq \mathbb{X}$ be Π^1_1 in the codes, in the sense of a fixed "good" universal Π^1_1 set $U \subseteq \mathbb{N} \times \mathbb{X}$. Then for any $X \in \mathcal{A}$ there is a Δ^1_1 set $Y \in \mathcal{A}$ with $Y \subseteq X$.

Σ^1_1 *form*: Let a collection $\mathcal{A}$ of Σ^1_1 sets $X \subseteq \mathbb{X}$ be Π^1_1 in the codes, in the sense of a fixed "good" universal Σ^1_1 set $V \subseteq \mathbb{N} \times \mathbb{X}$. Then for any $X \in \mathcal{A}$ there is a Δ^1_1 set $Y \in \mathcal{A}$ with $X \subseteq Y$.

One of (generally, irrelevant here) consequences of this principle is that the set of all codes of a properly Π^1_1 set or properly Σ^1_1 set is never Π^1_1.

1.d Polish–like families and the Gandy – Harrington topology

It will be often the case that, given a family $\mathcal{F}$ of sets (in a Polish space), we look for a useful sufficient condition for a decreasing sequence $F_0 \supseteq F_1 \supseteq F_2 \supseteq \dots$ of sets in $\mathcal{F}$ to have a non-empty intersection.

Definition 1 A family $\mathcal{F}$ is *Polish–like* if there exists a countable collection $\{\mathcal{D}_n : n \in \mathbb{N}\}$ of dense subsets $\mathcal{D}_n \subseteq \mathcal{F}$ such that we have $\bigcap_n F_n \neq \emptyset$ whenever $F_0 \supseteq F_1 \supseteq F_2 \supseteq \dots$ is a decreasing sequence of sets $F_n \in \mathcal{F}$ which intersects every $\mathcal{D}_n$. (A set $\mathcal{D} \subseteq \mathcal{F}$ is *dense* if $\forall F \in \mathcal{F}\, \exists D \in \mathcal{D}\, (D \subseteq F)$.) $\square$

For instance if $\mathcal{X}$ is a Polish space then the collection of all its non-empty closed sets is Polish–like, for take $\mathcal{D}_n$ to be all closed sets of diameter $\le n^{-1}$. The Polish–likeness is similar to the Choquet property but is somewhat more convenient to use in securing the nonemptiness of intersections.

Theorem 2 (Kanovei [25], Hjorth [17]) *The collection $\mathcal{F}$ of all non-empty Σ^1_1 subsets of $\mathbb{N}^{\mathbb{N}}$ is Polish–like.* $\square$

Proof For any $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ define $\mathtt{pr}\, P = \{x : \exists y\, P(x,y)\}$ (the projection). If $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ and $s, t \in \mathbb{N}^{<\omega}$ then let $P_{st} = \{\langle x,y\rangle \in P : s \subset x \wedge t \subset y\}$. Let $\mathcal{D}(P,s,t)$ be the collection of all Σ^1_1 sets $\emptyset \ne X \subseteq \mathbb{N}^{\mathbb{N}}$ such that either $X \cap \mathtt{pr}\, P_{st} = \emptyset$ or $X \subseteq \mathtt{pr}\, P_{s^\wedge i,\, t^\wedge j}$ for some i, j. (Note that in the "or" case i is unique but j may be not unique.) Let $\{\mathcal{D}_n : n \in \mathbb{N}\}$ be an arbitrary enumeration of all sets of the form $\mathcal{D}(P,s,t)$, where $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is Π^0_1. Note that in this case all sets of the form $\mathtt{pr}\, P_{st}$ are Σ^1_1 subsets of $\mathbb{N}^{\mathbb{N}}$, therefore, $\mathcal{D}(P,s,t)$ is easily a dense subset of $\mathcal{F}$, so that all $\mathcal{D}_n \subseteq \mathcal{F}$ are dense.

Now consider a decreasing sequence $X_0 \supseteq X_1 \supseteq ...$ of non-empty Σ^1_1 sets $X_k \subseteq \mathbb{N}^{\mathbb{N}}$, which intersects every $\mathcal{D}_n$; prove that $\bigcap_n X_n \ne \emptyset$. Call a set $X \subseteq \mathbb{N}^{\mathbb{N}}$ *positive* if there is n such that $X_n \subseteq X$. For any n, fix a Π^0_1 set $P^n \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $X_n = \mathtt{pr}\, P^n$. For any $s, t \in \mathbb{N}^{<\omega}$, if $\mathtt{pr}\, P^n_{st}$ is positive then, by the choice of the sequence of X_n, there is a unique i and some j such that $\mathtt{pr}\, P^n_{s^\wedge i,\, t^\wedge j}$ is also positive. It follows that there is a unique $x = x_n \in \mathbb{N}^{\mathbb{N}}$ and some $y = y_n \in \mathbb{N}^{\mathbb{N}}$ (perhaps not unique) such that $\mathtt{pr}\, P^n_{x\restriction k,\, y\restriction k}$ is positive for any k. As P^n is closed, we have $P^n(x,y)$, hence, $x_n = x \in X_n$.

It remains to show that $x_m = x_n$ for any two different m, n. But this follows from the observation that if both P_{st} and $Q_{s't'}$ are positive then we have either $s \subseteq s'$ or $s' \subseteq s$. $\square$

The collection of all non-empty Σ^1_1 subsets of $\mathbb{N}^{\mathbb{N}}$ is a base of the *Gandy – Harrington topology*, which has many remarkable applications in descriptive set theory. This topology is easily not Polish, even not metrizable at all, yet it shares the following important property of Polish topologies:

Corollary 3 *The Gandy – Harrington topology is Baire, i.e., every comeager set is dense.*

Proof This can be proved using Choquet property of the topology, see [13], however, the Polish–likeness (Theorem 5efs11pol) also immediately yields the result. $\square$

2 Reducibility

There are several reasonable ways to compare ERs, usually formalized in terms of existence of a *reduction*, *i.e.*, a map of certain kind which allows to derive one of the ERs from the other one. *Borel reducibility* $\le_{\mathrm{B}}$ is the key one, yet there are several special types of $\le_{\mathrm{B}}$, in particular, those induced by a low-level maps, useful in many cases Generally, the most of research on reducibility of Borel ERs or ideals is concentrated around the following notions of reducibility.

2.a Borel reducibility

If E and F are ERs on Polish spaces resp. $\mathbb{X}, \mathbb{Y}$, then

* $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$ means that there is a Borel map $\vartheta : \mathbb{X} \to \mathbb{Y}$ (called *reduction*) such that $x \mathrel{\mathsf{E}} y \iff \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all $x, y \in \mathbb{X}$;
* $\mathsf{E} \sim_{\mathrm{B}} \mathsf{F}$ iff $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$ and $\mathsf{F} \le_{\mathrm{B}} \mathsf{E}$, $\mathsf{E} <_{\mathrm{B}} \mathsf{F}$ iff $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$ but not $\mathsf{F} \le_{\mathrm{B}} \mathsf{E}$;
* $\mathsf{E} \sqsubseteq_{\mathrm{B}} \mathsf{F}$ means that there is a Borel *embedding*, *i.e.*, a $1-1$ reduction;
* $\mathsf{E} \sqsubseteq^{\mathrm{i}}_{\mathrm{B}} \mathsf{F}$ means that there is a Borel *invariant* embedding, *i.e.*, an embedding ϑ such that $\mathtt{ran}\,\vartheta = \{\vartheta(x) : x \in \mathbb{X}\}$ is an F-*invariant* set (meaning that the F-*saturation* $[\mathtt{ran}\,\vartheta]_{\mathsf{F}} = \{y' : \exists x\, (y \mathrel{\mathsf{F}} \vartheta(x))\}$ equals $\mathtt{ran}\,\vartheta$);
* $\mathsf{E} \le_{\mathrm{C}} \mathsf{F}$, $\mathsf{E} \sqsubseteq_{\mathrm{C}} \mathsf{F}$, $\mathsf{E} \sqsubseteq^{\mathrm{i}}_{\mathrm{C}} \mathsf{F}$ mean that there is a <u>continuous</u> resp. reduction, embedding, invariant embedding.

Sometimes they write $\mathbb{X}/\mathsf{E} \le_{\mathrm{B}} \mathbb{Y}/\mathsf{F}$ instead of $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$.

***Borel reducibility of ideals*:** $\mathcal{I} \le_{\mathrm{B}} \mathcal{J}$ iff $\mathsf{E}_{\mathcal{I}} \le_{\mathrm{B}} \mathsf{E}_{\mathcal{J}}$. Thus it is required that there is a Borel map $\vartheta : \mathcal{P}(A) \to \mathcal{P}(B)$ such that $x \,\Delta\, y \in \mathcal{I}$ iff $\vartheta(x) \,\Delta\, \vartheta(y) \in \mathcal{J}$. (Here $\mathcal{I}$ is an ideal on A and $\mathcal{J}$ is an ideal on B.)

Versions $\mathcal{I} \le_{\mathrm{C}} \mathcal{J}$, $\mathcal{I} \sqsubseteq_{\mathrm{B}} \mathcal{J}$, $\mathcal{I} \sqsubseteq_{\mathrm{C}} \mathcal{J}$ have the corresponding meaning.

2.b Ergodicity

There is a special method to prove Borel <u>non</u>-reducibility of ERs, useful in many cases. Let E, F be ERs on Polish spaces resp. $\mathbb{X}, \mathbb{Y}$. A map $\vartheta : \mathbb{X} \to \mathbb{Y}$ is

- E, F-*invariant* (or a *homomorphism* $\mathsf{E} \to \mathsf{F}$) if $x \mathrel{\mathsf{E}} y \Longrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all $x, y \in \mathbb{X}$, and *a. e.* E, F-*invariant* if this holds for all x, y in a fixed comeager subset of $\mathbb{X}$;
- F-*constant* if we have $\vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all $x, y \in \mathbb{X}$, and *a. e.* F-*constant* if this holds for all x, y in a fixed comeager subset of $\mathbb{X}$.

Finally, following Hjorth and Kechris, say that E is *generically* F*-ergodic* if every BM E,F-invariant map is a. e. F-constant.

Lemma 4 *E is generically F-ergodic if and only if every Borel a. e. E,F-invariant map is a. e. F-constant.*

Proof Let E,F live in resp. $\mathbb{X}$, $\mathbb{Y}$. Suppose that $\vartheta : \mathbb{X} \to \mathbb{Y}$ is a Borel a. e. E,F-invariant map. There is a Borel comeager set $D \subseteq \mathbb{X}$ on which ϑ is E,F-invariant. Then we can extend $\vartheta \restriction D$ to a BM map $\vartheta' : \mathbb{X} \to \mathbb{Y}$ which is still (everywhere) E,F-invariant. This proves implication $\Longrightarrow$ of the lemma. To prove the opposite implication, let $\vartheta : \mathbb{X} \to \mathbb{Y}$ be a BM E,F-invariant map. Then $\vartheta \restriction D$ is Borel for a suitable comeager Borel set $D \subseteq \mathbb{X}$. Let ϑ' be any Borel extension of $\vartheta \restriction D$ to the whole $\mathbb{X}$. $\square$

Interesting ERs normally do not have comeager equivalence classes, hence, generic F-ergodicity normally implies Borel non-reducibility to $\mathbb{F}$. This is exactly how the non-reducibility is often established, yet there are cases when E is neither F-ergodic nor Borel reducible to F, for instance, among the ERs of the form $\boldsymbol{\ell}^p$.

2.c "Algebraic" Borel reducibility

This is a more special version of Borel reducibility of ideals, characterized by the property that the reduction must respect a chosen algebraic structure. We shall be especially interested in the Boolean algebra structure and a weaker Δ-group structure of sets of the form $\mathcal{P}(A)$. Let $\mathcal{I}$, $\mathcal{J}$ be ideals on resp. A, B.

***Borel BA reducibility*:** $\mathcal{I} \le_{\text{B,BA}} \mathcal{J}$ if there is a Borel $\mathcal{J}$-approximate Boolean algebra homomorphism $\vartheta : \mathcal{P}(A) \to \mathcal{P}(B)$ such that $x \in \mathcal{I} \Longleftrightarrow \vartheta(x) \in \mathcal{J}$.

A version: $\mathcal{I} \le^{+}_{\text{B,BA}} \mathcal{J}$ if there is a set $A \in \mathcal{J}^+$ such that $\mathcal{I} \le_{\text{B,BA}} (\mathcal{J} \restriction A)$.

Here, $\vartheta : \mathcal{P}(A) \to \mathcal{P}(B)$ is an $\mathcal{J}$*-approximate* Boolean algebra homomorphism if the sets $(\vartheta(x) \cup \vartheta(y)) \,\Delta\, \vartheta(x \cup y)$ and $\vartheta(\complement x) \,\Delta\, \complement(\vartheta(x))$ always belong to $\mathcal{J}$ whenever $x, y \subseteq A$. Let further a $\mathcal{J}$-approximate Δ*-homomorphism* be any map $\vartheta : \mathcal{P}(A) \to \mathcal{P}(B)$ such that $(\vartheta(x) \,\Delta\, \vartheta(y)) \,\Delta\, \vartheta(x \,\Delta\, y)$ always belongs to $\mathcal{J}$. This leads to a weaker reducibility:

***Borel Δ-reducibility*:** $\mathcal{I} <_{\text{B},\Delta} \mathcal{J}$ iff there is a Borel $\mathcal{J}$-approximate Δ-homomorphism $\vartheta : \mathcal{P}(A) \to \mathcal{P}(B)$ such that $x \in \mathcal{I} \Longleftrightarrow \vartheta(x) \in \mathcal{J}$.

2.d Borel, continuous, and Baire measurable reductions

Many properties of Borel reductions hold for a bigger family of Baire measurable (BM, for brevity) maps. Any reducibility definition in §§ 2.a, 2.c admits a weaker BM version, which claims that the reduction postulated to exist is only BM, not

necessarily Borel. Such a version will be denoted with a subscript BM instead of B, for instance, $\mathsf{E} \le_{\text{BM}} \mathsf{F}$ means that there is a BM reduction, *i. e.*, a BM map $\vartheta : \mathbb{X} = \mathtt{dom}\,\mathsf{E} \to \mathbb{Y} = \mathtt{dom}\,\mathsf{F}$ such that $x \,\mathsf{E}\, y \Longleftrightarrow \vartheta(x) \,\mathsf{F}\, \vartheta(y)$ for all $x, y \in \mathbb{X}$.

On the other hand, a <u>continuous</u> reducibility can sometimes be derived.

Lemma 5 (Louveau ?) *If $\mathcal{I}$ and $\mathcal{J}$ are Borel ideals on $\mathbb{N}$, and $\mathcal{I} \le_{\text{BM}} \mathcal{J}$, then we have $\mathcal{I} \le_{\text{C}} \mathcal{J} \oplus \mathcal{J}$* (*i. e.*, via continuous reduction)*, that is* [1]*, there exist continuous maps ϑ_0 and $\vartheta_1 : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that, for any $x, y \in \mathcal{P}(\mathbb{N})$, $x \,\Delta\, y \in \mathcal{I}$ iff both $\vartheta_0(x) \,\Delta\, \vartheta_0(y) \in \mathcal{J}$ and $\vartheta_1(x) \,\Delta\, \vartheta_1(y) \in \mathcal{J}$.*

Proof Let $\vartheta : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ witness $\mathcal{I} \le_{\text{BM}} \mathcal{J}$. As any BM map, ϑ is continuous on a dense $\mathbf{G}_\delta$ set $D = \bigcap_i D_i \subseteq \mathcal{P}(\mathbb{N})$, all D_i being dense open and $D_{i+1} \subseteq D_i$. We can define a sequence $0 = n_0 < n_1 < n_2 < \dots$ and, for any i, a set $s_i \subseteq [n_i, n_{i+1})$ such that $x \cap [n_i, n_{i+1}) = s_i \Longrightarrow x \in D_i$. [2] Let

$$N_0 = \textstyle\bigcup_i [n_{2i}, n_{2i+1})\,, \quad N_1 = \bigcup_i [n_{2i+1}, n_{2i+2})\,, \quad S_0 = \bigcup_i s_{2i}\,, \quad S_1 = \bigcup_i s_{2i+1}\,.$$

Now set $\vartheta_0(x) = \vartheta((x \cap N_0) \cup S_1)$ and $\vartheta_1(x) = \vartheta((x \cap N_1) \cup S_0)$ for $x \subseteq \mathbb{N}$. $\square$

The following question should perhaps be answered in the negative in general and be open for some particular cases.

Question 6 Suppose that $\mathsf{E} \le_{\text{B}} \mathsf{F}$ are Borel ERs. Does there always exist a <u>continuous</u> reduction ? $\square$

2.e Reducibility via maps between the underlying sets

This is an even more special kind of Borel reducibility. Let $\mathcal{I}$, $\mathcal{J}$ be ideals on resp. A, B, as above.

***Rudin–Keisler order*:** $\mathcal{I} \le_{\text{RK}} \mathcal{J}$ iff there exists a function $b : \mathbb{N} \to \mathbb{N}$ (a *Rudin–Keisler* reduction) such that $x \in \mathcal{I} \Longleftrightarrow b^{-1}(x) \in \mathcal{J}$.

***Rudin–Blass order*:** $\mathcal{I} \le_{\text{RB}} \mathcal{J}$ iff there is a <u>finite-to-one</u> function $b : \mathbb{N} \to \mathbb{N}$ (a *Rudin–Blass* reduction) with the same property.

A version: $\mathcal{I} \le^{+}_{\text{RB}} \mathcal{J}$ allows b to be defined on a proper subset of $\mathbb{N}$, in other words, we have pairwise disjoint finite non-empty sets $w_k = b^{-1}(\{k\})$ such that $x \in \mathcal{I} \Longleftrightarrow w_x = \bigcup_{k \in x} w_k \in \mathcal{J}$.

Another version: $\mathcal{I} \le^{++}_{\text{RB}} \mathcal{J}$ requires that, in addition, the sets $w_k = b^{-1}(\{k\})$ satisfy $\mathtt{max}\, w_k < \mathtt{min}\, w_{k+1}$.

[1] See the definition of $\mathcal{I} \oplus \mathcal{J}$ in § 3.a. $\mathcal{J} \oplus \mathcal{J} \le_{\text{C}} \mathcal{J}$ for many practically interesting ideals.

[2] Sets like s_i are called *stabilizers*, they are of much help in study of Borel ideals.

There is a "clone" of the Rudin–Blass order which applies in a much more general situation. Suppose that $X = \prod_{k\in\mathbb{N}} X_k$ and $Y = \prod_{k\in\mathbb{N}} Y_k$, $0 = n_0 < n_1 < n_2 < ...$, and $H_i : X_i \to \prod_{n_i \le k < n_{i+1}} Y_k$ for any i. Then, we can define

$$\Psi(x) = H_0(x_0) \cup H_1(x_1) \cup H_2(x_2) \cup ... \in Y$$

for each $x = \{x_i\}_{i\in\mathbb{N}} \in X$. Maps Ψ of this kind were called *additive* by Farah [8]. More generally, if, in addition, $0 = m_0 < m_1 < m_2 < ...$, and $H_i : \prod_{m_i \le j < m_{i+1}} X_j \to \prod_{n_i \le k < n_{i+1}} Y_k$ for any i, then we can define

$$\Psi(x) = H_0(x \restriction [m_0, m_1)) \cup H_1(x \restriction [m_1, m_2)) \cup H_2(x \restriction [m_2, m_3)) \cup ... \in Y$$

for each $x \in X$. Farah calls maps Ψ of this kind *asymptotically additive*. All of them are Borel functions $X \to Y$, provided all sets X_j and Y_k are finite.

Suppose now that E and F are ERs on resp. $X = \prod_k X_k$ and $Y = \prod_k Y_k$.

***Additive reducibility*:** $\mathsf{E} \le_{\text{A}} \mathsf{F}$ if there is an additive reduction E to F. $\mathsf{E} \le_{\text{AA}} \mathsf{F}$ if there is an asymptotically additive reduction E to F.

Lemma 7 (Farah [8]) *Suppose that $\mathcal{I}$ and $\mathcal{J}$ are Borel ideals on $\mathbb{N}$. Then $\mathcal{I} \le_{\text{RB}}^{++} \mathcal{J}$ iff $\mathsf{E}_{\mathcal{I}} \le_{\text{A}} \mathsf{E}_{\mathcal{J}}$.*

(By definition $\mathsf{E}_{\mathcal{I}}$ and $\mathsf{E}_{\mathcal{J}}$ are ERs on $\mathcal{P}(\mathbb{N})$, yet we can consider them as ERs on $2^{\mathbb{N}} = \prod_{k\in\mathbb{N}}\{0,1\}$, as usual, which yields the intended meaning for $\mathsf{E}_{\mathcal{I}} \le_{\text{A}} \mathsf{E}_{\mathcal{J}}$.)

Proof If $\mathcal{I} \le_{\text{RB}}^{++} \mathcal{J}$ via a sequence of finite sets w_i with $\max w_i < \min w_{i+1}$ then we put $n_0 = 0$ and $n_i = \min w_i$ for $k \ge 1$, so that $w_i \subseteq [n_i, n_{i+1})$, and, for any i, put $H_i(0) = [n_i, n_{i+1}) \times \{0\}$ and let $H_i(1)$ be the characteristic function of w_i within $[n_i, n_{i+1})$. Conversely, if $\mathsf{E}_{\mathcal{I}} \le_{\text{A}} \mathsf{E}_{\mathcal{J}}$ via a sequence $0 = n_0 < n_1 < n_2 < ...$ and a family of maps $H_i : \{0,1\} \to 2^{[n_i, n_{i+1})}$ then $\mathcal{I} \le_{\text{RB}}^{++} \mathcal{J}$ via the sequence of sets $w_i = \{k \in [n_i, n_{i+1}) : H_i(0)(k) \ne H_i(1)(k)\}$. □

The following definition is taken from [22]. Let $\mathcal{I}$, $\mathcal{J}$ be ideals on $\mathbb{N}$.

***Reducibility via inclusion*:** $\mathcal{I} \le_{\text{I}} \mathcal{J}$ if there is a map $b : \mathbb{N} \to \mathbb{N}$ such that $x \in \mathcal{I} \Longrightarrow b^{-1}(x) \in \mathcal{J}$. (Note $\Longrightarrow$ instead of $\Longleftrightarrow$!)

In particular if $\mathcal{I} \subseteq \mathcal{J}$ then $\mathcal{I} \le_{\text{I}} \mathcal{J}$ via $b(k) = k$. It follows that this order is not fully compatible with $\le_{\text{B}}$ because $\mathcal{S}_{\{1/n\}} \subseteq \mathcal{Z}_0$ while the summable ideal $\mathcal{S}_{\{1/n\}}$ and the density-0 ideal $\mathcal{Z}_0$ are known to be $\le_{\text{B}}$-incomparable.

2.f Isomorphism

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on resp. A, B. *Isomorphism* $\mathcal{I} \cong \mathcal{J}$ means that there is a bijection $\beta : A \xrightarrow{\text{onto}} B$ such that we have $x \in \mathcal{I} \Longleftrightarrow \beta"x \in \mathcal{J}$ for all $x \subseteq A$.

Sometimes they use a weaker definition: let $\mathcal{I} \cong^* \mathcal{J}$ mean that there are sets $A' \in \mathcal{I}^{\complement}$ and $B' \in \mathcal{J}^{\complement}$ such that $\mathcal{I} \restriction A' \cong \mathcal{J} \restriction B'$. Yet this implies $\mathcal{I} \cong \mathcal{J}$ in most cases, the only notable exception (among nontrivial ideals), is produced by the ideals $\mathcal{I} = \mathsf{Fin}$ and $\mathcal{J} = \mathsf{Fin} \oplus \mathcal{P}(\mathbb{N}) \cong \{x \subseteq \mathbb{N} : x \cap D \in \mathsf{Fin}\}$, where D is an infinite and coinfinite set [3] : then $\mathcal{I} \cong^* \mathcal{J}$ but not $\mathcal{I} \cong \mathcal{J}$.

2.g Remarks

check this subsection once again

The following shows simple relationships between different reducibilities:

$$\mathcal{I} \le_{\text{RB}} \mathcal{J} \Rightarrow \mathcal{I} \le_{\text{RK}} \mathcal{J} \Rightarrow \mathcal{I} \le_{\text{BE}} \mathcal{J} \Rightarrow \mathcal{I} \le_{\text{BE}}^{+} \mathcal{J} \Rightarrow \mathcal{I} \le_{\Delta} \mathcal{J} \Rightarrow \mathcal{I} \le_{\text{B}} \mathcal{J}.$$

For instance if $b : \mathbb{N} \to \mathbb{N}$ witnesses $\mathcal{I} \le_{\text{RK}} \mathcal{J}$ then $\vartheta_b(X) = b^{-1}(X)$ witnesses $\mathcal{I} \le_{\text{BE}} \mathcal{J}$. Note that any ϑ_b is an exact Boolean algebra homomorphism $\mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$; moreover, it is known that any BM Boolean algebra homomorphism $\mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is ϑ_b for an appropriate $b : \mathbb{N} \to \mathbb{N}$. *Approximate* homomorphisms are liftings of homomorphisms into quotients of $\mathcal{P}(\mathbb{N})$, thus, any $\mathcal{J}$-approximate $\vartheta : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ induces the map $\Theta(X) = \{\vartheta(X) \Delta Y : Y \in \mathcal{J}\}$, which is a homomorphism $\mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{J}$. Farah [5], and Kanovei and Reeken [28] demonstrated that in some important cases (of "nonpatological" P-ideals and, generally, for all Fatou, or Fubini, ideals) we have $\mathcal{I} \le_{\text{RK}} \mathcal{J} \Longleftrightarrow \mathcal{I} \le_{\text{BE}} \mathcal{J}$. On the other hand $\mathcal{I} \le_{\text{RK}} \mathcal{J} \Longleftarrow \mathcal{I} \le_{\text{BE}} \mathcal{J}$ fails for rather artificial P-ideals.

The right-hand end is the most intrigueing: is there a pair of Borel ideals $\mathcal{I}$, $\mathcal{J}$ such that $\mathcal{I} \le_{\text{B}} \mathcal{J}$ but not $\mathcal{I} \le_{\Delta} \mathcal{J}$? If we actually have the equivalence then the whole theory of Borel reducibility for Borel ideals can be greatly simplified because reduction maps which are Δ-homomorphisms are much easier to tackle.

[3] Kechris [30] called ideals $\mathcal{J}$ of this kind *trivial variations of* Fin.

3 Introduction to ideals

As many interesting ERs appear as $\mathsf{E}_{\mathcal{I}}$ for a Borel ideal $\mathcal{I}$, we take space to discuss a few basic items related to Borel ideals. We begin with several examples and notation, and then continue with some important types of ideals.

- $\mathtt{Fin} = \{x \subseteq \mathbb{N} : x \text{ is finite}\}$, the ideal of all finite sets;
- $\mathcal{I}_1 = \mathtt{Fin} \times 0 = \{x \subseteq \mathbb{N}^2 : \{k : (x)_k \neq \emptyset\} \in \mathtt{Fin}\}$;
- $\mathcal{I}_2 = \mathcal{S}_{\{1/n\}} = \{x \subseteq \mathbb{N} : \sum_{n \in x} \frac{1}{n+1}\} < +\infty$, the *summable ideal*;
- $\mathcal{I}_3 = 0 \times \mathtt{Fin} = \{x \subseteq \mathbb{N}^2 : \forall\, k\, ((x)_k \in \mathtt{Fin})\}$;
- $\mathcal{Z}_0 = \mathcal{E}\mathcal{U}_{\{1\}} = \{x \subseteq \mathbb{N} : \mathtt{lim}_{n \to +\infty} \frac{\#(x \cap [0,n))}{n} = 0\}$, the *density ideal.*

3.a Notation

- For any ideal $\mathcal{I}$ on a set A, we define $\mathcal{I}^+ = \mathcal{P}(A) \setminus \mathcal{I}$ ($\mathcal{I}$*-positive* sets) and $\mathcal{I}^{\complement} = \{X : \complement X \in \mathcal{I}\}$ (*the dual filter*). Clearly $\emptyset \neq \mathcal{I}^{\complement} \subseteq \mathcal{I}^+$.
- If $B \subseteq A$, then we put $\mathcal{I} \restriction B = \{x \cap B : x \in \mathcal{I}\}$.
- If $\mathcal{I}$, $\mathcal{J}$ are ideals on resp. A, B, then $\mathcal{I} \oplus \mathcal{J}$ (the *disjoint sum*) is the ideal of all sets $x \subseteq C = (\{0\} \times A) \cup (\{1\} \times B)$ with $(x)_0 \in \mathcal{I}$ and $(x)_1 \in \mathcal{J}$ (where $(x)_i = \{c : \langle i, c\rangle \in x\}$, as usual).

  If the sets A, B are disjoint then $\mathcal{I} \oplus \mathcal{J}$ can be equivalently defined as the ideal of all sets $x \subseteq A \cup B$ with $x \restriction A \in \mathcal{I}$ and $x \restriction B \in \mathcal{J}$.
- The *Fubini product* $\prod_{a \in A} \mathcal{J}_a / \mathcal{I}$ of ideals $\mathcal{J}_a$ on sets B_a, over an ideal $\mathcal{I}$ on a set A is the ideal on the set $B = \{\langle a, b\rangle : a \in A \wedge b \in B_a\}$, which consists of all sets $y \subseteq B$ such that the set $\{a : (y)_a \notin \mathcal{J}_a\}$ belongs to $\mathcal{I}$, where $(y)_a = \{b : \langle a, b\rangle \in Y\}$ (the cross-section).
- In particular, if $\mathcal{I}$, $\mathcal{J}$ are ideals on resp. A, B, then $\mathcal{I} \otimes \mathcal{J} = \prod_{a \in A} \mathcal{J}_a / \mathcal{I}$, where each $\mathcal{J}_a$, $a \in A$, is equal to $\mathcal{J}$, thus, $\mathcal{I} \otimes \mathcal{J}$ is the ideal of all sets $y \subseteq A \times B$ such that the set $\{a : (y)_a \notin \mathcal{J}\}$ belongs to $\mathcal{I}$.

3.b P-ideals and submeasures

Many important Borel ideals belong to the class of P-ideals.

Definition 8 An ideal $\mathcal{I}$ on $\mathbb{N}$ is a *P-ideal* if for any sequence of sets $x_n \in \mathcal{I}$ there is a set $x \in \mathcal{I}$ such that $x_n \subseteq^* x$ (*i.e.*, $x_n \setminus x \in \mathtt{Fin}$) for all n ; □

For instance, the ideals $\mathtt{Fin}, \mathcal{I}_2, \mathcal{I}_3, \mathcal{Z}_0$ (but not $\mathcal{I}_1$!) are P-ideals.

This class admits several apparently different but equivalent characterizations, one of which is connected with submeasures.

- A *submeasure* on a set A is any map $\varphi : \mathcal{P}(A) \to [0, +\infty]$, satisfying $\varphi(\emptyset) = 0$, $\varphi(\{a\}) < +\infty$ for all a, and $\varphi(x) \le \varphi(x \cup y) \le \varphi(x) + \varphi(y)$.
- A submeasure φ on $\mathbb{N}$ is *lover semicontinuous*, or l. s. c. for brevity, if we have $\varphi(x) = \mathtt{sup}_n\, \varphi(x \cap [0, n))$ for all $x \in \mathcal{P}(\mathbb{N})$.

To be a *measure*, a submeasure φ has to satisfy, in addition, that $\varphi(x \cup y) = \varphi(x) + \varphi(y)$ whenever x, y are disjoint. Note that any σ-additive measure is l. s. c., but if φ is l. s. c. then φ_∞ is not necessarily l. s. c. itself.

Suppose that φ is a submeasure on $\mathbb{N}$. Define the *tailsubmeasure* $\varphi_\infty(x) = ||x||_\varphi = \mathtt{inf}_n(x \cap [n, \infty))$. The following ideals are considered:

$$\begin{array}{rcll} \mathtt{Fin}_\varphi & = & \{x \in \mathcal{P}(\mathbb{N}) : \varphi(x) < +\infty\} & ; \\ \mathtt{Null}_\varphi & = & \{x \in \mathcal{P}(\mathbb{N}) : \varphi(x) = 0\} & ; \\ \mathtt{Exh}_\varphi & = & \{x \in \mathcal{P}(\mathbb{N}) : \varphi_\infty(x) = 0\} \quad = \quad \mathtt{Null}_{\varphi_\infty} & . \end{array}$$

Example 9 $\mathtt{Fin} = \mathtt{Exh}_\varphi = \mathtt{Null}_\varphi$, where $\varphi(x) = 1$ for any $x \neq \emptyset$. We also have $0 \times \mathtt{Fin} = \mathtt{Exh}_\psi$, where $\psi(x) = \sum_k 2^{-k}\, \varphi(\{l : \langle k, l\rangle \in x\})$ is l. s. c.. □

It turns out (Solecki, see Theorem 25 below) that analytic P-ideals are the same as ideals of the form $\mathtt{Exh}_\varphi$, where φ is a l. s. c. submeasure on $\mathbb{N}$. It follows that any analytic P-ideal is $\mathbf{\Pi}^0_3$.

3.c Polishable ideals

There is one more characterization of Borel P-ideals. Let T be the ordinary Polish product topology on $\mathcal{P}(\mathbb{N})$. Then $\mathcal{P}(\mathbb{N})$ is a Polish group in the sense of T and the symmetric difference as the operation, and any ideal $\mathcal{I}$ on $\mathbb{N}$ is a subgroup of $\mathcal{P}(\mathbb{N})$.

Definition 10 An ideal $\mathcal{I}$ on $\mathbb{N}$ is *polishable* if there is a Polish group topology τ on $\mathcal{I}$ which produces the same Borel subsets of $\mathcal{I}$ as $T \restriction \mathcal{I}$. □

The same Solecki's theorem (Theorem 25) proves that, for analytic ideals, to be a P-ideal is the same as to be polishable. It follows (see Example 9) that, for instance, $\mathtt{Fin}$ and $\mathcal{I}_3 = 0 \times \mathtt{Fin}$ are polishable, but $\mathcal{I}_1 = \mathtt{Fin} \times 0$ is not. The latter will be shown directly after the next lemma.

Lemma 11 *Suppose that an ideal* $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ *is polishable. Then there is only one Polish group topology* τ *on* $\mathcal{I}$. *This topology refines* $T \restriction \mathcal{I}$ *and is metrizable by a* Δ*-invariant metric. If* $Z \in \mathcal{I}$ *then* $\tau \restriction \mathcal{P}(Z)$ *coincides with* $T \restriction \mathcal{P}(Z)$. *In addition,* $\mathcal{I}$ *itself is* T*-Borel.*

Proof Let τ witness that $\mathcal{I}$ is polishable. The identity map $f(x) = x : \langle \mathcal{I}\,; \tau \rangle \to \langle \mathcal{P}(\mathbb{N})\,; T \rangle$ is a Δ-homomorphism and is Borel-measurable because all $(T \restriction \mathcal{I})$-open sets are τ-Borel, hence, by the Pettis theorem (Kechris [29, ??]), f is continuous. It follows that all $(T \restriction \mathcal{I})$-open subsets of $\mathcal{I}$ are τ-open, and that $\mathcal{I}$ is T-Borel in $\mathcal{P}(\mathbb{N})$ because $1-1$ continuous images of Borel sets are Borel.

A similar "identity map" argument shows that τ is unique if exists.

It is known (Kechris [29,]) that any Polish group topology admits a left-invariant compatible metric, which, in this case, is right-invariant as well since Δ is an abelian operation.

Let finally $Z \in \mathcal{P}(\mathbb{N})$. Then $\mathcal{P}(Z)$ is T-closed, hence, τ-closed by the above, subgroup of $\mathcal{I}$, and $\tau \restriction \mathcal{P}(Z)$ is a Polish group topology on $\mathcal{P}(Z)$. Yet $T \restriction \mathcal{P}(Z)$ is another Polish group topology on $\mathcal{P}(Z)$, with the same Borel sets. It follows, by the same "identity map" argument, that T and τ coincide on $\mathcal{P}(Z)$. □

Example 12 $\mathcal{I}_1 = \mathtt{Fin} \times 0$ is not polishable. Indeed we have $\mathtt{Fin} \times 0 = \bigcup_n W_n$, where $W_n = \{x : x \subseteq \{0, 1, ..., n\} \times \mathbb{N}\}$. Let, on the contrary, τ be a Polish group topology on $\mathcal{I}_1$. Then τ and the ordinary topology T coincide on each set W_n by the lemma, in particular, each W_n remains τ-nowhere dense in W_{n+1}, hence, in $\mathcal{I}_1$, a contradiction with the Baire category theorem for τ. □

3.d Summable ideals

Any sequence $\{r_n\}_{n \in \mathbb{N}}$ of positive reals r_n with $\sum r_n = +\infty$ defines the ideal

$$\mathcal{S}_{\{r_n\}} = \{X \subseteq \mathbb{N} : \sum_{n \in X} r_n < +\infty\} = \{X : \mu_{\{r_n\}}(X) < +\infty\},$$

where $\mu_{\{r_n\}}(X) = \sum_{n \in X} r_n$. These ideals are called *summable ideals*. References [35, 37, 5]. Any summable ideal is easily a P-ideal: indeed, $\mathcal{S}_{\{r_n\}} = \mathtt{Exh}_\varphi$, where $\varphi(X) = \sum_{n \in X} r_n$ is a σ-additive measure.

Summable ideals are perhaps the easiest to study among all P-ideals. Further entries: 1) Farah [5, § 1.12] on summable ideals under $\le_{\mathrm{BE}}$, 2) Hjorth: $\le_{\mathrm{B}}$-structure of ideals $\le_{\mathrm{B}}$-reducible to summable ideals, in [17].

3.e Erdös – Ulam and density ideals

These are other types of Borel P-ideals. Any sequence $\{r_n\}_{n \in \mathbb{N}}$ of positive reals r_n with $\sum r_n = +\infty$ defines the ideal

$$\mathcal{EU}_{\{r_n\}} = \left\{ x \subseteq \mathbb{N} : \lim_{n \to +\infty} \frac{\sum_{i \in x \cap [0,n)} r_i}{\sum_{i \in [0,n)} r_i} = 0 \right\}.$$

These ideals are called *Erdös – Ulam* (or: EU) *ideals*. Examples: $\mathcal{Z}_0 = \mathcal{EU}_{\{1\}}$ and $\mathcal{Z}_{\mathtt{log}} = \mathcal{EU}_{\{1/n\}}$.

This definition can be generalized. Let $\mathtt{supp}\,\mu = \{n : \mu(\{n\}) > 0\}$, for any measure μ on $\mathbb{N}$. Measures μ, ν are *orthogonal* if we have $\mathtt{supp}\,\mu \cap \mathtt{supp}\,\nu = \emptyset$. Now suppose that $\vec{\mu} = \{\mu_n\}_{n \in \mathbb{N}}$ is a sequence of pairwise orthogonal measures on $\mathbb{N}$, with finite sets $\mathtt{supp}\,\mu_i$. Define $\varphi_{\vec{\mu}}(X) = \sup_n \mu_n(X)$: this is a l.s.c. submeasure on $\mathbb{N}$. Let finally $\mathcal{D}_{\vec{\mu}} = \mathtt{Exh}(\varphi_{\vec{\mu}}) = \{X : ||X||_{\varphi_\mu} = 0\}$. Ideals of this form are called *density ideals* by Farah [5, § 1.13]. This class includes all EU ideals (although this is not immediately transparent), and some other ideals: for instance, $0 \times \mathtt{Fin}$ is a density but non-EU ideal. Generally density ideals are more complicated than summables. We obtain an even wider class if the requirement, that the sets $\mathtt{supp}\,\mu_n$ are finite, is dropped: this wider family includes all summmable ideals, too.

References [24], [5, § 1.13].

Further entries: 1) Farah: structure of density ideals under $\le_{\mathrm{BE}}$, 2) Farah: c_0-equalities, 3) Relation to Banach spaces: Hjorth, SuGao.

Which ideals are both summable and density ?

3.f Some non-summable $\mathbf{F}_\sigma$ ideals

Farah [5, § 1.10] defines a non-summable $\mathbf{F}_\sigma$ P-ideal as follows. Let $I_k = [2^k, 2^{k+1})$ and $\psi_k(s) = k^{-2} \min\{k, \#s\}$ for all k and $s \subseteq I_k$, and then

$$\psi(X) = \sum_{k=0}^{\infty} \psi_k(X \cap I_k) \quad \text{and} \quad \mathcal{I} = \mathtt{Fin}_\psi\,;$$

it turns out that $\mathcal{I}$ is an $\mathbf{F}_\sigma$ P-ideal, but not summable. To show that $\mathcal{I}$ distincts from any $\mathcal{S}_{\{r_n\}}$, Farah notes that there is a set X (which depends on $\{r_n\}$) such that the differences $|\mu_{\{r_n\}}(X \cap I_k) - \psi_k(X \cap I_k)|$, $k = 0, 1, 2, \dots$, are unbounded.

Further entry: Farah [6, 7, 8] on *Tsirelson ideals*.

3.g Some transfinite sequences of Borel ideals

We consider three interesting families of Borel ideals (mainly, non-P-ideals), united by their relation to countable ordinals. Note that the underlying sets of the ideals below are countable sets different from $\mathbb{N}$.

Fréchet ideals. This family consists of ideals $\mathtt{Fr}_\xi$, $\xi < \omega_1$, obtained by inductive construction using Fubini products. We put $\mathtt{Fr}_1 = \mathtt{Fin}$ and $\mathtt{Fr}_{\xi+1} = \mathtt{Fin} \otimes \mathtt{Fr}_\xi$ for all ξ. Limit steps cause a certain problem. The most natural idea would be to define $\mathtt{Fr}_\lambda = \prod_{\xi<\lambda} \mathtt{Fr}_\xi \,/\, \mathtt{Fin}_\lambda$ for any limit λ, where $\mathtt{Fin}_\lambda$ is the ideal of all finite subsets of λ, or perhaps $\mathtt{Fr}_\lambda = \prod_{\xi<\lambda} \mathtt{Fr}_\xi \,/\, \mathtt{Bou}_\lambda$, where $\mathtt{Bou}_\lambda$ is the ideal of all bounted subsets of λ, or even $\mathtt{Fr}_\lambda = \prod_{\xi<\lambda} \mathtt{Fr}_\xi \,/\, 0$, where 0 is the ideal containing only the empty set, yet this appears not to be fully satisfactory in [22], where they define $\mathtt{Fr}_\lambda = \prod_{n \in \mathbb{N}} \mathtt{Fr}_{\xi_n} \,/\, \mathtt{Fin}$, where $\{\xi_n\}$ is a once and for all fixed cofinal increasing sequence of ordinals below λ, with understanding that the result is independent of the choice of ξ_n, modulo a certain equivalence.

Indecomposable ideals. Let $\mathtt{otp}\,X$ be the order type of $X \subseteq \mathtt{Ord}$. For any ordinals $\xi, \vartheta < \omega_1$ define:

$$\mathcal{I}^{\xi}_{\vartheta} = \{A \subseteq \vartheta : \mathtt{otp}\,A < \omega^{\xi}\} \quad (\text{nontrivial only if } \vartheta \geq \omega^{\xi}\,).$$

To see that the sets $\mathcal{I}^{\xi}_{\vartheta}$ are really ideals note that ordinals of the form ω^{ξ} and only those ordinals are *indecomposable*, *i.e.*, are not sums of a pair of smaller ordinals, hence, the set $\{A \subseteq \vartheta : \mathtt{otp}\,A < \gamma\}$ is an ideal iff $\gamma = \omega^{\xi}$ for some ξ.

Weiss ideals. Let $|X|_{\mathrm{CB}}$ be the *Cantor-Bendixson rank* of $X \subseteq \mathtt{Ord}$, *i.e.*, the least ordinal α such that $X^{(\alpha)} = \emptyset$. Here $X^{(\alpha)}$ is defined by induction on α: $X^{(0)} = X$, $X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}$ at limit steps λ, and finally $X^{(\alpha+1)} = (X^{(\alpha)})'$, where A', the Cantor-Bendixson derivative, is the set of all ordinals $\gamma \in x$ which are limit points of X in the interval topology. For any ordinals $\xi, \vartheta < \omega_1$ define:

$$\mathcal{W}^{\xi}_{\vartheta} = \{A \subseteq \vartheta : |A|_{\mathrm{CB}} < \omega^{\xi}\} \quad (\text{nontrivial only if } \vartheta \geq \omega^{\omega^{\xi}}\,).$$

It is less transparent that all $\mathcal{W}^{\xi}_{\vartheta}$ are ideals (Weiss, see Farah [5, § 1.14]) while $\{A \subseteq \vartheta : |A|_{\mathrm{CB}} < \gamma\}$ is not an ideal if γ is not of the form ω^{ξ}.

3.h "Other" ideals

This title intends to include those interesting ideals which have not yet been subject of comprehensive study. A common method to obtain interesting ideals is to consider a countable set bearing a nontrivial structure, as the underlying set. In principle, there is no difference between different countable set as which of them is taken as the underlying set for the ideals considered. Yet if the set bears a nontrivial structure (*i.e.*, more than just countability) then this gives additional insights as which ideals are meaningful. This is already transparent for the ideals defined in § 3.g.

We give two examples.

Ideals on finite sequences. The set $\mathbb{N}^{<\omega}$ of all finite sequences of natural numbers is countable, yet its own order structure is quite different from that of $\mathbb{N}$. We can exploit this in several ways, for instance, with ideals of sets $X \subseteq \mathbb{N}^{<\omega}$ which intersect every branch in $\mathbb{N}^{<\omega}$ by a set which belongs to a given ideal on $\mathbb{N}$.

4 Introduction to ERs

The structure of Borel and analytic ERs under $\leq_{\mathrm{B}}$ includes key ERs which play distinguished role. The plan of this section is to define some of them and outline their properties, then introduce some classes of ERs.

4.a Some key equivalence relations

Several interesting ERs appear in the form $\mathsf{E}_{\mathcal{I}}$, where $\mathcal{I}$ is a Borel ideal:

- $\mathsf{E}_0 = \mathsf{E}_{\mathrm{Fin}}$, thus, E_0 is a ER on $\mathcal{P}(\mathbb{N})$ and $x \,\mathsf{E}_0\, y$ iff $x \,\Delta\, y \in \mathrm{Fin}$;
- $\mathsf{E}_1 = \mathsf{E}_{\mathcal{I}_1}$, thus, E_1 is a ER on $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ and $x \,\mathsf{E}_0\, y$ iff $(x)_k = (y)_k$ for all but finite k, where, we recall, $(x)_k = \{n : \langle k,n\rangle \in x\}$ for $x \subseteq \mathbb{N}\times\mathbb{N}$. Alternatively, E_1 can be viewed as a ER on $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$ defined as $x \,\mathsf{E}_1\, y$ iff $x_k = y_k$ for all but finite k, for all $x = \{x_k\}_k$ and $y = \{y_k\}_k$ in $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$;
- $\mathsf{E}_2 = \mathsf{E}_{\mathcal{I}_2}$, thus, E_2 is a ER on $\mathcal{P}(\mathbb{N})$ and $x \,\mathsf{E}_{\infty}\, y$ iff $\sum_{k \in x \Delta y} k^{-1} < \infty$. E_2 is $\sim_{\mathrm{B}}$ to the ER: $x \,\boldsymbol{\ell}^1\, y$ iff $\sum_k |x_k - y_k| < +\infty$, on $\mathbb{R}^{\mathbb{N}}$;
- $\mathsf{E}_3 = \mathsf{E}_{\mathcal{I}_3}$, thus, E_3 is a ER on $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ and $x \,\mathsf{E}_0\, y$ iff $(x)_k \,\mathsf{E}_0\, (y)_k$ for all k. Alternatively, E_3 can be viewed as a ER on $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$ defined as $x \,\mathsf{E}_3\, y$ iff $x_k \,\mathsf{E}_0\, y_k$ for all k, for all $x = \{x_k\}_k$ and $y = \{y_k\}_k$ in $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$;
- $\mathsf{Z}_0 = \mathsf{E}_{\mathcal{Z}_0}$, thus, for x, y in $\mathcal{P}(\mathbb{N})$, $x \,\mathsf{Z}_0\, y$ iff $\mathtt{lim}_{n\to\infty} \frac{\#(x\Delta y)}{n} = 0$. Z_0 is $\sim_{\mathrm{B}}$ to the ER: $x \,\boldsymbol{c}_0\, y$ iff $\mathtt{lim}_{k\to\infty}(x_k - y_k) = 0$, on $\mathbb{R}^{\mathbb{N}}$; .

The simplest Borel ERs not of the form $\mathsf{E}_{\mathcal{I}}$ are

- the equalities $\mathsf{D}(\mathbb{X})$ on Polish spaces $\mathbb{X}$.

They can be characterized, modulo $\leq_{\mathrm{B}}$, even modulo Borel isomorphism between the domains, by the cardinality of the domain, which can be any finite $1 \leq n < \omega$, or $\aleph_0$, or $\mathfrak{c} = 2^{\aleph_0}$ (all uncountable Polish spaces are Borel isomorphic).

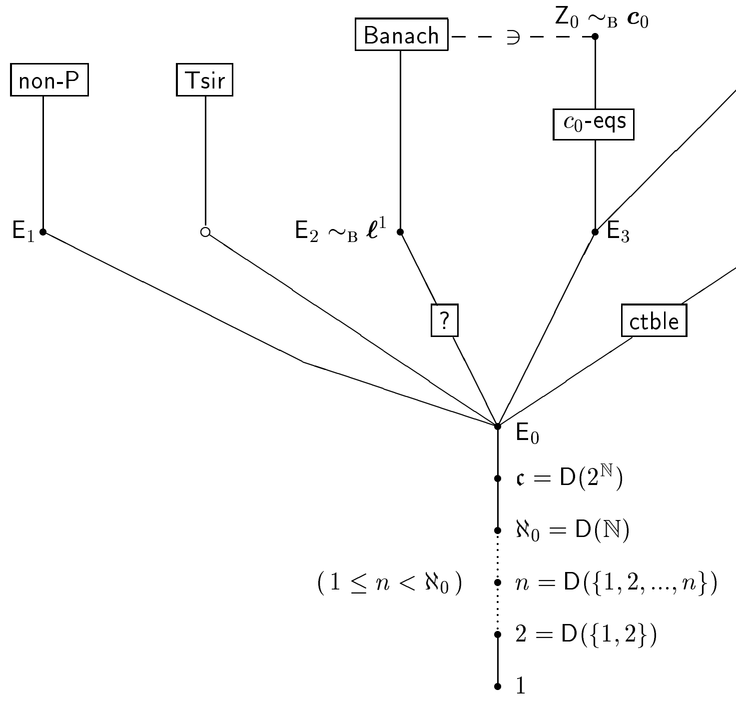
Another important ER hardly of the form $\mathsf{E}_{\mathcal{I}}$ is

- T_2, often called "the equality of countable sets of reals". *Prove that T_2 is not $\mathsf{E}_{\mathcal{I}}$ for a Borel $\mathcal{I}$*

There is no reasonable way to turn $\mathcal{P}_{\mathtt{ctbl}}(\mathbb{N}^{\mathbb{N}})$, the set of all at most countable subsets of $\mathbb{N}^{\mathbb{N}}$, into a Polish space, however, we can identify nonempty members of $\mathcal{P}_{\mathtt{ctbl}}(\mathbb{N}^{\mathbb{N}})$ with classes in $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}/\mathsf{T}_2$, where $g \,\mathsf{T}_2\, h$ iff $\mathtt{ran}\,g = \mathtt{ran}\,h$: for $g, h \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$. (See below on the relations T_α generally for all $\alpha < \omega_1$.)

- E_{∞}, the largest, or *universal countable Borel ER*. To define it let F_2 be the free group of two generators, which consists of (finite) words composed of the four symbols a, b, a^{-1}, b^{-1}, including the empty word (the neutral element), and the group operation of concatenation. The *shift action* of F_2 on 2^{F_2} is defined so that $(w\cdot x)(u) = x(w^{-1}u)$ for all $w, u \in F_2$ and $x \in 2^{F_2}$. Now, for $x, y \in 2^{F_2}$, put $x \,\mathsf{E}_{\infty}\, y$ iff $x = w\cdot y$ for some $w \in F_2$.

4.b Diagram of Borel reducibility of key ERs



4.c Comments: E_0 and below

This diagram begins, at the low end, with cardinals $1 \le n \in \mathbb{N}$, $\aleph_0$, $\mathfrak{c}$, which denote the ERs of equality on resp. finite, countable, uncountable Polish spaces. Any Borel ER E with at most countably many classes is $\sim_{\mathrm{B}}$ to $\aleph_0 = \mathsf{D}(\mathbb{N})$ or to one of numbers $n \ge 1$, accordingly with the number of classes. Note that $\mathfrak{c} = \mathsf{D}(2^{\mathbb{N}})$ is strictly $<_{\mathrm{B}} \mathsf{E}_0$: indeed, $\le_{\mathrm{B}}$ can be proved by a simple embedding while the strictness is a simple combination of classical theorems of DST.

1st dichotomy (Silver [39], Theorem 13 below)
Any Borel ER E *either has at most countable number of equivalence classes or* $\mathsf{D}(2^{\mathbb{N}}) \le_{\mathrm{B}} \mathsf{E}$, in other words, either $\mathsf{E} \le_{\mathrm{B}} \aleph_0$ or $\mathfrak{c} \le_{\mathrm{B}} \mathsf{E}$.

2nd dichotomy (Harrington, Kechris, Louveau [13], Theorem 17 below)
Any Bore ER E *satisfies either* $\mathsf{E} \le_{\mathrm{B}} \mathfrak{c}$ *or* $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E}$.

Thus the intervals $(\aleph_0, \mathfrak{c})$ and $(\mathfrak{c}, \mathsf{E}_0)$ are empty, moreover, the lower $\le_{\mathrm{B}}$-cone of E_0 contains only those $\sim_{\mathrm{B}}$-degrees which are mentioned in the diagram.

- A ER E is *smooth* if $\mathsf{E} \le_{\mathrm{B}} \mathfrak{c} = \mathsf{D}(2^{\mathbb{N}})$. (Then E is Borel, of course.)
- A ER E is *hyperfinite* if it is the union of a $\subseteq$-increasing ω-sequence of Borel *finite* ERs, *i.e.*, those with finite classes. (Then E is Borel, too.)

It turns out that E_0 is hyperfinite, and any hyperfinite ER is $\le_{\mathrm{B}} \mathsf{E}_0$.

4.d Comments: splitting at E_0

The linearity breaks above E_0 : every of the four ERs $\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3, \mathsf{E}_\infty$ of the next level is strictly $<_{\mathrm{B}}$-bigger than E_0, and the four are pairwise $\le_{\mathrm{B}}$-incomparable with each other (some of these incomparability results are quite difficult). Yet there is a tendency to prove "local" dichotomy theorems of the form: if $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_i$, $i \in \{1,2,3,\infty\}$, then either $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ or E is "rather complicated". Note that any $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ is either smooth or $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_0$, by the 2nd dichotomy.

Our brief review of ERs over E_0 begins with E_1. This relation is connected with one more important class of Borel ERs.

- A ER E is *hypersmooth* if $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_1$. Alternatively, hypersmooth ERs are exactly unions of $\subseteq$-increasing ω-sequences of smooth ERs.

3rd dichotomy (Kechris and Louveau [32], Theorem 17 below)
Any ER $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_1$ *satisfies either* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ *or* $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_1$. In other words, any hypersmooth ER either is essentially hyperfinite (*i.e.*, $\le_{\mathrm{B}} \mathsf{E}_0$) or is $\sim_{\mathrm{B}} \mathsf{E}_1$.

Yet another category of Borel ERs is connected with E_∞.

- A Borel ER E is *countable* if any E-class is (at most) countable. A Borel ER E is *essentially countable* if $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$ for a countable Borel ER F.

Despite the variety of different ways how a countable Borel ER can be defined, all of them are $\le_{\mathrm{B}}$ than E_∞, this is why the latter is called *universal* (countable Borel ER). This observation is marked by the framebox $\boxed{\text{ctble}}$ on the diagram. *reference*
It has only recently been established by Adams and Kechris [1] that there is continuum many pairwise $\le_{\mathrm{B}}$-incomparable countable Borel ERs.

4th dichotomy (Hjorth [17], Theorem 49 below)
Any Borel ER $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_2$ *either is essentially countable or* $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_2$.

The "either" case of this result remains somewhat misterious: any countable Borel ERs $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_2$ known so far are hyperfinite, *i.e.*, $\le_{\mathrm{B}} \mathsf{E}_0$. It is a hypothesis and a big problem to strengthen the "either" case here to "essentially hyperfinite": this is marked by the framebox $\boxed{?}$ on the diagram.

Terporarily missing 5th dichotomy, we arrive to

6th dichotomy (Hjorth and Kechris [19, 20], Theorem 34 below)
Any ER $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_3$ *satisfies either* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ *or* $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_3$.

4.e Comments: above the splitting level

1) It was once considered as a plausible hypothesis that any Borel ER which is not $\le_{\rm B} \mathsf{E}_\infty$, *i.e.*, not a countable ER, satisfies $\mathsf{E}_i \le_{\rm B} \mathsf{E}$ for at least one $i = 1, 2, 3$. This turns out to be not the case: Farah [6, 7] and Velickovic [47] found an independent family of uncountable Borel ERs, based on *Tsirelson ideals*, $\le_{\rm B}$-incomparable with $\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3$. We illustrated this result by the framebox $\boxed{\mathsf{Tsir}}$ on the diagram. $\circ$ in the break of the line means that there is no known "minimal" Tsirelson ideal. It is an interesting question whether $\{\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3, \text{Tsirelson ERs}\}$ is a "minimality basis" amongst uncountable Borel ERs ?

Is there a minimal tsirelson ?

2) Framebox $\boxed{\mathsf{non\text{-}P}}$ denotes the family of all ERs $\mathsf{E}_{\mathcal I}$, where $\mathcal I$ is a Borel ideal which is not a P-ideal. By Solecki [40, 41], for a Borel $\mathcal I$ to be not a P-ideal it is necessary and sufficient that $\mathcal I_1 \le_{\rm B} \mathcal I$, which means $\mathsf{E}_1 \le_{\rm B} \mathsf{E}_{\mathcal I}$ by definition.

E_1 and Polish grps action problem

3) Framebox $\boxed{\mathsf{Banach}}$ denotes a family of ERs induced by actions of (the additive groups of) some Banach spaces. This group includes, for any $p \ge 1$, the ER $\boldsymbol{\ell}^p$ on $\mathbb{R}^{\mathbb N}$ defined as follows: $x \mathrel{\boldsymbol{\ell}^p} y$ iff $\sum_k |x_x - y_k|^p < +\infty$ for all sequences $x = \{x_k\}_{k\in\mathbb N}$ and $y = \{y_k\}_k$ in $\mathbb{R}^{\mathbb N}$. Clearly $\boldsymbol{\ell}^p$ is the orbit relation of the (additive group of the) Banach space $\ell^p = \{x \in \mathbb{R}^{\mathbb N} : \sum_k |x_k|^p < +\infty\}$ acting on $\mathbb{R}^{\mathbb N}$ by componentwise addition. It is known (Dougherty and Hjorth [3]) that $\boldsymbol{\ell}^p <_{\rm B} \boldsymbol{\ell}^q$ whenever $1 \le p < q$, in particular, the relation $\boldsymbol{\ell}^1 \sim_{\rm B} \mathsf{E}_2$ is $<_{\rm B} \boldsymbol{\ell}^q$ for any $q > 1$. Yet not all "Banach" ERs are above E_2, for instance, the relation $\boldsymbol{c}_0 \sim_{\rm B} \mathsf{Z}_0 = \mathsf{E}_{\mathcal Z_0}$ is $\le_{\rm B}$-incomparable with E_2, but $\mathsf{E}_3 <_{\rm B} \boldsymbol{c}_0$.

Are tsirelsons Banach ?

Is any ctble Borel $\mathsf{E} \le_{\rm B} \boldsymbol{c}_0$ necessarily $\le_{\rm B} \mathsf{E}_0$?

4) Framebox $\boxed{c_0\text{-}\mathsf{eqs}}$ denotes c_0-*equalities*, a family of Borel ERs introduced by Farah [8], all of them are $\le_{\rm B}$-between E_3 and $\boldsymbol{c}_0 \sim_{\rm B} \mathsf{Z}_0$, and there is continuum-many $\le_{\rm B}$-incomparable among them.

Are $\boldsymbol{\ell}^p$ and $\boldsymbol{c}_0$ cmprble with tsirelsons ?

4) T_2, "the equality on countable sets of reals",

4.f Operations on ERs

The following operations over ERs are in part parallel to the operations on ideals in § 3.a.

(o1) countable union (if it results in a ER) and countable intersection of ERs on one and the same space;

(o2) countable disjoint union $\mathsf{E} = \bigvee_k \mathsf{E}_k$ of ERs E_k on Polish spaces $\mathbb{S}_k$, that is, a ER on $\mathbb{S} = \bigcup_k(\{k\} \times \mathbb{S}_k)$ (with the topology generated by sets of the form $\{k\} \times U$, where $U \subseteq \mathbb{S}_k$ is open) defined as follows: $\langle k, x\rangle \mathrel{\mathsf{E}} \langle l, y\rangle$ iff $k = l$ and $x \mathrel{\mathsf{E}_k} y$. (If $\mathbb{S}_k$ are pairwise disjoint and open in $\mathbb{S}' = \bigcup_k \mathbb{S}_k$ then we can equivalently define $\mathsf{E} = \bigvee_k \mathsf{E}_k$ on $\mathbb{S}'$ so that $x \mathrel{\mathsf{E}} y$ iff x, y belong to the same $\mathbb{S}_k$ and $x \mathrel{\mathsf{E}_k} y$.);

(o3) product $\mathsf{E} = \prod_k \mathsf{E}_k$ of ERs E_k on spaces $\mathbb{S}_k$, that is, the ER on the product space $\prod_k \mathbb{S}_k$ defined by: $x \mathrel{\mathsf{E}} y$ iff $x_k \mathrel{\mathsf{E}_k} y_k$ for all k.

(o4) the *Fubini product* or *ultraproduct* $\prod_{k\in\mathbb N} \mathsf{E}_k / \mathcal I$ of ERs E_k on spaces $\mathbb{S}_k$, modulo an ideal $\mathcal I$ on $\mathbb N$, that is, the ER on the product space $\prod_{k\in\mathbb N} \mathbb{S}_k$ defined as follows: $x \mathrel{\mathsf{E}} y$ iff $\{k : x_k \not\mathrel{\mathsf{E}}_k y_k\} \in \mathcal I$;

(o5) *countable power* ER E^∞ of a ER E on a space $\mathbb S$ is a ER on $\mathbb{S}^{\mathbb N}$ defined as follows: $x \mathrel{\mathsf{E}^\infty} y$ iff $\{[x_k]_{\mathsf E} : k \in \mathbb N\} = \{[y_k]_{\mathsf E} : k \in \mathbb N\}$, so that for any k there is l with $x_k \mathrel{\mathsf{E}} y_l$ and for any l there is k with $x_k \mathrel{\mathsf{E}} y_l$.

These operations allow us to obtain a lot of interesting ERs starting just with very primitive ones. For instance, we can define the sequence of ERs T_α, $\alpha < \omega_1$, of H. Friedman [9] as follows. Let $\mathsf{T}_0 = \Delta_{\mathbb N}$, the equality relation on $\mathbb N$. Let $\mathsf{T}_{\alpha+1} = {\mathsf{T}_\alpha}^\infty$. If $\lambda < \omega_1$ is a limit ordinal, then put $\mathsf{T}_\lambda = \bigvee_{\alpha<\lambda} \mathsf{T}_\alpha$, where all T_α are assumed to be redefined so that their domains are disjoint.

In particular $\mathtt{dom}\,\mathsf{T}_1 = \mathbb{N}^{\mathbb N}$ and $x \mathrel{\mathsf{T}_1} y$ iff $\mathtt{ran}\,x = \mathtt{ran}\,y$ for all $x, y \in \mathbb{N}^{\mathbb N}$. Thus the map $\vartheta(x) = \mathtt{ran}\,x$ witnesses that $\mathsf{T}_1 \le_{\rm B} \Delta_{\mathcal P(\mathbb N)}$. To show the converse, define, for any infinite $u \subseteq \mathbb N$, $\beta(u)$ be the increasing bijection $\mathbb N \xrightarrow{\text{onto}} u$, while if $u = \{k_0, ..., k_n\}$ is finite, put $\beta(u)(i) = k_i$ for $i < n$ and $\beta(u)(i) = k_n$ for $i \ge n$. Then β witnesses $\Delta_{\mathcal P(\mathbb N)} \le_{\rm B} \mathsf{T}_1$, thus, $\mathsf{T}_1 \sim_{\rm B} \Delta_{\mathcal P(\mathbb N)}$.

It easily follows that $\mathsf{T}_2 \sim_{\rm B} {\Delta_{\mathcal P(\mathbb N)}}^\infty$, in fact, $\mathsf{T}_2 \sim_{\rm B} {\Delta_{\mathbb X}}^\infty$ for any uncountable Polish space $\mathbb X$ as any such $\mathbb X$ is Borel isomorphic to $\mathcal P(\mathbb N)$ (or to $2^{\mathbb N}$, which is essentially the same). With $\mathbb X = \mathbb{N}^{\mathbb N}$ we obtain the definition of T_2 in § 4.a.

5 ERs induced by group actions

Suppose that $\mathbb{G}$ is a group acting on a space $\mathbb{X}$, which means that there is a map $\langle a,x\rangle \mapsto a\cdot x : \mathbb{G}\times\mathbb{X}\to\mathbb{X}$ (called: *an action*) such that, first, for any $a\in\mathbb{G}$ the map $x\mapsto a\cdot x$ is a bijection $\mathbb{X}$ onto $\mathbb{X}$, and second, $a\cdot(b\cdot x)=(ab)\cdot x$. In this case, the *orbit equivalence relation* $\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}$ is defined on $\mathbb{X}$ so that $x\,\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}\,y$ iff there is $a\in\mathbb{G}$ with $y=a\cdot x$. Clearly the $\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}$-classes are the same as $\mathbb{G}$-orbits, *i.e.*,

$$[x]_{\mathbb{G}}=[x]_{\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}}=\{y:\exists\, g\in\mathbb{G}\,(g\cdot x=y)\}\,.$$

They usually consider the case when the space $\mathbb{X}$ is Polish and the group $\mathbb{G}$ is Polish or at least Borel. In this case if the action is continuous, resp. Borel, then the space $\mathbb{X}$ is called a *Polish*, resp. *Borel* G*-space*. Then $\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}$ is, generally speaking, an analytic ER. Sometimes $\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}$ is even Borel, for instance, when $\mathbb{G}$ is a countable group and the action is Borel.

For instance if $\mathbb{G}=\mathcal{I}\subseteq\mathcal{P}(\mathbb{N})$ is an ideal considered as a group with Δ as the operation, which acts on $\mathbb{X}=\mathcal{P}(\mathbb{N})$ by Δ, then clearly $\mathsf{E}^{\mathbb{G}}_{\mathcal{P}(\mathbb{N})}=\mathsf{E}_{\mathcal{I}}$, *i.e.*, $x\,\mathsf{E}^{\mathbb{G}}_{\mathcal{P}(\mathbb{N})}\,y$ iff $x\,\Delta\,y\in\mathcal{I}$, in particular, $\mathsf{E}^{\mathbb{G}}_{\mathcal{P}(\mathbb{N})}$ is Borel.

5.a Actions of the infinite symmetric group

Let S_∞ be the group of all permutations (*i.e.*, 1–1 maps $\mathbb{N}\xrightarrow{\text{onto}}\mathbb{N}$) of $\mathbb{N}$, with the superposition as the group operation. In the topology inherited from $\mathbb{N}^{\mathbb{N}}$, S_∞ is not Polish, however, there is a metric on S_∞ which gives it a Polish topology compatible with the inherited Borel structure.

Give a defn of the Polish topol. of S_∞

For instance isomorphism relations of various kinds of countable structures (*i.e.*, a countable domain, which can be identified with $\mathbb{N}$, and countably many atomic relations) are induced by the natural action of S_∞ (*i.e.*, $g\cdot n=g(n)$).

The operations (o2) of countable disjoint union and (o5) of countable power easily preserve the property of being induced by a continuous action of S_∞ and the property of being induced by a Borel action of S_∞. Thus all ERs T_α of §4.f are induced by continuous actions of S_∞ because so is the initial one $\mathsf{T}_0=\Delta_{\mathbb{N}}$.

5.b Turbulent actions

This is an entirely different class of ERs. Suppose that $\mathbb{G}$ is a group which acts on a space $\mathbb{X}$. If $G\subseteq\mathbb{G}$ and $X\subseteq\mathbb{X}$ then let

$$\mathsf{R}^G_X=\{\langle x,y\rangle\in U^2:\exists\, g\in G\,(x=g\cdot y)\}$$

and let $\sim^G_X$ denote the ER-hull of R^G_X, *i.e.*, the $\subseteq$-least ER on X such that $x\,\mathsf{R}^G_X\,y\Longrightarrow x\sim^G_X y$. In particular $\sim^{\mathbb{G}}_{\mathbb{X}}=\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}$, but generally we have $\sim^G_X\subsetneqq\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}\restriction X$. Finally, define $\mathcal{O}(U,X,x)=\{y\in X: x\sim^U_X y\}$ for $x\in X$: the *local orbit* of x. In particular, $[x]_{\mathbb{G}}=\mathcal{O}(\mathbb{X},\mathbb{G},x)$: the full $\mathbb{G}$-orbit of $x\in\mathbb{X}$.

The following definitions, taken from Kechris [31, § 8], usually are considered when $\mathbb{X}$ is a Polish space and $\mathbb{G}$ is a Polish group acting on $\mathbb{X}$ continuously, but formally they make sense for all spaces, groups, and actions.

(t1) A point $x\in\mathbb{X}$ is *turbulent* if for any open $X\subseteq\mathbb{X}$ containing x and any open symmetric ($G=G^{-1}$) set $G\subseteq\mathbb{G}$ (not necessarily a subgroup) containing $1_{\mathbb{G}}$, the local orbit $\mathcal{O}(X,G,x)$ is somewhere dense (*i.e.*, not a nowhere dense set) in $\mathbb{X}$.

(t2) The orbit $[x]_{\mathbb{G}}$ is *turbulent* if x is such. (In this case every $y\in[x]_{\mathbb{G}}$ is turbulent, too.)

(t3) The action (of $\mathbb{G}$ on $\mathbb{X}$) is *generically turbulent* if there is a dense and $\mathbb{G}$-invariant $\mathbf{G}_\delta$ set $D\subseteq\mathbb{X}$ such that, for all $x\in D$, the orbit $[x]_{\mathbb{G}}$ is dense, turbulent, and meager. (For this it is necessary and sufficient [31, 8.7] that the union of all dense, turbulent, and meager orbits $[x]_{\mathbb{G}}$ is comeager.)

The action is *turbulent* if all orbits are dense, turbulent, and meager.

6 First dichotomy

Theorem 13 (Silver [39]) *Any* $\mathbf{\Pi}^1_1$ *ER* E *on* $\mathbb{N}$ *either has at most countably many equivalence classes or admits a perfect set of pairwise* E*-inequivalent reals.*

Thus, formally, either $\mathsf{E} \le_{\mathsf{B}} \mathsf{D}(\mathbb{N})$ or $\mathsf{D}(2^{\mathbb{N}}) \le_{\mathsf{B}} \mathsf{E}$.

Proof [4] We can, as usual, suppose that E is a lightface Π^1_1.

Case 1: any $x \in \mathbb{N}^{\mathbb{N}}$ belongs to a Δ^1_1 E-equivalent set X (*i.e.*, all elements of X are E-equivalent to each other, in other words, the saturation $[X]_{\mathsf{E}}$ is an equivalence class). Then E has at most countably many equivalence classes.

Case 2: otherwise. Then the set H of all x, which do <u>not</u> belong to a Δ^1_1 pairwise E-equivalent set (*the domain of nontriviality*), is non-empty.

Claim 13.1 *H is Σ^1_1. Any Σ^1_1 set $\emptyset \ne X \subseteq H$ is not pairwise E-equivalent.*

Proof $x \in H$ iff for any $e \in \mathbb{N}$: <u>if</u> e codes a Δ^1_1 set, say, $W_e \subseteq \mathbb{N}^{\mathbb{N}}$ and $x \in W_e$ <u>then</u> W_e is not E-equivalent. The "if" part of this characterization is Π^1_1 while the "then" part is Σ^1_1, by Δ^1_1 $\mathsf{Enumeration}$ (see § 1.c).

If $X \ne \emptyset$ is a pairwise E-equivalent Σ^1_1 set then $B = \bigcap_{x \in X} [x]_{\mathsf{E}}$ is a Π^1_1 E-equivalence class and $X \subseteq B$. By $\mathsf{Separation}$, there is a Δ^1_1 set C with $X \subseteq C \subseteq B$. Then, if $X \subseteq H$ then $C \subseteq H$ is a Δ^1_1 pairwise E-equivalent set, a contradiction to the definition of H. □ (*Claim*)

Let us fix a countable transitive model $\mathfrak{M}$ of a big enought fragment of **ZFC**, and an elementary submodel of the universe w.r.t. all analytic formulas [5]. Consider $\mathbb{P} = \{X \subseteq \mathbb{N}^{\mathbb{N}} : X$ is non-empty and $\Sigma^1_1\}$ as a forcing to extend $\mathfrak{M}$ (smaller sets are stronger conditions), the *Gandy – Harrington forcing*. We have $\mathbb{P} \notin$ and $\not\subseteq \mathfrak{M}$, of course, but clearly $\mathbb{P}$ can be adequately coded in $\mathfrak{M}$, say, via a universal Σ^1_1 set.

Corollary 13.2 (from Theorem 2) *If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic, over $\mathfrak{M}$, set, then $\bigcap G$ contains a single real, denoted x_G.* □

Reals of the form x_G, G as in the Corollary, are called $\mathbb{P}$-*generic* (over $\mathfrak{M}$). Let $\dot{x}$ be the name for x_G. Then any $A \in \mathbb{P}$ forces that $\dot{x} \in A$.

Let $\mathbb{P}^2$ consist of all "rectangles" $X \times Y$, with $X, Y \in \mathbb{P}$. It follows from the above by the product forcing lemmas that any $\mathbb{P}^2$-generic, over $\mathfrak{M}$, set $G \subseteq \mathbb{P}^2$ produces a pair of reals (a $\mathbb{P}^2$-*generic pair*), say, $x^G_{\mathtt{left}}$ and $x^G_{\mathtt{right}}$, so that $\langle x^G_{\mathtt{left}}, x^G_{\mathtt{right}} \rangle \in W$ for any $W \in G$. Let $\dot{x}_{\mathtt{left}}$ and $\dot{x}_{\mathtt{right}}$ be their names.

[4] We present a forcing proof of Miller [38], with some simplifications. See [34] for another proof, based on the Gandy – Harrington topology.

[5] For instance, $\mathfrak{M}$ models the first one million of **ZFC** axioms plus the schemata for Σ_{100} $\in$-formulas. Being an elementary submodel is required to guarantee that relations like the inclusion order of $\mathbb{P}$ are absolute for $\mathfrak{M}$.

Lemma 13.3 $H \times H$ $\mathbb{P}^2$*-forces* $\dot{x}_{\mathtt{left}} \not\mathsf{E}\, \dot{x}_{\mathtt{right}}$.

Proof Otherwise a "condition" $X \times Y \in \mathbb{P}^2$ with $X \cup Y \subseteq H$ $\mathbb{P}^2$-forces $\dot{x}_{\mathtt{left}} \mathsf{E}\, \dot{x}_{\mathtt{right}}$, so that any $\mathbb{P}^2$-generic pair $\langle x, y \rangle \in X \times Y$ satisfies $x \mathsf{E} y$. By the product forcing lemmas for any pair of $\mathbb{P}$-generic $x', x'' \in X$ there is $y \in Y$ such that both $\langle x, y \rangle$ and $\langle x', y \rangle$ are $\mathbb{P}^2$-generic pairs, hence, we have

($*$) If $x', x'' \in X$ are $\mathbb{P}$-generic over $\mathfrak{M}$ then $x' \mathsf{E} x''$.

The set $\mathbb{P}_2$ of all non-empty Σ^1_1 subsets of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is just a copy of $\mathbb{P}$ (not of $\mathbb{P}^2$!) as a forcing, in particular, if $G \subseteq \mathbb{P}_2$ is $\mathbb{P}_2$-generic over $\mathfrak{M}$ then there is a unique pair of reals ($\mathbb{P}_2$-*generic pair*) $\langle x^G_{\mathtt{left}}, x^G_{\mathtt{right}} \rangle$ which belongs to every W in G, and in this case, *both $x^G_{\mathtt{left}}$ and $x^G_{\mathtt{right}}$ are $\mathbb{P}$-generic*, because if $G \subseteq \mathbb{P}_2$ is $\mathbb{P}_2$-generic then the sets G' and G'' of all projections of sets $W \in G$ to resp. 1st and 2nd co-ordinate, are easily $\mathbb{P}$-generic. Now let $G \subseteq \mathbb{P}_2$ be a $\mathbb{P}_2$-generic set, over $\mathfrak{M}$, containing the Σ^1_1 set $P = X^2 \setminus \mathsf{E}$. (Note that $P \ne \emptyset$ by Lemma 13.1.) Then $\langle x^G_{\mathtt{left}}, x^G_{\mathtt{right}} \rangle \in P$, hence, $x^G_{\mathtt{left}} \not\mathsf{E}\, x^G_{\mathtt{right}}$, however, as we observed, both $x^G_{\mathtt{left}}$ and $x^G_{\mathtt{right}}$ are $\mathbb{P}$-generic elements of X (because $P \subseteq X \times X$), which contradicts ($*$). □ (*Lemma 13.3*)

Fix enumerations $\{\mathcal{D}(n)\}_{n \in \mathbb{N}}$ and $\{\mathcal{D}^2(n)\}_{n \in \mathbb{N}}$ of all dense subsets of resp. $\mathbb{P}$ and $\mathbb{P}^2$ which are coded in $\mathfrak{M}$. Then there is a system $\{X_u\}_{u \in 2^{<\omega}}$ of sets X_u, satisfying

(i) $X_u \in \mathbb{P}$, moreover, $X_\Lambda \subseteq H$ and $X_u \in \mathcal{D}(n)$ whenever $u \in 2^n$;

(ii) $X_{u^\wedge i} \subseteq X_u$ for all $u \in 2^{<\omega}$ and $i = 0, 1$;

(iii) if $u \ne v \in 2^n$ then $X_u \times X_v \in \mathcal{D}^2(n)$.

It follows from (i) that, for any $a \in 2^{\mathbb{N}}$, the set $\{X_{a \restriction m} : m \in \mathbb{N}\}$ is $\mathbb{P}$-generic over $\mathfrak{M}$, hence, $\bigcap_m X_{a \restriction m}$ is a singleton, say, x_a, by Corollary 13.2. Moreover the map $a \mapsto x_a$ is continuous as diameters of X_u converge to 0 uniformly with $\mathtt{lh}\, u \to 0$, by (i). In addition, by (iii) and Lemma 13.3, $x_a \not\mathsf{E}\, x_b$ whenever $a \ne b$, in particular, $x_a \ne x_b$, hence, we have a perfect E-inequivalent set $Y = \{x_a : a \in 2^{\mathbb{N}}\}$.

□ (*Theorem 13*)

7 Ideal $\mathtt{Fin}$ and the equivalence relation E_0

After a rather simple theorem which shows that $\mathtt{Fin}$ is the least ideal in the sense of $\le_{\rm RB}^{++}$, $\le_{\rm RB}$, $\le_{\rm B}$, we prove the "Glimm–Effros", or second, dichotomy which asserts that $\mathsf{E}_0 = \mathsf{E}_{\mathtt{Fin}}$ is the $\le_{\rm B}$-least among all non-smooth Borel ERs. Finally, we present a characterization, in terms of the existence of transversals, of those Borel sets X for which $\mathsf{E}_0 \restriction X$ is smooth.

7.a $\mathtt{Fin}$ is the least !

Theorem 14 (Jalali–Naini [23], Mathias [36], Talagrand [43].)
If $\mathcal{I}$ is a (nontrivial) *ideal on* $\mathbb{N}$, *with the Baire property in the topology of* $\mathcal{P}(\mathbb{N})$, *then* $\mathtt{Fin} \le_{\rm RB}^{++} \mathcal{I}$ *and* $\mathtt{Fin} \le_{\rm RB} \mathcal{I}$.

Proof The proof is based on a short high quality argument that also appears in many other results. First of all we note that $\mathcal{I}$ must be meager in $\mathcal{P}(\mathbb{N})$. (Here the Baire property is applied: otherwise $\mathcal{I}$ would be comeager somewhere, easily leading to contradiction.) Thus, all $x \in \mathcal{P}(\mathbb{N})$ which are "generic" (over a certain countable family of dense open sets and everything relevant) do not belong to $\mathcal{I}$. Now it suffices to define non-empty finite sets $w_i \subseteq \mathbb{N}$ with $\max w_i < \min w_{i+1}$ such that any union of infinitely many of them is "generic". Clearly the following observation yields the result: if D is an open dense subset of $\mathcal{P}(\mathbb{N})$ and $n \in \mathbb{N}$ then there is $m > n$ and a set $u \subseteq [n, m]$ with $m, n \in u$ such that any $x \in \mathcal{P}(\mathbb{N})$ satisfying $x \cap [n, m] = u$ belongs to D.

Thus we have $\mathtt{Fin} \le_{\rm RB}^{++} \mathcal{I}$. To derive $\mathtt{Fin} \le_{\rm RB} \mathcal{I}$, we can embed each w_k in a finite set u_k such that $\bigcup_{k\in\mathbb{N}} u_k = \mathbb{N}$ and still $u_k \cap u_l = \emptyset$ for $k \ne l$. □

Corollary 15 *Let $\mathcal{J} \le_{\rm RB}^{+} \mathcal{I}$ be Borel ideals, and there is an infinite set $Z \subseteq \mathtt{dom}\,\mathcal{J}$ such that $\mathcal{J} \restriction Z = \mathcal{P}_{\mathtt{fin}}(Z)$. Then $\mathcal{J} \le_{\rm RB} \mathcal{I}$.*

Proof Let pairwise disjoint finite sets $w_k \subseteq \mathbb{N}$ witness $\mathcal{J} \le_{\rm RB}^{+} \mathcal{I}$. Put $Z' = \mathbb{N} \setminus Z$, $X = \bigcup_{k\in Z} w_k$, and $Y = \bigcup_{k\in Z'} w_k$. The reduction via $\{w_k\}$ reduces $\mathcal{P}_{\mathtt{fin}}(Z)$ to $\mathcal{I} \restriction X$ and $\mathcal{J} \restriction Z'$ to $\mathcal{I} \restriction Y$. Keeping the latter, replace the former by a $\le_{\rm RB}$-like reduction $\mathcal{P}_{\mathtt{fin}}(z)$ to $\mathcal{I} \restriction Y'$, where $Y' = \mathbb{N} \setminus Y$, which exists by Theorem 14. □

7.b Second dichotomy

Despite of Theorem 14, $\mathsf{E}_0 = \mathsf{E}_{\mathtt{Fin}}$ is <u>not</u> the $\le_{\rm B}$-least among Borel ERs.

Lemma 16 $\mathsf{D}(2^{\mathbb{N}}) <_{\rm B} \mathsf{E}_0$ *strictly.*

Proof That $\mathsf{D}(2^{\mathbb{N}}) \le_{\rm B} \mathsf{E}_0$ is witnessed by any perfect set $X \subseteq 2^{\mathbb{N}}$ which is a transversal for E_0 (*i.e.*, any $x \ne y$ in X are E_0-inequivalent). On the other hand, $\mathsf{D}(2^{\mathbb{N}})$ is smooth but E_0 is non-smooth, since otherwise it would have a full Borel transversal, which is known to be impossible. □

Thus, $\mathsf{D}(2^{\mathbb{N}})$ is not a ER generated by a Borel ideal, even modulo $\sim_{\rm B}$.

Theorem 17 ("Glimm–Effros dichotomy" of Harrington, Kechris, Louveau [13])
If E is a Borel ER then either E is smooth or $\mathsf{E}_0 \sqsubseteq_{\rm C} \mathsf{E}$.

Proof As usual, we can suppose that E is a lightface Δ^1_1 ER on $\mathbb{N}^{\mathbb{N}}$. Consider an auxiliary ER $x \,\overline{\mathsf{E}}\, y$ iff $x, y \in \mathbb{N}^{\mathbb{N}}$ belong to the same E-invariant Δ^1_1 sets. (A set X is E-*invariant* iff $X = [X]_{\mathsf{E}}$.) Easily $\mathsf{E} \subseteq \overline{\mathsf{E}}$. To see that $\overline{\mathsf{E}}$ is the closure of E in the Gandy – Harrington topology, prove

Lemma 17.1 *Suppose that F is a Σ^1_1 ER on $\mathbb{N}^{\mathbb{N}}$. If $X, Y \subseteq \mathbb{N}^{\mathbb{N}}$ are disjoint F-invariant Σ^1_1 sets then there exists an F-invariant Δ^1_1 set X with $X \subseteq X'$ and $Y \cap X' = \emptyset$.*

Proof By $\mathsf{Separation}$, for any Σ^1_1 set A with $A \cap Y = \emptyset$ there is a Δ^1_1 set A' with $A \subseteq A'$ and $A' \cap Y = \emptyset$ — note that then $[A']_{\mathsf{F}} \cap Y = \emptyset$ because Y is F-invariant. It follows that that there is a sequence $X = A_0 \subseteq A'_0 \subseteq A_1 \subseteq A'_1 \subseteq \ldots$, where A'_i are Δ^1_1 sets, accordingly, $A_{i+1} = [A'_i]_{\mathsf{F}}$ are Σ^1_1 sets, and $A_i \cap Y = \emptyset$. Then $X' = \bigcup_n A_n = \bigcup_n A'_n$ and is an F-invariant Borel set which separates X from Y. To make X' Δ^1_1 we have to maintain the choice of sets A_n effectively.

Let $U \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ be a rather "good" universal set. The following obviously suffices to get X' as above in Δ^1_1:

(∗) There exists a Δ^1_1 function $\gamma : \mathbb{N} \to \mathbb{N}$ such that, for any n, if $[U_n]_{\mathsf{F}} \cap Y = \emptyset$ then $U_{\gamma(n)}$ is Δ^1_1, $[U_n]_{\mathsf{F}} \subseteq U_{\gamma(n)}$, and still $[U_{\gamma(n)}]_{\mathsf{F}} \cap Y = \emptyset$.

The sets $P = \{\langle n, x\rangle : x \in [U_n]_{\mathsf{F}}\}$ and $Q = \mathbb{N} \times Y$ are Σ^1_1, hence, by $\mathsf{Reduction}$ there is a Σ^1_1 set $R \supseteq P$ such that, for any n, if $[U_n]_{\mathsf{F}} \cap Y = \emptyset$ then $[U_n]_{\mathsf{F}} \subseteq R_n$ and $R_n \cap Y = \emptyset$. Yet, assuming U is "good" enough there is a Δ^1_1 function $\gamma : \mathbb{N} \to \mathbb{N}$ such that $R_n = U_{\gamma(n)}$ for all n, as required. □ (*Lemma*)

Lemma 17.2 $\overline{\mathsf{E}}$ *is a Σ^1_1 relation.*

Proof Let $C \subseteq \mathbb{N}$ and $W, W' \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ be as in Δ^1_1 $\mathsf{Enumeration}$ (§1.c). The formula $\mathtt{inv}(e)$ saying that $e \in C$ and $W_e = W'_e$ is E-invariant, *i.e.*,

$$e \in C \wedge \forall a, b\,(a \in W_e \wedge b \notin W'_e \Longrightarrow a \not\mathsf{E}\, b)$$

is obviously Π^1_1, however $x \,\overline{\mathsf{E}}\, y$ iff

$$\forall e\,(\mathtt{inv}(e) \Longrightarrow (x \in W_e \Longrightarrow y \in W'_e) \wedge (y \in W_e \Longrightarrow x \in W'_e)) \qquad \square\ (Lemma)$$

Let us return to the proof of the theorem. We have two cases.

Case 1: $\mathsf{E} = \overline{\mathsf{E}}$, *i.e.*, E is Gandy – Harrington closed.

Lemma 17.3 *If $\mathsf{E} = \overline{\mathsf{E}}$ then there is a Δ^1_1 reduction of E to $\mathsf{D}(2^{\mathbb{N}})$.*

Proof Let $C \subseteq \mathbb{N}$ and $W, W' \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ be as in the Δ^1_1 Enumeration of § 1.c. By Kreisel Selection there is a Δ^1_1 function $\varphi : X^2 \to C$ such that $W_{\varphi(x,y)} = W'_{\varphi(x,y)}$ is a E-invariant Δ^1_1 set containing x but not y whenever $x, y \in X$ are E-inequivalent. Then $R = \mathtt{ran}\,\varphi$ is a Σ^1_1 subset of C, hence, by Separation, there is a Δ^1_1 set N with $R \subseteq N \subseteq C$. The map $\vartheta(x) = \{n \in N : x \in D_n\}$ is a Δ^1_1 reduction of E to $\mathsf{D}(2^{\mathbb{N}})$. □ (*Lemma and Case 1*)

Case 2: $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$. Then the Σ^1_1 set $H = \{x : [x]_{\mathsf{E}} \subsetneqq [x]_{\overline{\mathsf{E}_0}}\}$ (the union of all $\overline{\mathsf{E}}$-classes containing more than one E-class) is non-empty.

Lemma 17.4 *If $X \subseteq H$ is a Σ^1_1 set then $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$ on X.*

Proof Suppose that $\mathsf{E} \restriction X = \overline{\mathsf{E}} \restriction X$. Then $\mathsf{E} = \overline{\mathsf{E}}$ on $Y = [X]_{\mathsf{E}}$ as well. (If $y, y' \in Y$ then there are $x, x' \in X$ such that $x \mathrel{\mathsf{E}} y$ and $x' \mathrel{\mathsf{E}} y'$, so that if $y \mathrel{\overline{\mathsf{E}}} y'$ then $x \mathrel{\overline{\mathsf{E}}} x'$ by transitivity, hence, $x \mathrel{\mathsf{E}} x'$, and $y \mathrel{\mathsf{E}} y'$ again by transitivity.) It follows that $\mathsf{E} = \overline{\mathsf{E}}$ on an even bigger set, $Z = [X]_{\overline{\mathsf{E}}}$. (Otherwise the Σ^1_1 set $Y' = Z \setminus Y = \{z : \exists x \in X\,(x \mathrel{\overline{\mathsf{E}}} y \wedge x \not\mathrel{\mathsf{E}} y)\}$ is non-empty and E-invariant, together with Y, hence, there is a E-invariant Δ^1_1 set B with $Y \subseteq B$ and $Y' \cap B = \emptyset$ (Lemma 17.1), which implies that no point in Y is E_0-equivalent to a point in Y', contradiction.) Then by definition $Z \cap H = \emptyset$. □ (*Lemma*)

Lemma 17.5 *If $A, B \subseteq H$ are non-empty Σ^1_1 sets with $A \mathrel{\mathsf{E}} B$ then there exist non-empty* <u>*disjoint*</u> *Σ^1_1 sets $A' \subseteq A$ and $B' \subseteq B$ still satisfying $A' \mathrel{\mathsf{E}} B'$.*

Proof We assert that there are points $a \in A$ and $b \in B$ with $a \neq b$ and $a \mathrel{\mathsf{E}} b$.

(Otherwise E is the equality on $X = A \cup B$. Prove that then $\mathsf{E} = \overline{\mathsf{E}}$ on X, a contradiction to Lemma 17.4. Take any $x \neq y$ in X. Let U be a clopen set containing x but not y. Then $A = [U \cap X]_{\mathsf{E}}$ and $C = [X \setminus U]_{\mathsf{E}}$ are two disjoint E-invariant Σ^1_1 sets containing resp. x, y. Then $x \mathrel{\overline{\mathsf{E}}} y$ fails by Lemma 17.1.)

Thus let a, b be as indicated. Let U be a clopen set containing a but not b. Put $A' = A \cap U \cap [U^{\complement}]_{\mathsf{E}}$ and $B' = B \cap U^{\complement} \cap [U]_{\mathsf{E}}$. □ (*Lemma*)

7.c Restricted product forcing

In continuation of the proof of Theorem 17 (Case 2), let $\mathbb{P}^2 \restriction \mathsf{E}$ be the collection of all sets of the form $X \times Y$, where $X, Y \subseteq \mathbb{N}^{\mathbb{N}}$ are non-empty Σ^1_1 sets and $X \mathrel{\mathsf{E}} Y$ (which means here that $[X]_{\mathsf{E}} = [Z]_{\mathsf{E}}$). Easily $\mathbb{P}_2 \subseteq \mathbb{P}^2 \restriction \mathsf{E} \subseteq \mathbb{P}^2$.

Below, $\mathbb{P}^2 \restriction \mathsf{E}$ will be used as a forcing over a countable model $\mathfrak{M}$ chosen in accordance with the requirements in Footnote 5.

The forcing $\mathbb{P}^2 \restriction \mathsf{E}$ is not really a product, yet if $X \times Z \in \mathbb{P}^2 \restriction \mathsf{E}$ and $\emptyset \neq X' \subseteq X$ is Σ^1_1 then $Z' = Z \cap [X']_{\mathsf{E}}$ is Σ^1_1 and $X' \times Z' \in \mathbb{P}^2 \restriction \mathsf{E}$. It follows that any $\mathbb{P}^2 \restriction \mathsf{E}$-generic set $G \subseteq \mathbb{P}^2 \restriction \mathsf{E}$ produces a pair of $\mathbb{P}$-generic sets $G_{\mathtt{left}} = \{\mathtt{dom}\,P : P \in G\}$ and $G_{\mathtt{right}} = \{\mathtt{ran}\,P : P \in G\}$, hence, produces a pair of $\mathbb{P}$-generic reals $x^G_{\mathtt{left}}$ and $x^G_{\mathtt{right}}$, whose names will be $\dot{x}_{\mathtt{left}}$ and $\dot{x}_{\mathtt{right}}$.

Lemma 17.6 *Any $P = X \times Z \in \mathbb{P}^2 \restriction \mathsf{E}$ forces $\langle \dot{x}_{\mathtt{left}}, \dot{x}_{\mathtt{right}} \rangle \in P$ and forces $\dot{x}_{\mathtt{left}} \mathrel{\overline{\mathsf{E}}} \dot{x}_{\mathtt{right}}$, but "condition" $H \times H$ $\mathbb{P}^2 \restriction \mathsf{E}$-forces $\dot{x}_{\mathtt{left}} \not\mathrel{\mathsf{E}} \dot{x}_{\mathtt{right}}$.*

Proof To see that $\dot{x}_{\mathtt{left}} \mathrel{\overline{\mathsf{E}}} \dot{x}_{\mathtt{right}}$ is forced suppose otherwise. Then, by the definition of $\overline{\mathsf{E}}$, there is a condition $P = X \times Z \in \mathbb{P}^2 \restriction \mathsf{E}$ and an E-invariant Δ^1_1 set B such that P forces $\dot{x}_{\mathtt{left}} \in B$ but $\dot{x}_{\mathtt{right}} \notin B$. Then easily $X \subseteq B$ but $Z \cap B = \emptyset$, a contradiction with $[X]_{\mathsf{E}} = [Z]_{\mathsf{E}}$.

To see that $H \times H$ forces $\dot{x}_{\mathtt{left}} \not\mathrel{\mathsf{E}} \dot{x}_{\mathtt{right}}$ suppose towards the contrary that some $P = X \times Z \in \mathbb{P}^2 \restriction \mathsf{E}$ with $X \cup Z \subseteq H$ forces $\dot{x}_{\mathtt{left}} \mathrel{\mathsf{E}} \dot{x}_{\mathtt{right}}$, thus,

(1) We have $x \mathrel{\mathsf{E}} z$ for each $\mathbb{P}^2 \restriction \mathsf{E}$-generic pair $\langle x, z \rangle \in P$.

Claim 17.7 *If $x, y \in X$ are $\mathbb{P}$-generic over $\mathfrak{M}$, and $x \mathrel{\overline{\mathsf{E}}} y$, then $x \mathrel{\mathsf{E}} y$.*

Proof We assert that

(2) $x \in A \iff y \in A$ holds for each E-invariant Σ^1_1 set A.

Indeed, if, say, $x \in A$ but $y \notin A$ then by the genericity of y there is a Σ^1_1 set C with $y \in C$ and $A \cap C = \emptyset$. As A is E-invariant, Lemma 17.1 yields an E-invariant Δ^1_1 set B such that $C \subseteq B$ but $A \cap B = \emptyset$. Then $x \notin B$ but $y \in B$, a contradiction to $x \mathrel{\overline{\mathsf{E}}} y$.

Let $\{\mathcal{D}_n\}_{n \in \mathbb{N}}$ be an enumeration of all dense subsets of $\mathbb{P}^2 \restriction \mathsf{E}$ which are coded in $\mathfrak{M}$. We define two sequences $P_0 \supseteq P_1 \supseteq \ldots$ and $Q_0 \supseteq Q_1 \supseteq \ldots$ of conditions $P_n = X_n \times Z_n$ and $Q_n = Y_n \times Z_n$ in $\mathbb{P}^2 \restriction \mathsf{E}$, so that $P_0 = Q_0 = P$, $x \in X_n$ and $y \in Y_n$ for any n, and finally $P_n, Q_n \in \mathcal{D}_{n-1}$ for $n \geq 1$. If this is done then we have a real z (the only element of $\bigcap_n Z_n$) such that both $\langle x, z \rangle$ and $\langle y, z \rangle$ are $\mathbb{P}^2 \restriction \mathsf{E}$-generic, hence, $x \mathrel{\mathsf{E}} z$ and $y \mathrel{\mathsf{E}} z$ by (1), hence, $x \mathrel{\mathsf{E}} y$.

Suppose that P_n and Q_n have been defined. As x is generic, there is (we leave details for the reader) a condition $P' = A \times C \in \mathcal{D}_n$ and $\subseteq P_n$ such that $x \in A$. Let $B = Y_n \cap [A]_{\mathsf{E}}$: then $y \in B$ by (2), and easily $[B]_{\mathsf{E}} = [C]_{\mathsf{E}} = [A]_{\mathsf{E}}$ (as $[X_n]_{\mathsf{E}} = [Z_n]_{\mathsf{E}} = [Y_n]_{\mathsf{E}}$), thus, $B \times C \in \mathbb{P}^2 \restriction \mathsf{E}$, so there is a condition $Q' = V \times W \in \mathcal{D}_n$ and $\subseteq B \times C \subseteq Q_n$ such that $y \in V$. Put $Y_{n+1} = V$, $Z_{n+1} = W$, and $X_{n+1} = A \cap [W]_{\mathsf{E}}$. □ (*Claim*)

It follows that $\mathsf{E} = \overline{\mathsf{E}}$ on X. (Otherwise $S = \{\langle x, y \rangle \in X^2 : x \mathrel{\overline{\mathsf{E}}} y \wedge x \not\mathrel{\mathsf{E}} y\}$ is a non-empty Σ^1_1 set, and any $\mathbb{P}_2$-generic pair $\langle x, y \rangle \in S$ implies a contradiction to Claim 17.7. Recall that $\mathbb{P}_2 =$ all non-empty Σ^1_1 subsets of $(\mathbb{N}^{\mathbb{N}})^2$.) But this implies $X \cap H = \emptyset$ by Lemma 17.4, contradiction. □ (*Lemma 17.6*)

7.d Splitting system

Let us fix enumerations $\{\mathcal{D}(n)\}_{n \in \mathbb{N}}$, $\{\mathcal{D}_2(n)\}_{n \in \mathbb{N}}$, $\{\mathcal{D}^2(n)\}_{n \in \mathbb{N}}$ of all dense subsets of resp. $\mathbb{P}, , \mathbb{P}_2, \mathbb{P}^2 \restriction \mathsf{E}$, which belong to $\mathfrak{M}$; we assume that $\mathcal{D}(n+1) \subseteq \mathcal{D}(n)$, $\mathcal{D}_2(n+1) \subseteq \mathcal{D}_2(n)$, and $\mathcal{D}^2(n+1) \subseteq \mathcal{D}^2(n)$. If $u, v \in 2^m$ (binary sequences of length m) have the form $u = 0^k{}^\wedge 0^\wedge w$ and $v = 0^k{}^\wedge 1^\wedge w$ for some $k < m$ and $w \in 2^{m-k-1}$ then we call $\langle u, v\rangle$ a *crucial pair*. It can be proved, *e.g.*, by induction on m, that 2^m is a connected tree (*i.e.*, a connected graph without cycles) of crucial pairs, with sequences beginning with 1 as the endpoints of the graph. We define a system of sets X_u ($u \in 2^{<\omega}$) and R_{uv}, $\langle u, v\rangle$ being a crucial pair, so that the following conditions are satisfied:

(i) $X_u \in \mathbb{P}$, moreover, $X_\Lambda \subseteq H$, and $X_u \in \mathcal{D}(n)$ whenever $u \in 2^n$;

(ii) $X_{u^\wedge i} \subseteq X_u$ for all u and i;

(iii) $\mathsf{R}_{uv} \in \mathbb{P}_2$, moreover, $X_u \in \mathcal{D}_2(n)$ whenever $u \in 2^n$;

(iv) $\mathsf{R}_{uv} \subseteq \mathsf{E}$ and $X_u \,\mathsf{R}_{uv}\, X_v$ (for any crucial pair $\langle u, v\rangle$);

(v) $\mathsf{R}_{u^\wedge i,\, v^\wedge i} \subseteq \mathsf{R}_{uv}$;

(vi) if $u, v \in 2^n$ and $u(n-1) \neq v(n-1)$ then $X_u \times X_v \in \mathcal{D}^2(n)$ and also $X_u \cap X_v = \emptyset$.

Note that (iv) implies that $X_u \mathsf{E} X_v$ for any crucial pair $\langle u, v\rangle$, hence, also for any pair in 2^n because any two $u, v \in 2^n$ are connected by a unique chain of crucial pairs. It follows that $X_u \times X_v \in \mathbb{P}^2 \restriction \mathsf{E}$ for any pair of $u, v \in 2^n$, for any n.

Assume that such a system has been defined. Then for any $a \in 2^{\mathbb{N}}$ the sequence $\{X_{a\restriction n}\}_{n\in\mathbb{N}}$ is $\mathbb{P}$-generic over $\mathfrak{M}$, hence, $\bigcap_n X_{a\restriction n} = \{x_a\}$, where x_a is $\mathbb{P}$-generic, and the map $a \mapsto x_a$ is continuous since diameters of X_u converge to 0 uniformly with $\mathtt{lh}\, u \to 0$ by (i), and is $1-1$ by the last condition of (vi).

Let $a, b \in 2^{\mathbb{N}}$. If $a \not\mathsf{E}_0 b$ then, by (vi), $\langle x_a, x_b\rangle$ is a $\mathbb{P}^2 \restriction \mathsf{E}$-generic pair, hence, $x_a \not\mathsf{E}\, x_b$ by Lemma 17.6. Now suppose that $a \,\mathsf{E}_0\, b$, prove that then $x_a \,\mathsf{E}\, x_b$. We can suppose that $a = w^\wedge 0^\wedge c$ and $b = w^\wedge 0^\wedge c$, where $w \in 2^{<\omega}$ and $c \in 2^{\mathbb{N}}$ (indeed if $a \,\mathsf{E}_0\, b$ then a, b can be connected by a finite chain of such special pairs). Then $\langle x_a, x_b\rangle$ is $\mathbb{P}_2$-generic, actually, the only member of the intersection $\bigcap_n \mathsf{R}_{w^\wedge 0^\wedge (c\restriction n),\, w^\wedge 1^\wedge (c\restriction n)}$ by (iii) and (iv), in particular, $x_a \,\mathsf{E}\, x_b$ because we have $R_{uv} \subseteq \mathsf{E}$ for all u, v.

Thus we have a continuous $1-1$ reduction of E_0 to E.

$\square$ (*Case 2 in Theorem 17*)

7.e Construction of a splitting system

Let X_Λ be any member of $\mathcal{D}(0)$ satisfying $X_\Lambda \subseteq H$. Now suppose that X_s and R_{st} have been defined for all $s \in 2^n$ and all crucial pairs in 2^n, and extend the construction on 2^{n+1}. Temporarily, define $X_{s^\wedge i} = X_s$ and $\mathsf{R}_{s^\wedge i,\, t^\wedge i} = \mathsf{R}_{st}$: this leaves $\mathsf{R}_{0^n{}^\wedge 0,\, 0^n{}^\wedge 1}$ still undefined, so we put $\mathsf{R}_{0^n{}^\wedge 0,\, 0^n{}^\wedge 1} = \mathsf{E} \cap X_{0^n} \times X_{0^n}$. Note that the such defined system of sets X_u and relations R_{uv} at level $n+1$ satisfies all requirements of (i) – (vi) except for the requirement of membership in the dense sets – say in this case that the system is "coherent". It remains to produce a still "coherent" system of smaller sets and relations which also satisfies the membership in the dense sets. This will be achieved in several steps.

Step 1: achieve that $X_u \in \mathcal{D}(n+1)$ for any $u \in 2^{n+1}$. Take any particular $u_0 \in 2^{n+1}$. There is, by the density, $X'_{u_0} \in \mathcal{D}(n+1)$ and $\subseteq X_{u_0}$. Suppose that $\langle u_0, v\rangle$ is a crucial pair. Put $\mathsf{R}'_{u_0,v} = \{\langle x, y\rangle \in \mathsf{R}_{u_0,v} : x \in X'_{u_0}\}$ and $X'_v = \mathtt{ran}\, \mathsf{R}'_{u_0,v}$. This shows how the change spreads along the whole set 2^{n+1} viewed as the tree of crucial pairs. Finally we obtain a coherent system with the additional requirement that $X'_{u_0} \in \mathcal{D}(n+1)$. Do this consecutively for all $u_0 \in 2^{n+1}$. The total result – we re-denote it as still X_u and R_{uv} – is a "coherent" system with $X_u \in \mathcal{D}(n+1)$ for all u. Note that still $X_{0^n{}^\wedge 0} = X_{0^n{}^\wedge 1}$ and

$$\mathsf{R}_{0^n{}^\wedge 0,\, 0^n{}^\wedge 1} = \mathsf{E} \cap (X_{0^n{}^\wedge 0} \times X_{0^n{}^\wedge 1}). \qquad (*)$$

Step 2: achieve that $X_{s^\wedge 0} \times X_{t^\wedge 1} \in \mathcal{D}^2(n+1)$ for all $s, t \in 2^{n+1}$. Consider a pair of $u_0 = s_0{}^\wedge 0$ and $v_0 = t_0{}^\wedge 1$ in 2^{n+1}. By the density there is a set $X'_{u_0} \times X'_{v_0} \in \mathcal{D}^2(n+1)$ and $\subseteq X_{u_0} \times X_{v_0}$. By definition we have $X'_{u_0} \mathsf{E} X'_{v_0}$, but, due to Lemma 17.5 we can maintain that $X'_{u_0} \cap X'_{v_0} = \emptyset$. The two "shockwaves", from the changes at u_0 and v_0, as in Step 1, meet only at the pair $0^m{}^\wedge 0$, $0^m{}^\wedge 1$, where the new sets satisfy $X'_{0^m{}^\wedge 0} \,\mathsf{E}\, X'_{0^m{}^\wedge 1}$ just because E-equivalence is everywhere kept and preserved though the changes. Now, in view of $(*)$, we can define $\mathsf{R}'_{0^n{}^\wedge 0,\, 0^n{}^\wedge 1} = \mathsf{E} \cap (X'_{0^n{}^\wedge 0} \times X'_{0^n{}^\wedge 1})$, preserving $(*)$ as well. All pairs considered, we will be left with a coherent system of sets and relations, re-denoted as X_u and R_{uv}, which satisfies the $\mathcal{D}(n+1)$-requirements in (i) and (vi).

Step 3: achieve that $\mathsf{R}_{uv} \in \mathcal{D}_2(n+1)$ for any crucial pair at level $n+1$, and also that $X'_{0^n{}^\wedge 0} \cap X'_{0^n{}^\wedge 1} = \emptyset$. Consider any crucial pair $\langle u_0, v_0\rangle$. If this is not $\langle 0^n{}^\wedge 0, 0^n{}^\wedge 1p\rangle$ then let $\mathsf{R}'_{u_0v_0} \subseteq \mathsf{R}_{u_0v_0}$ be any set in $\mathcal{D}_2(n+1)$. If this is $u_0 = 0^n{}^\wedge 0$ and $v_0 = 0^n{}^\wedge 1$ then first we choose (Lemma 17.5) disjoint non-empty Σ^1_1 sets $U \subseteq X_{0^n{}^\wedge 0}$ and $V \subseteq X_{0^n{}^\wedge 1}$ still with $U \,\mathsf{E}\, V$, and only then a set $\mathsf{R}'_{u_0v_0} \subseteq \mathsf{E} \cap (U \times V)$ which belongs to $\in \mathcal{D}_2(n+1)$. In both cases, put $X'_{u_0} = \mathtt{dom}\, \mathsf{R}'_{u_0v_0}$ and $X'_{v_0} = \mathtt{ran}\, \mathsf{R}'_{u_0v_0}$. It remains to spread the changes, along the chain of crucial pairs, to the left of u_0 and to the right of v_0, exactly as in Case 1. Executing such a reduction for all crucial pairs $\langle u_0, v_0\rangle$ at level $n+1$ one by one, we end up with a system of sets fully satisfying (i) – (vi).

$\square$ (*Theorem 17*)

7.f When $\mathsf{E}_0 \restriction X$ is smooth ?

It follows from the 2nd dichotomy that if $X \subseteq 2^{\mathbb{N}}$ is a Borel set then either $\mathsf{E}_0 \restriction X$ is smooth or $\mathsf{E}_0 \restriction X \sim_{\mathrm{B}} \mathsf{E}_0$. Let is show that the existence of a full Borel transversal is necessary and sufficient for the smoothness. Recall that a E-*transversal* is a set having exactly one element in common with every E-class. Let a *partial transversal* be a set having ≤ 1 element in common with every E-class, in other words, it is required that the set is pairwise E-inequivalent.

Theorem 18 *Let $X \subseteq 2^{\mathbb{N}}$ be Σ^1_1. Then exactly one of the following two conditions is satisfied:*

(I) *there is a Σ^1_1 transversal S for $\mathsf{E}_0 \restriction X$;*

(II) *there is a closed set $F \subseteq X$ such that $\mathsf{E}_0 \sim_{\mathrm{B}} \mathsf{E}_0 \restriction F$.*

Proof To see that (I) and (II) are incompatible, prove

Lemma 18.1 *Any Σ^1_1 partial transversal S for E_0 can be covered by a Δ^1_1 partial transversal D .*

Proof $P = \{x \in 2^{\mathbb{N}} : x \notin [S \setminus \{x\}]_{\mathsf{E}_0}\}$ is a Π^1_1 set and $S \subseteq P$: P is the union of S and all E_0-classes disjoint from S. If U is a Δ^1_1 set with $S \subseteq U \subseteq P$ then $D = \{x \in U : \mathtt{card}([x]_{\mathsf{E}_0} \cap U) = 1\}$ is as required, that D remains Δ^1_1 follows from the countability of E_0-classes. □ (*Lemma*)

Now, if F is as in (II) then $\mathsf{E}_0 \restriction F$ has a Δ^1_1 transversal by the lemma, hence, E_0 itself has a Borel transversal D, then E_0 would be smooth via the map sending any x to the only element of $[x]_{\mathsf{E}_0} \cap D$, contradiction.

To see that (I) $\vee$ (II), define $x \,\overline{\mathsf{E}_0}\, y$ iff x, y belong to the same E_0-invariant Δ^1_1 sets. The rest depends on the relationships between the given X and the Σ^1_1 set $H_0 = \{x : [x]_{\mathsf{E}_0} \subsetneqq [x]_{\overline{\mathsf{E}}}\}$.

Case 1: $X \cap H_0 = \emptyset$. Following the proof of Lemma 17.3, we obtain a Δ^1_1 set Y with $X \subseteq Y \subseteq {H_0}^{\complement}$ and a Δ^1_1 reduction $\vartheta : Y \to 2^{\mathbb{N}}$ of $\mathsf{E}_0 \restriction Y$ to $\mathsf{D}(2^{\mathbb{N}})$. The set $P = \{\langle a, y\rangle : y \in Y \wedge \vartheta(y) = a\}$ is then a Δ^1_1 set whose cross-sections $P_a = \vartheta^{-1}(a)$ are just classes of $\mathsf{E}_0 \restriction Y$. Then by Countable-to-1 Enumeration of § 1.c there is Δ^1_1 set $T \subseteq \mathbb{N} \times Y$ such that each $T_n = \{y : T(n, y)\}$ is a transversal for $\mathsf{E}_0 \restriction Y$ and $Y = \bigcup_n T_n$. Now it takes some efforts, which we leave for the reader (in particular Kreisel Selection is applied) to assemble a Σ^1_1 transversal for $\mathsf{E}_0 \restriction X$ from parts of the transversals T_n .

Case 2: $Z = X \cap H_0 \neq \emptyset$. Then the argument of Case 2 of the proof of Theorem 17 (begin with Z rather than with the whole H !) shows that there is a perfect set $F \subseteq Z$ such that $\mathsf{E}_0 \leq_{\mathrm{B}} \mathsf{E}_0 \restriction F$.

□ (*Theorem 18*)

8 Assembling equivalence relations

8.a Countable unions of domains

Theorem 19 *Let E be a Borel ER on a Borel set $X = \bigcup_k X_k$, with all X_k also Borel. Suppose that each $\mathsf{E} \restriction X_k$ is smooth. Then E is smooth.*

Proof [6] First consider the case of a union $X = Y \cup Z$ of just two Borel sets, so that a Borel ER E is smooth on both Y and Z. We can assume that $Y \cap Z = \emptyset$. Let this be witnessed by Borel reductions $f : Y \to Q$ and $g : Z \to R$, with Q, R being disjoint Borel sets. The set

$$F = \{\langle q, r\rangle : \exists y \in Y\ \exists z \in Z\ (f(y) = q \wedge g(z) = r \wedge y \mathbin{\mathsf{E}} z)\} \subseteq Q \times R$$

is a partial $\mathbf{\Sigma}^1_1$ map $Q \to R$. Let $G : Q \to R$ be any Borel map with $F \subseteq G$, and $H : R \to Q$ be any Borel map with $F^{-1} \subseteq H$. Then $\Phi = G \cap H^{-1}$ is a $1-1$ Borel partial map $P \to Q$ with $F \subseteq \Phi$. Now the $\mathbf{\Pi}^1_1$ set

$$P = \{\langle q, r\rangle \in \Phi : \forall y \in Y\ \forall z \in Z\ (f(y) = q \wedge g(z) = r \Longrightarrow y \mathbin{\mathsf{E}} z)\},$$

satisfies $F \subseteq P \subseteq \Phi$, hence, there is a Borel function Ψ with $F \subseteq \Psi \subseteq P$. The sets $A = \mathtt{dom}\,\Psi$ and $B = \mathtt{ran}\,\Psi$ are Borel subsets of resp. Q', R', and it follows from the construction that $\Psi \cap (\mathtt{dom}\,F \times \mathtt{ran}\,F) = F$. Finally, put

$$D = \Psi \cup \{\langle q, q\rangle : q \in Q \setminus A\} \cup \{\langle r, r\rangle : r \in R \setminus B\},$$

then, for any $y \in Y$ there is unique $h(y) = \langle q, r\rangle \in D$ with $q = g(y)$, correspondingly, for any $z \in Z$ there is unique $h(z) = \langle q, r\rangle \in D$ with $r = g(z)$, and if $y \mathbin{\mathsf{E}} z$ then $h(y) = h(z) = \langle f(y), g(z)\rangle$, hence, h witnesses that E is smooth.

As for the general case, we can now assume that $X_k \subseteq X_{k+1}$ for all k. Then there are disjoint Borel sets B_k and Borel maps $f_k : X_k \to B_k$ which witness that $\mathsf{E} \restriction X_k$ are smooth ERs. Let $R_k = \mathtt{ran}\,f_k$ (a $\mathbf{\Sigma}^1_1$ set) and

$$F_k = \{\langle a, b\rangle \in R_k \times R_{k+1} : \exists x \in X_k\ (f_k(x) = a \wedge f_{k+1}(x) = b)\},$$

this is a $\mathbf{\Sigma}^1_1$ set and a $1-1$ map $R_k \to R_{k+1}$. For each k there is a Borel $1-1$ map G_k with $F_k \subseteq G_k$. Let $A_k = \mathtt{dom}\,G_k$ and $\mathtt{ran}\,G_k = B_k$: these are Borel sets with $R_k \subseteq A_k$. We can assume that $B_k \subseteq A_{k+1}$ (otherwise G_k can be reduced in a certain iterative manner to achieve this property). Then, for any k and $b \in A_k$ there is the least $n = n(b) \leq k$ such that the application

$$h(b) = G_n^{-1}(G_{n+1}^{-1}(G_{n+2}^{-1}(...G_{k-1}^{-1}(b)...)))$$

is possible, for instance, $n(b) = k$ and $h(b) = b$ whenever $b \in A_k \setminus B_{k-1}$. Then, $h(f_k(x)) = h(f_{k+1}(x))$ holds for any $x \in X_k$ because $F_k \subseteq G_k$, so that the map $g(x) = h(f_k(x))$ for $x \in X_k \setminus X_{k-1}$ witnesses the smoothness of E . □

[6] The shortest proof is to note that otherwise $\mathsf{E}_0 \leq_{\mathrm{B}} \mathsf{F}$ by the 2-nd dichotomy. Let this be held via a Borel reduction $\vartheta : \mathcal{P}(\mathbb{N}) \to X$. Then, for some k, the preimage $\vartheta^{-1}(X_k)$ is not meager in $\mathcal{P}(\mathbb{N})$, which implies $\mathsf{E}_0 \leq_{\mathrm{B}} \mathsf{E}_0 \restriction \vartheta^{-1}(X_k)$, hence $\mathsf{E}_0 \leq_{\mathrm{B}} \mathsf{F} \restriction X_k$, which contradicts the smoothness. However we prefer to give a more elementary proof.

Theorem 20 *Let* E *be a Borel ER on a Borel set* $X = \bigcup_k X_k$, *with all* X_k *also Borel. Suppose that* $\mathsf{E} \restriction X_k \le_{\mathrm{B}} \mathsf{E}_0$ *for each* k. *Then* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$.

Proof We consider only the case when $X_k \subseteq X_{k+1}$ for all k (the result will be used below only for this particular case), the general case needs to consider separately the two–sets case, as in Theorem 19, which we leave to the reader.

There are disjoint Borel sets $B_k \subseteq \mathcal{P}(\mathbb{N})$ and Borel maps $f_k : X_k \to B_k$ which witness that $\mathsf{E} \restriction X_k \le_{\mathrm{B}} \mathsf{E}_0$. We shall assume that the sets B_k are E_0-incompatible in the sense that if $k \ne n$ then $a \mathrel{\mathsf{E}_0} b$ does not hold for any $a \in B_k$ and $b \in B_n$. Let $R_k = \mathtt{ran}\, f_k$ (a $\mathbf{\Sigma}^1_1$ subset of B_k). Then

$$F_k = \{\langle a, b\rangle \in R_k \times R_{k+1} : \exists x \in X_k\, (f_k(x) = a \wedge f_{k+1}(x) = b)\},$$

is a $\mathbf{\Sigma}^1_1$ set, $1-1$ modulo E_0 in the sense that if $\langle a, b\rangle$ and $\langle a', b'\rangle$ belong to F_k then $a \mathrel{\mathsf{E}_0} a' \iff b \mathrel{\mathsf{E}_0} b'$. As "to be $1-1$ modulo E_0" is a Π^1_1 property in the codes (of Σ^1_1 subsets of $\mathcal{P}(\mathbb{N})^2$), there is, by Reflection, a Δ^1_1 set F'_k with $F_k \subseteq F'_k \subseteq B_k \times B_{k+1}$ and still $1-1$ modulo E_0. The following Δ^1_1 set

$$G_k = \{\langle a', b'\rangle : \exists \langle a, b\rangle \in F'_k\, (a \mathrel{\mathsf{E}_0} a' \wedge b \mathrel{\mathsf{E}_0} b')\}$$

is still $1-1$ modulo E_0, hence, both "vertical" and "horisontal" cross-sections of G_k are countable, thus, $A_k = \mathtt{dom}\, G_k$ and $B_k = \mathtt{ran}\, G_k$ are E_0-invariant Borel sets (and $R_k = \mathtt{dom}\, F_k \subseteq A_k$), and there are Borel maps $h_k : B_k \to A_k$ such that $\langle h_k(b), b\rangle \in G_k$ whenever $b \in B_k$. It follows still from the "$1-1$ modulo E_0" property that if $b \in B_k$ and $b' \mathrel{\mathsf{E}_0} b$ then $b' \in B_k$ and $h_k(b) \mathrel{\mathsf{E}_0} h_k(b')$.

We can assume that $B_{k+1} \subseteq A_k$ for all k. Then, for any k and $b \in A_k$ there is the least $n = n(b) \le k$ such that the application

$$h(b) = h_n(h_{n+1}(h_{n+2}(\ldots h_{k-1}(b)\ldots)))$$

is possible, for instance, $n(b) = k$ and $h(b) = b$ whenever $b \in A_k \setminus B_{k-1}$. As in the proof of Theorem 19, the map $g(x) = h(f_k(x))$ for $x \in X_k \setminus X_{k-1}$ witnesses $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$. □

8.b Assembling smooth equivalence relations

The following theorem shows that, in some cases, if a two-dimensional Borel ER is smooth on each vertical cross-section then the associated Borel cardinality can be bounded by the "trace" of the ER on the horisontal axis.

Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are Polish spaces, and $P \subseteq \mathbb{X} \times \mathbb{Y}$ is a Borel set. For any $x \in \mathbb{X}$, define $P(x) = \{\langle x', y\rangle \in P : x' = x\}$.

Theorem 21 *Let* $\mathbb{X}, \mathbb{Y}, P$ *be as indicated. Suppose that* E *is a Borel ER on* P, *and* $\mathbb{G}$ *is a countable group acting on* $\mathbb{X}$ *in a Borel way so that* $\langle x, y\rangle \mathrel{\mathsf{E}} \langle x', y'\rangle$ *implies* $x \mathrel{\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}} x'$. *Finally, assume that* $\mathsf{E} \restriction P(x)$ *is smooth for each* $x \in \mathbb{X}$. *Then* E *is Borel-reducible to a Borel action of* $\mathbb{G}$.

Proof We can assume that $\mathbb{X} = \mathbb{Y} = 2^{\mathbb{N}}$ and both P and E are Δ^1_1. We can also assume that the action of $\mathbb{G}$ (a countable group) is Δ^1_1. Then clearly $x \mathrel{\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}} x' \Longrightarrow \Delta^1_1(x) = \Delta^1_1(x')$. Define $P^*(x) = \bigcup_{a \in \mathbb{G}} P(a \cdot x)$ for $x \in \mathbb{X}$.

Claim 21.1 *Suppose that* $\langle x, y\rangle$ *and* $\langle x', y'\rangle$ *belong to* P *and* $x \mathrel{\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}} x'$. *Then* $\langle x, y\rangle \mathrel{\mathsf{E}} \langle x', y'\rangle$ *iff the equivalence* $\langle x, y\rangle \in U \iff \langle x', y'\rangle \in U$ *holds for any* $\mathsf{E} \restriction P^*(x)$*–invariant* $\Delta^1_1(x)$ *set* $U \subseteq P^*(x)$.

Proof Note that $\mathsf{E} \restriction P^*(x)$ is still smooth by Theorem 19 because $\mathbb{G}$ is countable. In addition $\mathsf{E} \restriction P^*(x)$ is $\Delta^1_1(x)$. This observation yields the result, because otherwise, *i.e.*, if the ER, defined om $P^*(x)$ by intersections with $\mathsf{E} \restriction P^*(x)$–invariant $\Delta^1_1(x)$ sets, is coarser than $\mathsf{E} \restriction P^*(x)$, then it is known from the proof of the 2nd dichotomy theorem (Theorem 17) that we would have $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E} \restriction P^*(x)$, a contradiction with the smoothness. □ (*Claim*)

For any $x \in \mathbb{X}$ let $E(x)$ be the set of all $e \in \mathbb{N}$ which code a $\Delta^1_1(x)$ subset of P, and, for $e \in E(x)$, let $\mathbf{W}^e_x$ be the $\Delta^1_1(x)$ subset of P coded by e. (It is known that $\{\langle x, e\rangle : e \in E(x)\}$ is Π^1_1.) Let $\mathtt{inv}(x, e)$ be the formula

$$x \in \mathbb{X} \wedge e \in E(x) \wedge \mathbf{W}^e_x \subseteq P^*(x) \wedge \mathbf{W}^e_x \text{ is } \mathsf{E} \restriction P^*(x)\text{–invariant}.$$

Corollary 21.2 *Let* $\langle x, y\rangle$, $\langle x', y'\rangle$ *be as in Claim 21.1. Then* $\langle x, y\rangle \mathrel{\mathsf{E}} \langle x', y'\rangle$ *iff* $\langle x, y\rangle \in \mathbf{W}^e_x \iff \langle x', y'\rangle \in \mathbf{W}^e_x$ *holds for any* e *with* $\mathtt{inv}(x, e)$. □

Implication $\Longleftarrow$ of the "iff" in this Corollary can be considered as a property of the Π^1_1 set $C = \{\langle x, e\rangle : \mathtt{inv}(x, e)\}$, *i.e.*, the property that

- for all pairs $\langle x, y\rangle$ and $\langle x', y'\rangle$ in P with $x \mathrel{\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}} x'$, we have:
  if $\forall \langle x, e\rangle \in C\, (\langle x, y\rangle \in \mathbf{W}^e_x \Leftrightarrow \langle x', y'\rangle \in \mathbf{W}^e_x)$ then $\langle x, y\rangle \mathrel{\mathsf{E}} \langle x', y'\rangle$.

This is easily a Π^1_1 property in the codes, hence, by the Π^1_1 Reflection, there is a Δ^1_1 set $B \subseteq C$ satisfying the same property, that is, we have

Corollary 21.3 *Let* $\langle x, y\rangle$, $\langle x', y'\rangle$ *be as in Claim 21.1. Then* $\langle x, y\rangle \mathrel{\mathsf{E}} \langle x', y'\rangle$ *iff* $\langle x, y\rangle \in \mathbf{W}^e_x \iff \langle x', y'\rangle \in \mathbf{W}^e_x$ *holds for any* e *with* $\langle x, e\rangle \in B$. □

To continue the proof of the theorem, define, for any $\langle x, y\rangle \in P$,

$$D_{xy} = \{\langle a, e\rangle : a \in \mathbb{G} \wedge \langle a \cdot x, e\rangle \in B \wedge \langle x, y\rangle \in \mathbf{W}^e_{a \cdot x}\}.$$

Clearly $\langle x, y\rangle \mapsto D_{x,y}$ is a Δ^1_1 map $P \to \mathcal{P}(\mathbb{G} \times \mathbb{N})$.

If $D \subseteq \mathbb{G} \times \mathbb{N}$ and $b \in \mathbb{G}$ then put $b \circ D = \{\langle ab^{-1}, e\rangle : \langle a, e\rangle \in D\}$.

Claim 21.4 *Suppose that* $\langle x, y\rangle$ *and* $\langle x', y'\rangle$ *belong to* P, $b \in \mathbb{G}$, *and* $x' = b \cdot x$. *Then* $\langle x, y\rangle \mathrel{\mathsf{E}} \langle x', y'\rangle$ *iff* $b \circ D_{xy} = D_{x'y'}$.

Proof Assume that $b \circ D_{xy} = D_{x'y'}$. According to Corollary 21.3, to prove $\langle x, y\rangle \mathsf{E} \langle x', y'\rangle$ it suffices to show that $\langle x, y\rangle \in \mathbf{W}^e_x \iff \langle x', y'\rangle \in \mathbf{W}^e_x$ holds whenever $\langle x, e\rangle \in B$. We have

$$\langle x, y\rangle \in \mathbf{W}^e_x \Leftrightarrow \langle \Lambda, e\rangle \in D_{xy} \Leftrightarrow \langle b^{-1}, e\rangle \in D_{x'y'} \Leftrightarrow \langle x', y'\rangle \in \mathbf{W}^e_{b^{-1}\cdot x'} = \mathbf{W}^e_x\,,$$

as required. Conversely, let $\langle x, y\rangle \mathsf{E} \langle x', y'\rangle$. If $\langle a, e\rangle \in D_{xy}$ then $\langle a\cdot x, e\rangle \in B$ and $\langle x, y\rangle \in \mathbf{W}^e_{a\cdot x}$, hence, $\langle x', y'\rangle \in \mathbf{W}^e_{a\cdot x}$, too, because the set $\mathbf{W}^e_{a\cdot x}$ is invariant and $\langle x, y\rangle \mathsf{E} \langle x', y'\rangle$. Yet $a\cdot x = ab^{-1}\cdot x'$, therefore, by definition, $\langle ab^{-1}, e\rangle \in D_{x'y'}$. The same argument can be carried out in the opposite direction, so that $\langle a, e\rangle \in D_{xy}$ iff $\langle ab^{-1}, e\rangle \in D_{x'y'}$, that means $b \circ D_{xy} = D_{x'y'}$. □ (*Claim*)

To end the proof of the theorem, consider $\mathbb{Z} = \mathbb{X} \times \mathcal{P}(\mathbb{G} \times \mathbb{N})$, a Polish space. Define a Borel action $b\cdot\langle x, D\rangle = \langle b\cdot x, b \circ D\rangle$ of $\mathbb{G}$ on $\mathbb{Z}$. We assert that $\vartheta(x, y) = \langle x, D_{xy}\rangle$ is a Borel reduction of $\mathsf{E} \restriction P$ to the action $\mathsf{E}^{\mathbb{G}}_{\mathbb{Z}}$. Indeed, let $\langle x, y\rangle$ and $\langle x', y'\rangle$ belong to P. Suppose that $\langle x, y\rangle \mathsf{E} \langle x', y'\rangle$. Then $x \mathsf{E}^{\mathbb{G}}_{\mathbb{X}} x'$, so that $x' = b\cdot x$ for some $b \in \mathbb{G}$. Moreover, $b \circ D_{xy} = D_{x'y'}$ by Claim 21.4, hence, $\vartheta(x', y') = b\cdot\vartheta(x, y)$. Let, conversely, $\vartheta(x', y') = b\cdot\vartheta(x, y)$, so that $x' = b\cdot x$ and $D_{x'y'} = b \circ D_{xy}$. Then $\langle x, y\rangle \mathsf{E} \langle x', y'\rangle$ by Claim 21.4, as required. □

9 Ideal $\mathcal{I}_1$ and the P-ideals

The ideal $\mathtt{Fin} \times 0$, often denoted traditionally as $\mathcal{I}_1$, is the first amongst the "key ideals" other than $\mathtt{Fin}$, which we consider. Recall that the ideal $\mathcal{I}_1 = \mathtt{Fin} \times 0$ consists of all sets $x \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ such that all, except for finitely many, cross-sections $(x)_n = \{k : \langle n, k\rangle \in x\}$ are empty.

9.a Ideals below $\mathcal{I}_1$

It turns out that there exist only three different ideals Borel reducible to $\mathcal{I}_1$, they are $\mathtt{Fin}$, the disjoint sum $\mathtt{Fin} \oplus \mathcal{P}(\mathbb{N})$, and $\mathcal{I}_1$ itself.

Lemma 22 $\mathtt{Fin} <_{\mathrm{B}} \mathcal{I}_1$.

Proof That $\mathtt{Fin} \le_{\mathrm{B}} \mathcal{I}_1$ is witnessed by the map $f(x) = \{\langle 0, n\rangle : n \in x\}$. That $\mathcal{I}_1$ is not Borel reducible to $\mathtt{Fin}$ can be easily derived from some results below in several ways, but we would like to give a direct proof. Let, on the contrary, $\vartheta : \mathcal{P}(\mathbb{N} \times \mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be a Borel reduction, so that $x \,\Delta\, y \in \mathcal{I}_1$ iff $\vartheta(x) \,\Delta\, \vartheta(y) \in \mathtt{Fin}$. By the Baire category argument, there is a finite set s and a pair of non-empty basic clopen sets $U, V \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ such that $\vartheta(x) \,\Delta\, \vartheta(y) = s$ for comeager-many pairs $\langle x, y\rangle \in U \times V$. We may express this differently: $\vartheta(x) \,\Delta\, \vartheta(y) = s$ for any "generic" enough pair $\langle x, y\rangle \in U \times V$. We conclude that if $x, x' \in U$ are sufficiently "generic" then $\vartheta(x) = \vartheta(x')$ simply because there exists $y \in V$ such that both $\langle x, y\rangle$ and $\langle x', y\rangle$ are "generic". Thus ϑ is constant on a comeager set $D \subseteq U$. But any non-meager $D \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ contains a pair of elements x, x' with $x \,\Delta\, x' \notin \mathcal{I}_1$, contradiction. □

In this Subection $\mathcal{I} \cong \mathcal{J}$ will mean: isomorphism via a bijection between the underlying sets. Following Kechris [30], let us say that an ideal $\mathcal{I}$ is a *trivial variation* of $\mathcal{J}$ if there is an infinite set D such that $I \restriction D \cong \mathcal{J}$ while $\mathcal{I} \restriction \complement D = \mathcal{P}(\complement D)$. (The last condition is equivalent to $\mathcal{I} = \{x : x \cap D \in \mathcal{I} \restriction D\}$.)

Exercise 23 Prove that any trivial variation of $\mathcal{I}_1$ is isomorphic to $\mathcal{I}_1$. Prove that any trivial variation of $\mathtt{Fin}$ is isomorphic either to $\mathtt{Fin}$ or to the disjoint sum $\mathtt{Fin} \oplus \mathcal{P}(\mathbb{N})$, *e.g.*, realized in the form of $\{x \subseteq \mathbb{N} : x \cap \textsc{odd} \in \mathtt{Fin}\}$. □

Theorem 24 (Kechris [30]) *If $\mathcal{I} \le_{\mathrm{B}} \mathcal{I}_1$ is a Borel (nontrivial) ideal on $\mathbb{N}$ then either $\mathcal{I} \cong \mathcal{I}_1$ or $\mathcal{I}$ is a trivial variation of $\mathtt{Fin}$.*

Proof We begin with another version of the smart method used in the proof of Theorem 14. Suppose that $\{\mathcal{B}_k\}_{k\in\mathbb{N}}$ is a fixed system of Borel subsets of $\mathcal{P}(\mathbb{N})$. (It will be specified later.) Then there exists an increasing sequence of integers $0 = n_0 < n_1 < n_2 < \dots$ and sets $s_k \subseteq [n_k, n_{k+1})$ such that

(1) any $x \subseteq \mathbb{N}$ with $\forall^\infty k\,(x \cap [n_k, n_{k+1}) = s_k)$ is "generic" [7] ;

(2) if $k' \geq k$ and $u \subseteq [0, n_{k'})$ then $u \cup s_{k'}$ decides $\mathcal{B}_k$ in the sense that either any "generic" $x \in \mathcal{P}(\mathbb{N})$ with $x \cap [0, n_{k'+1}) = u \cup s_{k'}$ belongs to $\mathcal{B}_k$ or any "generic" x with $x \cap [0, n_{k'+1}) = u \cup s_{k'}$ does not belong to $\mathcal{B}_k$.

Now put $\mathcal{D}_0 = \{x \cup S_1 : x \subseteq Z_0\}$ and $\mathcal{D}_1 = \{x \cup S_0 : x \subseteq Z_1\}$, where

$$S_0 = \bigcup_k s_{2k} \subseteq Z_0 = \bigcup_k [n_{2k}, n_{2k+1}), \quad S_1 = \bigcup_k s_{2k+1} \subseteq Z_1 = \bigcup_k [n_{2k+1}, n_{2k+2}).$$

Clearly any $x \in \mathcal{D}_0 \cup \mathcal{D}_1$ is "generic" by (1), hence, by (2),

(3) each $\mathcal{B}_k$ is clopen on both $\mathcal{D}_0$ and $\mathcal{D}_1$.

Now let us come closer to the theorem. It follows from Lemma 5 (and the trivial fact that $\mathcal{I}_1 \oplus \mathcal{I}_1 \cong \mathcal{I}_1$) that there exists a *continuous* reduction $\vartheta : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N} \times \mathbb{N})$, so that $\mathsf{E}_{\mathcal{I}}$ is the union of an increasing sequence of (topologically) closed ERs $\mathsf{R}_m \subseteq \mathsf{E}_{\mathcal{I}}$ just because $\mathcal{I}_1$ admits such a form. We now require that $\{\mathcal{B}_k\}$ includes all sets $B^m_l = \{x \in \mathcal{P}(\mathbb{N}) : \forall s \subseteq [0,l)\; x \,\mathsf{R}_m\, (x \,\Delta\, s)\}$. Then by (3) and the compactness of $\mathcal{D}_i$ for any l there is $m(l) \geq l$ satisfying

(4) $\forall x \in \mathcal{D}_0 \cup \mathcal{D}_1\, \forall s \subseteq [0,l)\; (x \,\mathsf{R}_{m(l)}\, (x \,\Delta\, s))$.

To prove the theorem it suffices to obtain a sequence $x_0 \subseteq x_1 \subseteq x_2 \subseteq \dots$ of sets $x_k \in \mathcal{I}$ with $\mathcal{I} = \bigcup_n \mathcal{P}(x_n)$: that in this case $\mathcal{I}$ is as required is an easy exercise. As any topologically closed ideal is easily $\mathcal{P}(x)$ for some $x \subseteq \mathbb{N}$, it suffices to show that $\mathcal{I}$ is a union of a countable sequence of closed subideals. (In this argument, "ideal" means only a subset of $\mathcal{P}(\mathbb{N})$ closed under $\cup$ and hereditary, *i.e.*, closed down under $\subseteq$, but not necessarily containing all singletons.) It suffices to demonstrate this fact separately for $\mathcal{I} \restriction Z_0$ and $\mathcal{I} \restriction Z_1$. Without any restriction of generality we show that $\mathcal{I} \restriction Z_0$ is a countable union of closed subideals, ending the proof of the theorem.

If $m \in \mathbb{N}$ and $s \subseteq u \subseteq Z_0$ are finite then let

$$I^m_{us} = \{A \subseteq Z_0 : \forall x \in \mathcal{D}_0\, (x \cap u = s \Longrightarrow (x \cup (A \setminus u))\, \mathsf{R}_m\, x)\}.$$

Lemma 24.1 *Sets I^m_{us} are closed topologically and under $\cup$, and $I^m_{us} \subseteq \mathcal{I}$.*

Proof I^m_{us} are topologically closed because so are R_m.

Suppose that $A, B \in I^m_{us}$. To prove that $A \cup B \in I^m_{us}$, let $x \in \mathcal{D}_0$ satisfy $x \cap u = s$. Then $x' = x \cup (A \setminus u) \in \mathcal{D}_0$ satisfies $x' \cap u = s$, too, hence, as $B \in I^m_{us}$, we have $(x' \cup (B \setminus u))\, \mathsf{R}_m\, x'$, thus, $(x \cup ((A \cup B) \setminus u))\, \mathsf{R}_m\, x'$. However $x' \,\mathsf{R}_m\, x$ just because $A \in I^m_{us}$. It remains to recall that R_m is a ER.

To prove that any $A \in I^m_{us}$ belongs to $\mathcal{I}$ take $x = s \cup S_1$. Then we have $x \cup (A \setminus u)\, \mathsf{R}_m\, x$, thus, $A \in \mathcal{I}$ as s is finite and $\mathsf{R}_m \subseteq \mathsf{E}_{\mathcal{I}}$. □ (*Lemma*)

[7] We mean, Cohen generic over a certain fixed countable transitive model $\mathfrak{M}$ of a big enough fragment of **ZFC**, which contains Borel codes for all sets $\mathcal{B}_k$.

Lemma 24.2 $\mathcal{I} \restriction Z_0 = \bigcup_{m,u,s} I^m_{us}$.

Proof Let $A \in \mathcal{I}$, $A \subseteq Z_0$. The sets $Q_m = \{x \in \mathcal{D}_0 : (x \cup A)\, \mathsf{R}_m\, x\}$ are closed and satisfy $\mathcal{D}_0 = \bigcup_m Q_m$. It follows that one of them has a non-empty interior in $\mathcal{D}_0$, thus, there exist finite sets $s \subseteq u \subseteq Z_0$ and some m_0 with

$$\forall x \in \mathcal{D}_0\, (x \cap u = s \Longrightarrow (x \cup A)\, \mathsf{R}_{m_0}\, x).$$

This is not exactly what we need, however, by (4), there exists a number $m = \mathtt{max}\{m_0, m(\mathtt{sup}\, u)\}$ big enough for

$$\forall x \in \mathcal{D}_0 : (x \cup A)\, \mathsf{R}_m\, (x \cup (A \setminus u)).$$

It follows that $A \in I^m_{su}$, as required. □ (*Lemma*)

Let J^m_{su} be the hereditary hull of I^m_{su} (*i.e.*, all subsets of sets in I^m_{su}). It follows from Lemma 24.1 that any J^m_{su} is a topologically closed subideal of $\mathcal{I} \restriction Z_0$, however, $\mathcal{I} \restriction Z_0$ is the union of those ideals by Lemma 24.2, as required. □

9.b $\mathcal{I}_1$ and P-ideals

Thus $\mathcal{I}_1$ is a $\leq_{\mathrm{B}}$-minimal ideal over $\mathtt{Fin}$: we have $\mathtt{Fin} <_{\mathrm{B}} \mathcal{I}_1$ and the $<_{\mathrm{B}}$-interval $(\mathtt{Fin}, \mathcal{I}_1)$ is empty. Although $\mathcal{I}_1$ is not *the least* over $\mathtt{Fin}$, still it turns out that $\mathcal{I}_1$ is the least among all Borel ideals which are not P-ideals.

We need another definition. For any ideal $\mathcal{I}$, a set $X \subseteq \mathcal{P}(\mathbb{N})$ is *$\mathcal{I}$-small* if there is $A \in \mathcal{I}$ such that $X \restriction A = \{x \cap A : x \in X\} \subseteq \mathcal{P}(A)$ is meager in $\mathcal{P}(A)$.

The next theorem is of great importance for the whole theory of Borel ideals.

Theorem 25 (Solecki [40, 41]) *The following families of ideals on $\mathbb{N}$ coincide:*

(i) *ideals of the form $\mathtt{Exh}_\varphi$, where φ is a l. s. c. submeasure on $\mathbb{N}$;*

(ii) *polishable ideals.*

(iii) *analytic P-ideals;*

(iv) *analytic ideals $\mathcal{I}$ with $\mathcal{I}_1 \not\leq_{\mathrm{RB}} \mathcal{I}$;*

(v) *analytic ideals $\mathcal{I}$ such that all countable unions of $\mathcal{I}$-small sets are $\mathcal{I}$-small.*

It follows that all analytic P-ideals actually belong to $\mathbf{\Pi}^0_3$, just because any ideal of type (i) is easily $\mathbf{\Pi}^0_3$.

Proof The formal scheme of the proof is: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (i). The hard part will be (v) $\Longrightarrow$ (i), the rest is rather elementary but tricky in some points. The elementary part of the proof is organized so that the proofs that (i) $\Longleftrightarrow$ (ii) and (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) and that the first group implies the second, are obtained independently of the hard part.

(i) $\Longrightarrow$ (ii) If $\varphi(\{n\}) > 0$ for all n then the required metric on $\mathcal{I} = \mathtt{Exh}_\varphi$ can be defined by $d_\varphi(x, y) = \varphi(x \,\Delta\, y)$. Then any set $U \subseteq \mathcal{I}$ open in the sense of the ordinary topology (the one inherited from $\mathcal{P}(\mathbb{N})$) is d_φ-open, while any d_φ-open set is Borel in the ordinary sense. In the general case we assemble the required metric of d_φ on the domain $\{n : \varphi(\{n\}) > 0\}$ and the ordinary Polish metric on $\mathcal{P}(\mathbb{N})$ on the complementary domain.

Give corollaries of Thm 25

(ii) $\Longrightarrow$ (i) Let τ be a Polish group topology on $\mathcal{I}$, generated by a Δ-invariant compatible metric d. It can be shown (Solecki [41, p. 60]) that $\varphi(x) = \mathtt{sup}_{y \in \mathcal{I},\, y \subseteq x}\, d(\emptyset, x)$ is a l.s.c. submeasure with $\mathcal{I} = \mathtt{Exh}_\varphi$. The key observation is that for any $x \in \mathcal{I}$ the sequence $\{x \cap [0, n)\}_{n \in \mathbb{N}}$ d-converges to x by the last statement of Lemma 11, which implies both that φ is l.s.c. (because the supremum above can be restricted to finite sets y) and that $\mathcal{I} = \mathtt{Exh}_\varphi$ (where the inclusion $\supseteq$ needs another "identity map" argument).

(i) $\Longrightarrow$ (iii) That any $\mathcal{I} = \mathtt{Exh}_\varphi$, φ being l.s.c., is a P-ideal, is an easy exercise: if $x_1, x_2, x_3, \ldots \in \mathcal{I}$ then define an increasing sequence of numbers $n_i \in x_i$ with $\varphi(x \cap [n_i, \infty)) \le 2^{-n}$ and put $x = \bigcup_i (x \cap [n_i, \infty))$.

Any of (iii), (i), (ii), (v) $\Longrightarrow$ (iv) This is because $\mathcal{I}_1$ easily does not satisfy any of the four properties indicated. For the formal purpose to complete the proof of Theorem 25, we need here only the implication (iii) $\Longrightarrow$ (iv).

(iv) $\Longrightarrow$ (v) Suppose that sets $X_n \subseteq \mathcal{P}(\mathbb{N})$ are $\mathcal{I}$-small, so that $X_n \restriction A_n$ is meager in $\mathcal{P}(A_n)$ for some $A_n \in \mathcal{I}$, but $X = \bigcup_n X_n$ is not $\mathcal{I}$-small, and prove $\mathcal{I}_1 \le_{\mathrm{RB}} \mathcal{I}$. Arguing as in the proof of Theorem 14, we use the meagerness to find, for any n, a sequence of pairwise disjoint non-empty finite sets $w^n_k \subseteq x_n$, $k \in \mathbb{N}$, and subsets $u^n_k \subseteq w^n_k$, such that

(a) if $x \subseteq \mathbb{N}$ and $\exists^\infty k\, (x \cap w^n_k = u^n_k)$ then $x \notin X_n$.

Dropping some sets w^n_k away and reenumerating the rest, we can strengthen the disjointness to the following: $w^n_k \cap w^m_l = \emptyset$ unless both $n = m$ and $k = l$.

Now put $w^n_{ij} = w^n_{2^i(2j+1)-1}$. The sets $\overline{w}_{ij} = \bigcup_{n \le i} w^n_{ij}$ are still pairwise disjoint, and satisfy the following two properties:

(b) $\bigcup_j \overline{w}_{ij} \subseteq x_n$, hence, $\in \mathcal{I}$, for any i;

(c) if a set $Z \subseteq \mathbb{N} \times \mathbb{N}$ does not belong to $\mathcal{I}_1$, *i.e.*, $\exists^\infty i\, \exists j\, (\langle i, j\rangle \in Z)$, then $\forall n\, \exists^\infty k\, (w^n_k \subseteq \overline{w}_Z)$, where $\overline{w}_Z = \bigcup_{\langle i,j\rangle \in K} \overline{w}_{ij})$.

We assert that the map $\langle i, j\rangle \mapsto \overline{w}_{ij}$ witnesses $\mathcal{I}_1 \le^+_{\mathrm{RB}} \mathcal{I}$. (Then a simple argument, as in the proof of Theorem 14, gives $\mathcal{I}_1 \le_{\mathrm{RB}} \mathcal{I}$.)

Indeed if $Z \subseteq \mathbb{N} \times \mathbb{N}$ belongs to $\mathcal{I}_1$ then $\overline{w}_Z \in \mathcal{I}$ by (b). Suppose that $Z \notin \mathcal{I}_1$. It suffices to show that $X_n \restriction \overline{w}_Z$ is meager in $\mathcal{P}(\overline{w}_Z)$ for any n. Note that by (c) the set $K = \{k : w^n_k \subseteq \overline{w}_Z\}$ is infinite and in fact $\overline{w}_Z \cap x_n = \bigcup_{k \in K} w^n_k$.

Therefore, any $x \subseteq \overline{w}_Z$ satisfying $x \cap w^n_k = u^n_k$ for infinitely many $k \in K$, does not belong to X_n by (a). Now the meagerness of $X_n \restriction \overline{w}_Z$ is clear.

(v) $\Longrightarrow$ (iii) This also is quite easy: if a sequence of sets $Z_n \in \mathcal{I}$ witnesses that $\mathcal{I}$ is not a P-ideal, then the union of $\mathcal{I}$-small sets $\mathcal{P}(Z_n)$ is not $\mathcal{I}$-small.

9.c The hard part

We prove (v) $\Longrightarrow$ (i), the hard part of Theorem 25. A couple of definitions before the key lemma.

- Let $C(\mathcal{I})$ be the collection of all hereditary (*i.e.*, $y \subseteq x \in K \Longrightarrow y \in K$) compact $\mathcal{I}$-large sets $K \subseteq \mathcal{P}(\mathbb{N})$.
- Given sets $A, B \subseteq \mathcal{P}(\mathbb{N})$, let $A + B = \{x \cup y : x \in A \wedge y \in B\}$.

Lemma 26 *Assuming that $\mathcal{I}$ is of type* (v), *there is a countable sequence of sets $K_m \in C(\mathcal{I})$ such that for any set $K \in C(\mathcal{I})$ there are m, n with $K_m + K_n \subseteq K$.*

Proof Fix a continuous map $f : \mathbb{N}^\mathbb{N} \overset{\text{onto}}{\longrightarrow} \mathcal{I}$. For any $s \in \mathbb{N}^{<\omega}$, we define

$$N_s = \{a \in \mathbb{N}^\mathbb{N} : s \subset a\} \quad \text{and} \quad B_s = f\,\text{"}N_s \quad (\text{the } f\text{-image of } N_s).$$

Consider the set $T = \{s : B_s \text{ is } \mathcal{I}\text{-large}\}$. As $\mathcal{I}$ itself is clearly $\mathcal{I}$-large, $\Lambda \in T$. On the other hand, the assumption (v) easily implies that T has no endpoints and no isolated branches, hence, $P = \{a \in \mathbb{N}^\mathbb{N} : \forall n\, (a \restriction n \in T)\}$ is a perfect set. Moreover, $A_s = f\,\text{"}(P \cap N_s)$ is $\mathcal{I}$-large for any $s \in T$ because $B_s \setminus A_s$ is a countable union of $\mathcal{I}$-small sets.

Now consider any set $K \in C(\mathcal{I})$. By definition, if $x, y \in \mathcal{I}$ then $Z = x \cup y \in \mathcal{I}$, thus, $K \restriction Z$ is not meager in $\mathcal{P}(Z)$, hence, by the compactness, $K \restriction Z$ includes a basic nbhd of $\mathcal{P}(Z)$, hence, by the hereditarity, there is a number n such that $Z \cap [n, \infty) \in K$. We conclude that $P^2 = \bigcup_n Q_n$, where each $Q_n = \{\langle a, b\rangle \in P^2 : (f(a) \cup f(b)) \cap [n, \infty) \in K\}$ is closed in P because so is K and f is continuous. Thus, there are $s, t \in T$ such that $P^2 \cap (N_s \times N_t) \subseteq Q_n$, in other words, $(A_s + A_t) \restriction [n, \infty) \subseteq K$, hence, $(\overline{A_s} + \overline{A_t}) \restriction [n, \infty) \subseteq K$, where $\overline{\ldots}$ denotes the topological closure of the hereditary hull. Thus we can take, as $\{K_m\}$, all sets of the form $K_{sn} = \overline{A_s} \restriction n$. □

Using the fact that $C(\mathcal{I})$ is a filter (as easy exercise which makes main use if the hereditarity), we can define (still in the assumption that $\mathcal{I}$ is of type (v)) a $\subseteq$-decreasing sequence of sets $K_n \in C(\mathcal{I})$ such that

(1) for any $K \in C(\mathcal{I})$ there is n with $K_n \subseteq K$,

and $K_{n+1} + K_{n+1} \subseteq K_n$ for any n. Taking any other term of the sequence, we can sharpen the latter requirement to

(2) for any $n:\ K_{n+1}+K_{n+1}+K_{n+1}\subseteq K_n$.

This is the starting point for the construction of a l.s.c. submeasure φ with $\mathcal{I}=\mathtt{Exh}_\varphi$. Assuming that, in addition, $K_0=\mathcal{P}(\mathbb{N})$, let, for any $x\in\mathcal{P}_{\mathtt{fin}}(\mathbb{N})$,

$$\begin{aligned}\varphi_1(x) &= \mathtt{inf}\{\,2^{-n}: x\in K_n\,\} &&, \text{ and}\\ \varphi_2(x) &= \mathtt{inf}\{\,\textstyle\sum_{i=1}^m\varphi_1(x_i): m\ge 1\wedge x_i\in\mathcal{P}_{\mathtt{fin}}(\mathbb{N})\wedge x\subseteq\bigcup_{i=1}^m x_i\,\} &&.\end{aligned}$$

Then set $\varphi(x)=\mathtt{sup}_n\,\varphi_2(x\cap[0,n))$ for any $x\subseteq\mathbb{N}$. A routine verification shows that φ submeasure and that $\mathcal{I}=\mathtt{Exh}_\varphi$. (See Solecki [41]. To check that any $x\in\mathtt{Exh}_\varphi$ belongs to $\mathcal{I}$ we use the following observation: $x\in\mathcal{I}$ iff for any $K\in C(\mathcal{I})$ there is n such that $x\cap[0,n)\in K$.)

□ (*Theorem 25*)

Corollary 27 *Suppose that $\mathcal{J}$ is an analytic P-ideal. Then any ideal $\mathcal{I}\le_{\mathrm{B}}\mathcal{J}$ is an analytic P-ideal, too.*

Proof Use equivalence (iv) $\Longleftrightarrow$ (iii) of the theorem. (The result can be obtained via a more direct argument, of course.) □

10 Ideal $\mathcal{I}_3$

$\mathcal{I}_3$ is how the ideal $0\times\mathtt{Fin}$ is denoted traditionally. It consists of all sets $x\subseteq\mathcal{P}(\mathbb{N}\times\mathbb{N})$ such that all cross-sections $(x)_n=\{k:\langle n,k\rangle\in x\}$ are finite.

Lemma 28 $\mathtt{Fin}<_{\mathrm{B}}\mathcal{I}_3$. *$\mathcal{I}_3$ and $\mathcal{I}_1$ are $\le_{\mathrm{B}}$-incomparable.*

Proof To see that $\mathtt{Fin}<_{\mathrm{B}}\mathcal{I}_3$ take $\vartheta(x)=\{\langle n,0\rangle:n\in x\}$. That $\mathcal{I}_3\not\le_{\mathrm{B}}\mathcal{I}_1$ can be shown as follows: otherwise by Theorem 24 $\mathcal{I}_3$ would be isomorphic either to one of $\mathtt{Fin}$, $\mathcal{I}_1$, or to a trivial variation of $\mathtt{Fin}$, which can be easily shown to be not the case. To see that $\mathcal{I}_1\not\le_{\mathrm{B}}\mathcal{I}_3$ recall that $\mathcal{I}_3=0\times\mathtt{Fin}$ is of the form $\mathtt{Exh}_\psi$ for a l.s.c. submeasure ψ (Example 9) and apply Theorem 25. □

10.a Ideals below $\mathcal{I}_3$

The following theorem is analogous to Theorem 24, yet the method of its proof is absolutely different.

Theorem 29 (Kechris [30]) *If $\mathcal{I}\le_{\mathrm{B}}\mathcal{I}_3$ is a Borel (nontrivial) ideal on $\mathbb{N}$ then either $\mathcal{I}\cong\mathcal{I}_3$ or $\mathcal{I}$ is a trivial variation of $\mathtt{Fin}$.*

Proof First of all we make use of Theorem 25: $\mathcal{I}_1\not\le_{\mathrm{B}}\mathcal{I}$ according to Lemma 28, therefore, $\mathcal{I}=\mathtt{Exh}_\varphi$ for a l.s.c. submeasure φ on $\mathbb{N}$. We can *w.l.o.g.* suppose that $\varphi(x)\le 1$ for any $x\in\mathcal{P}(\mathbb{N})$. Now put $U_n=\{k:\varphi(\{k\})\le\frac{1}{n}\}$.

Case 1: $\mathtt{lim}_{n\to\infty}\varphi(U_n)=0$. Then clearly a set $x\in\mathcal{P}(\mathbb{N})$ belongs to $\mathcal{I}$ iff $x\cap(U_n\setminus U_{n+1})$ is finite for any m, which easily implies that $\mathcal{I}$ is as required.

Case 2: $\{\varphi(U_n)\}\not\to 0$, so that $\varphi(U_n)>\varepsilon$ for some $\varepsilon>0$ and all n .

We are going to show that this cannot be.

Definition 29.1 A Borel ideal $\mathcal{Z}\subseteq\mathcal{P}(\mathbb{N})$ is *special* if there is a sequence of reals $r_k>0$ with $\{r_k\}\to 0$, such that $\mathcal{S}_{\{r_k\}}\subseteq\mathcal{I}$. □

Lemma 29.2 *In Case 2 there is a special ideal $\mathcal{Z}\le_{\mathrm{B}}\mathcal{I}$.*

Proof As φ is l.s.c. we can choose a sequence of numbers $n_1<n_2<n_3<\ldots$ and for any l a finite set $w_l\subseteq U_{n_l}\setminus U_{n_{l+1}}$ with $\varphi(w_l)>\varepsilon$. Then $W=\bigcup_l w_l\notin\mathcal{I}$ and obviously $\{\varphi(\{k\})\}_{k\in W}\to 0$. Note that the Borel ideal $\mathcal{Z}=\mathcal{I}\restriction W$ satisfies $\mathcal{Z}\le_{\mathrm{B}}\mathcal{I}$ (via the identity map), because $W\notin\mathcal{I}$. It remains to put $r_k=\varphi(\{k\})$ for $k\in W$ and then apply any bijection W onto $\mathbb{N}$. □ (*Lemma*)

This the next proposition proves that Case 2 does not take place, hence, ends the proof of the theorem.

Proposition 30 *If $\mathcal{Z}$ is a special ideal then $\mathcal{Z}\not\le_{\mathrm{B}}\mathcal{I}_3$.*

It will not take much more effort to prove a more general result. Let $\mathfrak{F}$ be the least family of ideals $\mathcal{J}$ over finite and countable sets, containing all ideals on finite sets and closed under the Fubini product in the form $\prod_{i\in\mathbb{N}}\mathcal{J}_i\,/\,\mathtt{Fin}$. Then $\mathfrak{F}$ contains $\mathtt{Fin}=\mathtt{Fr}_1$ ($\mathtt{Fin}$ is the same as $\prod_{i\in\mathbb{N}}\mathcal{J}_i\,/\,\mathtt{Fin}$, where each $\mathcal{J}_i=\{\emptyset,\{j_i\}\}$ is an ideal on a 1-element set $J_i=\{j_i\}$), $\mathtt{Fin}\otimes\mathtt{Fin}=\mathtt{Fr}_2$, *etc.* As obviously $\mathcal{I}_3=0\times\mathtt{Fin}\le_{\mathrm{B}}\mathtt{Fin}\otimes\mathtt{Fin}$, the following theorem implies the proposition.

Theorem 31 (Kanovei [27]) *If* $\mathcal{J}\in\mathfrak{F}$ *and* $\mathcal{Z}$ *is a special ideal then* $\mathcal{Z}\not\le_{\mathrm{B}}\mathcal{J}$.

The proof involves methods of Hjorth's [18] *turbulence theory*, applied here in a somewhat reduced form, just to the extent necessary to get the result.

10.b Structure of the proof

The proof of Theorem 31 will go on by induction on the conctruction of ideals in $\mathfrak{F}$. Yet we consider another property to be preserved.

Definition 32 Let E, F be Borel ERs on Polish spaces $\mathbb{X}$, $\mathbb{Y}$.

- A map $\vartheta:\mathbb{X}\to\mathbb{Y}$ is *a. e.* E,F*-invariant* if there is a co-meager set $D\subseteq\mathbb{X}$ such that $x\,\mathsf{E}\,x'\Longrightarrow\vartheta(x)\,\mathsf{F}\,\vartheta(x')$ for all $x,\,x'\in D$.
- A map $\vartheta:\mathbb{X}\to\mathbb{Y}$ is *a. e.* F*-constant* if there is a co-meager set $D\subseteq\mathbb{S}$ and $y\in\mathbb{Y}$ such that $\vartheta(x)\,\mathsf{F}\,y$ for all $x\in D$.

Following Kechris [31, 12.1] and Hjorth [18, 3.6], we say that E is *generically* F*-ergodic* if every Borel a.e. E,F-invariant map is a.e. F-constant.

- If $\mathcal{I}$, $\mathcal{J}$ are ideals then a.e. $\mathcal{I},\mathcal{J}$-invariant means a.e. $\mathsf{E}_{\mathcal{I}},\mathsf{E}_{\mathcal{J}}$-invariant, a.e. $\mathcal{J}$-constant means a.e. $\mathsf{E}_{\mathcal{J}}$-constant, generically $\mathcal{J}$-ergodic means generically $\mathsf{E}_{\mathcal{J}}$-ergodic. □

Note also that if $\mathsf{E}=\mathsf{E}_{\mathcal{Z}}$ for a nontrivial Borel ideal $\mathcal{Z}$, and E is generically F-ergodic, then $\mathsf{E}\not\le_{\mathrm{B}}\mathsf{F}$, because in this case there is no comeager set of pairwise $\mathsf{E}_{\mathcal{Z}}$-equivalent reals $x\in 2^{\mathbb{N}}$. Note finally that if $\mathcal{Z}$ is a Borel special ideal and this is witnessed by a sequence $\{r_n\}$, and $\mathcal{S}_{\{r_n\}}$ is generically $\mathcal{J}$-ergodic, then so is $\mathcal{Z}$, simply because any a.e. $\mathcal{Z},\mathcal{J}$-invariant map is a.e. $\mathcal{S}_{\{r_n\}},\mathcal{J}$-invariant as well. Thus the next lemma implies Theorem 31:

Lemma 33 *If* $\mathcal{J}\in\mathfrak{F}$, $r_n>0$ *for all* n, *and* $\{r_n\}\to 0$, *then* $\mathcal{S}_{\{r_n\}}$ *is generically* $\mathcal{J}$*-ergodic.*

The lemma will be proved by induction on the $\mathfrak{F}$-construction of $\mathcal{J}$. The basis of the induction is trivial: if $\mathcal{J}$ is an ideal on a finite set then there is only finitely many $\mathcal{J}$-classes, while any ideal $\mathcal{S}_{\{r_n\}}$ as considered produces infinitely (and uncountably) many equivalence classes.

10.c Induction step

Let us fix a sequence $\{r_n\}$ of reals $r_n>0$ with $\{r_n\}\to 0$. Suppose that $\mathcal{J}_k$, $k\in\mathbb{N}$, are ideals in $\mathfrak{F}$, and $\mathcal{S}_{\{r_n\}}$ is generically $\mathcal{J}_k$-ergodic for all k. The aim is to prove that $\mathcal{S}_{\{r_n\}}$ is generically $\mathcal{J}$-ergodic, where $\mathcal{J}=\prod_{k\in\mathbb{N}}\mathcal{J}_k\,/\,\mathtt{Fin}$.

Suppose, for the sake of simplicity, that each $\mathcal{J}_k$ is an ideal on $\mathbb{N}$, then, by definition, $\mathcal{J}$ is an ideal on $\mathbb{N}\times\mathbb{N}$. Let $\vartheta:\mathcal{P}(\mathbb{N})\to\mathcal{P}(\mathbb{N}\times\mathbb{N})$ be a Borel function. It splits in the sequence of Borel functions

$$\vartheta_k(x)=(\vartheta(x))_k=\{l:\langle k,l\rangle\in\vartheta(x)\}:\quad\mathcal{P}(\mathbb{N})\to\mathcal{P}(\mathbb{N})\,.$$

Suppose that ϑ is $\mathcal{S}_{\{r_n\}},\mathcal{J}$-invariant on a dense $\mathbf{G}_\delta$ set $D\subseteq\mathcal{P}(\mathbb{N})$, so that

$$a\,\Delta\,b\in\mathcal{S}_{\{r_n\}}\implies\exists\,k_0\;\forall\,k\ge k_0\;(\vartheta_k(a)\,\Delta\,\vartheta_k(b)\in\mathcal{J}_k)$$

for all $a,\,b\in D$. Our plan is to show that almost all ϑ_k are a.e. $\mathcal{S}_{\{r_n\}},\mathcal{J}_k$-invariant. In that we'll make use of two topologies. The first one is the ordinary product topology on $\mathcal{P}(\mathbb{N})$. The other one is the topology on the set $\mathcal{S}_{\{r_n\}}$, generated by the metric $d_{\{r_n\}}(a,b)=\varphi_{\{r_n\}}(a\,\Delta\,b)$ on $\mathcal{S}_{\{r_n\}}$, where

- $\varphi_{\{r_n\}}(x)=\sum_{n\in x}r_n$ for $x\in\mathcal{P}(\mathbb{N})$, so that $\mathcal{S}_{\{r_n\}}=\{x:\varphi_{\{r_n\}}(x)<+\infty\}$;

It is easy to verify (even in a much more general case, see [40]) that $d_{\{r_n\}}\restriction\mathcal{S}_{\{r_n\}}$ is a Polish (*i.e.*, complete separable [8]) metric on $\mathcal{S}_{\{r_n\}}$. The $d_{\{r_n\}}$-topology is stronger than the product topology of $\mathcal{P}(\mathbb{N})$ on $\mathcal{S}_{\{r_n\}}$, yet it yields the same Borel subsets of $\mathcal{S}_{\{r_n\}}$ as the product topology. Sets of the form

$$\mathcal{O}_\varepsilon(t)=\{z\in\mathcal{S}_{\{r_n\}}:d_{\{r_n\}}(z,t)<\varepsilon\},$$

where $t,\,\varepsilon$ are as above, provide a base of the $d_{\{r_n\}}$-topology.

Below, let $\mathbb{P}=2^{<\omega}$ be the ordinary Cohen forcing for $\mathcal{P}(\mathbb{N})$. Let $\mathbb{P}_{\{r_n\}}$ be the Cohen forcing for $\langle\mathcal{S}_{\{r_n\}};d_{\{r_n\}}\rangle$, which consists of all sets of the form $\mathcal{O}_\varepsilon(t)$, where $t\in\mathcal{P}_{\mathtt{fin}}(\mathbb{N})$ and $\varepsilon\in\mathbb{Q}^+$. (Smaller sets are stronger conditions.) Let us fix a countable transitive model $\mathfrak{M}$ of a big enought fragment of **ZFC** (see Footnote 5) which contains all relevant objects or their codes, in particular, the sequence $\{r_n\}$ and a code of the Borel map ϑ.

Claim 33.1 *Suppose that* $\langle a,z\rangle\in\mathcal{P}(\mathbb{N})\times\mathcal{S}_{\{r_n\}}$ *is* $\mathbb{P}\times\mathbb{P}_{\{r_n\}}$*-generic over* $\mathfrak{M}$. *Then* $b=a\,\Delta\,z$ *is* $\mathbb{P}$*-generic over* $\mathfrak{M}$.

Proof b is $\mathbb{P}$-generic even over $\mathfrak{M}[z]$: a is such by the product forcing lemma, and, for any fixed z, the map $a\mapsto a\,\Delta\,z$ is a homeomorphism. □ (*Claim*)

Fix $u\in 2^{<\omega}$. Then by the invariance of ϑ and Claim 33.1 there is another sequence $v\in 2^{<\omega}$ with $u\subset v$, a number k_0, and a non-empty $d_{\{r_n\}}$-nbhd $\mathcal{O}_\varepsilon(t)$ in $\mathcal{S}_{\{r_n\}}$ (a condition in $\mathbb{P}_{\{r_n\}}$), where $\varepsilon>0$ and $t\in\mathcal{P}_{\mathtt{fin}}(\mathbb{N})$, such that we have $\vartheta_k(a)\,\Delta\,\vartheta_k(a\,\Delta\,z)\in\mathcal{J}_k$ for any $\mathbb{P}\times\mathbb{P}_{\{r_n\}}$-generic over $\mathfrak{M}$ pair $\langle a,z\rangle$ of $a\in I_v$ [9] and $z\in\mathcal{O}_\varepsilon(t)$, and any $k\ge k_0$. We can assume that the length $\mathtt{lh}\,v$

[8] The set of all finite sets $u\subseteq\mathcal{P}(\mathbb{N})$ is dense in $\langle\mathcal{S}_{\{r_n\}};d_{\{r_n\}}\rangle$.

[9] $I_v=\{x\in\mathcal{P}(\mathbb{N}):x\cap[0,n)=\{j<n:v(j)=1\}\}$, where $n=\mathtt{lh}\,v$: a basic clopen set.

is big enough for $i \geq \mathtt{lh}\, v \Longrightarrow r_i < \varepsilon$.

Claim 33.2 *If* $a, b \in I_v$ *are* $\mathbb{P}$*-generic over* $\mathfrak{M}$ *and* $a \,\Delta\, b \in \mathcal{S}_{\{r_n\}}$ *then we have* $\vartheta_k(a) \,\Delta\, \vartheta_k(b) \in \mathcal{J}_k$ *for all* $k \geq k_0$.

Proof First consider the case when $\varphi_{\{r_n\}}(a \,\Delta\, b) < \varepsilon$. Take any $z \in Z = \mathcal{O}_\varepsilon(t)$ with $\varphi_{\{r_n\}}(z \Delta t) < \varepsilon - \varphi_{\{r_n\}}(a \Delta b)$, $\mathbb{P}_{\{r_n\}}$-generic over $\mathfrak{M}[a, b]$, [10] (This is possible as $\{r_n\} \to 0$.) Then z is $\mathbb{P}_{\{r_n\}}$-generic over $\mathfrak{M}[a]$, hence, $\langle a, z\rangle$ is $\mathbb{P} \times \mathbb{P}_{\{r_n\}}$-generic over $\mathfrak{M}$ by the product forcing lemma, thus, $\vartheta_k(a) \,\Delta\, \vartheta_k(a \,\Delta\, z) \in \mathcal{J}_k$. Moreover, $z' = z \Delta (a \Delta b)$ still belongs to Z and is $\mathbb{P}_{\{r_n\}}$-generic over $\mathfrak{M}[a, b]$, so that $\vartheta_k(b) \,\Delta\, \vartheta_k(b \,\Delta\, z') \in \mathcal{J}_k$ by the same argument. Yet we have $a \,\Delta\, z = b \,\Delta\, z'$.

Now consider the general case. By definition $x = a \,\Delta\, b$ satisfies $\sum_{n \in x} r_n = \varphi_{\{r_n\}}(x) < +\infty$, moreover, $\mathtt{min}\, x \geq \mathtt{lh}\, v$, hence, by the choice of v, all r_j with $j \in x$ satisfy $r_j < \varepsilon$. In this case x has the form $x = \{j_1, ..., j_n\} \cup x'$, where $\varphi_{\{r_n\}}(x') < \varepsilon$, and $r_{j_m} < \varepsilon$ for all m. Define $a_m = a \,\Delta\, \{j_1, ..., j_m\}$ for $m = 1, ..., n$. Then $a = a_0, a_1, a_2, ..., a_m, a_{m+1} = b$ is a chain of $\mathbb{P}$-generic, over $\mathfrak{M}$, elements of I_v with $d_{\{r_n\}}(a_m, a_{m+1}) < \varepsilon$ for all m, so that we can apply the particular case considered above. □ (*Claim*)

To summarize, we have shown that for any $u \in 2^{<\omega}$ there exist k_0 and $v \in 2^{<\omega}$ with $u \subset v$ such that $\vartheta_k(a)$ is $\mathcal{S}_{\{r_n\}}, \mathcal{J}_k$-invariant for all $k \geq k_0$ and all sufficiently $\mathbb{P}$-generic $a \in I_v$, so that ϑ_k is a.e. $\mathcal{S}_{\{r_n\}}, \mathcal{J}_k$-invariant on I_v. Therefore, by the inductive hypothesis, ϑ_k is a.e. $\mathcal{J}_k$-constant on I_v, for each $k \geq k_0$. (Indeed, we can trivially extend $\vartheta_k \restriction I_v$ on the whole domain $\mathcal{P}(\mathbb{N})$ so that the invariance is preserved.) Thus ϑ is a.e. $\mathcal{J}$-constant on I_v as well. Thus is enough, using the supposed a.e. $\mathcal{S}_{\{r_n\}}, \mathcal{J}$-invariance of ϑ, to derive that ϑ is then a.e. $\mathcal{J}$-constant on the whole of $\mathcal{P}(\mathbb{N})$.

□ (*Lemma 33, Theorem 31, Proposition 30, and Theorem 29*)

[10] By $\mathfrak{M}[a, b]$, we understand a countable transitive model of the same fragment of **ZFC** as mentioned in Footnote 5, which contains a, b, and all sets in $\mathfrak{M}$. This model may contain more ordinals than $\mathfrak{M}$ because the pair $\langle a, b\rangle$ of two generic elements is not necessarily generic. On the contrary, the models $\mathfrak{M}[a]$ and $\mathfrak{M}[b]$ are ordinary generic extensions of $\mathfrak{M}$, containing the same ordinals as $\mathfrak{M}$.

11 Equivalence relation E_3

Ideal $\mathcal{I}_3 = 0 \times \mathtt{Fin}$ defines the ER $\mathsf{E}_3 = \mathsf{E}_{\mathcal{I}_3}$ on $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ by $x \mathsf{E}_3 y$ iff $x \Delta y \in \mathcal{I}_3$. We can consider E_3 as an ER on $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$ defined by $x \,\mathsf{E}_3\, y$ iff $x_n \,\mathsf{E}_0\, y_n$ for all n: here $x = \{x_n\}_{n \in \mathbb{N}}$ and $y = \{y_n\}_{n \in \mathbb{N}}$, and all x_n, y_n belong to $\mathcal{P}(\mathbb{N})$.

Theorem 34 (Hjorth and Kechris [19, 20]) *If* $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_3$ *is a Borel ER then either* $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_0$ *or* $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_3$.

Proof (a modification of the proof in [20]). We may assume that E is a Δ^1_1 ER on a recursively presented Polish space $\mathbb{X}$, and there is a Δ^1_1 reduction $\vartheta : \mathbb{X} \to \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ of E to E_3. Let $Q = \mathtt{ran}\, \vartheta$, a Σ^1_1 subset of $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$.

For $x, y \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ and $n \in \mathbb{N}$, define $x \equiv_n y$ iff $x \,\mathsf{E}_3\, y$ and $x \restriction_{<n} = y \restriction_{<n}$ (the latter requirement means $x_k = y_k$ for all $k < n$). For $n, k, p \in \mathbb{N}$ put

$$\mathcal{A}_{nkp} = \{A \subseteq \mathcal{P}(\mathbb{N})^{\mathbb{N}} : A \text{ is } \Sigma^1_1 \wedge \forall x, y \in A\, (x \equiv_n y \Longrightarrow x_k \,\Delta\, y_k \subseteq [0, p))\}.$$ [11]

Claim 34.1 *If* $A \in \mathcal{A}_{nkp}$ *then there is a* Δ^1_1 *set* $B \in \mathcal{A}_{nkp}$ *with* $A \subseteq B$.

Proof (Reflection) □ (*Claim*)

Put $A_0 = \bigcup_n \bigcap_{k \geq n} \bigcup_p \{A : A \in \mathcal{A}_{nkp}\}$.

***Case 1*:** $Q \subseteq A_0$. ***Case 2*:** otherwise.

11.a Case 1

We are going to prove that in this case $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_0$.

As easily A_0 is Π^1_1 by Claim 34.1 and a standard computation, there is a Δ^1_1 set R such that $Q \subseteq R \subseteq B_0$. Thus, for $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_0$ it suffices now to prove

Lemma 35 $\mathsf{E}_3 \restriction R \leq_{\mathrm{B}} \mathsf{E}_0$ *for any* Δ^1_1 *set* $R \subseteq A_0$.

Proof By Kreisel selection there exists a Δ^1_1 map $\nu : R \to \mathbb{N}$ such that

$$\forall k \geq \nu(x)\, \exists p\, \exists B \in \mathcal{A}_{\nu(x), k, p}\, (x \in B \in \Delta^1_1)$$

for any $x \in R$. Let $R_n = \{x \in R : \nu(x) \leq n\}$, these are increasing Δ^1_1 subsets of R, and $R = \bigcup_n R_n$. According to Theorem 20, it suffices to prove that $\mathsf{E}_3 \restriction R_n \leq_{\mathrm{B}} \mathsf{E}_0$ for any n. Thus let us fix n. By definition we have

$$\forall x \in R_n\, \forall k \geq n\, \exists p\, \exists B \in \mathcal{A}_{nkp}\, (x \in B \in \Delta^1_1). \qquad (*)$$

Recall that C is the least class of sets containing all open sets and closed under the A-operation and the complement. A map f is called *C-measurable* iff all f-preimages of open sets belong to C.

[11] Hjorth and Kechris [20] define $\mathcal{A}_{nkp}$ with $\forall x, y \in Q \cap A$ instead of $\forall x, y \in A$. Let us use $\mathcal{A}'_{nkp}$ to denote their version, thus, $\mathcal{A}_{nkp} \subseteq \mathcal{A}'_{nkp}$. However if Case 1 holds in the sense of $\mathcal{A}'_{nkp}$ then it also holds in the sense of $\mathcal{A}_{nkp}$ because $A \in \mathcal{A}'_{nkp}$ iff $A \cap Q \in \mathcal{A}_{nkp}$.

Claim 35.1 *There is a C-measurable map* $f : R_n \to \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ *such that* $f(x) = f(y) \equiv_n x$ *whenever* $x, y \in R_n$ *satisfy* $x \equiv_n y$ *.*

Proof Let $E \subseteq \mathbb{N}$ be the Π^1_1 set of all codes of Δ^1_1 subsets of $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$, and let $W_e \subseteq \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ be the Δ^1_1 set coded by $e \in E$. We have, by $(*)$,

$$\forall x \in R_n \, \forall k \geq n \, \exists p \, \exists e \in E \, (x \in W_e \in \mathcal{A}_{nkp}) \,,$$

and an ordinary application of the Kreisel selection yields a pair of Δ^1_1 maps $\pi, \varepsilon : R_n \times \mathbb{N} \to \mathbb{N}$ such that $\varepsilon(x,k) \in E$ and $x \in W_{\varepsilon(x,k)} \in \mathcal{A}_{n,k,\pi(x,k)}$ hold whenever $x \in R_n$ and $k \geq n$. Let $\tilde\pi(x,k)$ and $\tilde\varepsilon(x,k)$ to be the least, in the sense of any fixed recursive ω-long wellordering of $\mathbb{N} \times \mathbb{N}$, of all possible pairs $\pi(x',k)$ and $\varepsilon(x',k)$ with $x' \in R_n \cap [x]_{\equiv_n}$. Then $\tilde\pi$ and $\tilde\varepsilon$ are $\equiv_n$-invariant in the 1st argument. In addition, we have $W_{\tilde\varepsilon(x,k)} \in \mathcal{A}_{n,k,\tilde\pi(x,k)}$ and the set $Z_{xk} = R_n \cap [x]_{\equiv_n} \cap W_{\tilde\varepsilon(x,k)}$ is nonempty, whenever $x \in R_n$ and $k \geq n$.

Let $x \in R_n$. For any $k \geq n$, the set $Y_{xk} = \{y_k : y \in Z_{xk}\} \subseteq \mathcal{P}(\mathbb{N})$ is finite (and nonempty) by the definition of $\mathcal{A}_{nkp}$, thus, let $f_k(x)$ be the least member of Y_{xk} in the sense of the lexicographical order of $\mathcal{P}(\mathbb{N})$. Define $f(x) \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ so that $f(x)_k = x_k$ for $k < n$ and $f(x)_k = f_k(x)$ for $k \geq n$.

That $f(x) = f(y)$ whenever $x \equiv_n y$ follows from the invariance of ε and π. To see that $f(x) \equiv_n x$ note that by definition $f_k(x) \mathsf{E}_0 \, x_k$ for $k \geq n$: indeed, $f_k(x) = y_k$ for some $y \in [x]_{\equiv_n}$, but $x \equiv_n y$ implies $x_k \mathsf{E}_0 \, y_k$ for all k. Finally, the C-measurability needs a routine check. □ (*Claim*)

For any $u \in \mathcal{P}(\mathbb{N})^n$ let $R_n(u) = \{x \in R_n : x \restriction_{<n} = u\}$.

Claim 35.2 *If* $u \in \mathcal{P}(\mathbb{N})^n$ *then* $\mathsf{E}_3 \restriction R_n(u)$ *is smooth.*

Proof As E_3 and $\equiv_n$ coincide on $R_n(u)$, the relation $\mathsf{E}_3 \restriction R_n(u)$ is smooth via a C-measurable, hence, a Baire-measurable map. Suppose, towards the contrary, that it is not really smooth, *i.e.*, via a Borel map. Then, by the 2-nd dichotomy theorem, we have $\mathsf{E}_0 \leq_{\mathrm{B}} \mathsf{E}_3 \restriction R_n(u)$, hence, E_0 turns out to be smooth via a Baire-measurable map, which is easily impossible. □ (*Claim*)

To complete the proof of the lemma, let $\mathbb{G} = \mathcal{P}_{\mathtt{fin}}(\mathbb{N})^n$, acting on $\mathbb{X} = \mathcal{P}(\mathbb{N})^n$ componentwise and by Δ at each of the n co-ordinates, so that, for $u, v \in \mathbb{X}$, we have $u \, \mathsf{E}^{\mathbb{G}}_{\mathbb{X}} \, v$ iff $u_k \, \mathsf{E}_0 \, v_k$ for all $k < n$. Let us apply Theorem 21 with $\mathbb{G}$ and $\mathbb{X}$ as indicated, and $P = R_n$ and $\mathsf{E} = \mathsf{E}_3 \restriction R_n$, Claim 35.2 witnesses the principal requirement. We obtain: $\mathsf{E}_3 \restriction R_n$ is Borel reducible to a ER induced by a Borel action of $\mathbb{G}$. Yet $\mathbb{G}$ is the increasing union of a countable sequence of its finite subgroups, hence, any ER induced by a Borel action of $\mathbb{G}$ is hyperfinite, hence, Borel reducible to E_0 .

□ (*Lemma 35 and Case 1 in Theorem 34*)

11.b Case 2

Then the Σ^1_1 set $H = Q \setminus A_0$ is non-empty. Our idea will be to define a Borel subset X of H such that $\mathsf{E}_3 \restriction X \sim_{\mathrm{B}} \mathsf{E}_3$, getting the "or" alternative of Theorem 34.

□ (*Theorem 34*)

12 Equivalence relation E_1

Ideal $\mathcal{I}_1$ naturally defines the ER E_1 on $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ by $x\,\mathsf{E}_1\,y$ iff $x \,\Delta\, y \in \mathcal{I}_1$. We can also consider E_1 as a ER on $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$ defined by $x\,\mathsf{E}_1\,y$ iff $x_n = y_n$ for almost all n : here we assume that $x = \{x_n\}_{n\in\mathbb{N}}$ and $y = \{y_n\}_{n\in\mathbb{N}}$, and all x_n, y_n belong to $\mathcal{P}(\mathbb{N})$.

12.a Above E_1

Recall that an embedding is a $1-1$ reduction, and an invariant embedding is an embedding ϑ such that its range is an invariant set, see Subsection 2.e above.

Theorem 36 (Kechris and Louveau [32]) *Suppose that* $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{F}$, *where* F *is an analytic ER on a Polish space* $\mathbb{Y}$. *Then both* $\mathsf{E}_1 \sqsubseteq_{\mathrm{C}} \mathsf{F}$ *and* $\mathsf{E}_1 \sqsubseteq^{\mathrm{i}}_{\mathrm{B}} \mathsf{F}$.

Proof To prove the first statement, let $\preccurlyeq$ be the inverted order on $\mathbb{N}$, *i.e.*, $m \preccurlyeq n$ iff $n \le m$. Let $\mathfrak{P}$ be the collection of all sets $P \subseteq \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ such that there is a continuous $1-1$ map $\eta : \mathcal{P}(\mathbb{N})^{\mathbb{N}} \xrightarrow{\text{onto}} P$ such that we have

$$x\restriction_{\preccurlyeq n} = y\restriction_{\preccurlyeq n} \iff \eta(x)\restriction_{\preccurlyeq n} = \eta(y)\restriction_{\preccurlyeq n}$$

for all n and $x, y \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$, where $x\restriction_{\preccurlyeq n} = \{x_i\}_{i\preccurlyeq n}$ for any $x = \{x_i\} \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$. Clearly any such a map is a continuous embedding of E_1 into itself.

This set $\mathfrak{P}$ is a forcing notion to extend the universe by a sequence of reals x_i so that each x_n is Sacks–generic over $\{x_i\}_{i\preccurlyeq n}$, an example of iterated Sacks extensions with an ill-founded "skeleton" of iteration, which we defined in [26]. Here, the "skeleton" is $\mathbb{N}$ with the inverted order $\preccurlyeq$.

The method of [26] contains a study of behaviour of continuous and Borel functions on sets in $\mathfrak{P}$. In particular it is shown there that Borel maps admit the following *cofinal classification* on sets in $\mathfrak{P}$: if $\mathbb{Y}$ is Polish, $P' \in \mathfrak{P}$, and $\vartheta : P' \to \mathbb{Y}$ is Borel then there is a set $P \in \mathfrak{P}$, $P \subseteq P'$, on which ϑ is continuous, and either a constant or, for some n, $1-1$ on $P\restriction_{\preccurlyeq n}$ in the sense that,

$$\text{for all } x, y \in P: \quad x\restriction_{\preccurlyeq n} = y\restriction_{\preccurlyeq n} \iff \vartheta(x) = \vartheta(y). \qquad (*)$$

Let us apply this to a Borel map $\vartheta : \mathcal{P}(\mathbb{N})^{\mathbb{N}} \to \mathbb{Y}$ which reduces E_1 to F. We begin with $P' = \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ and find a set $P \in \mathfrak{P}$ as indicated. Since ϑ cannot be a constant on P (indeed, any $P \in \mathfrak{P}$ contains many pairwise E_1-inequivalent elements), we have $(*)$ for some n. In other words, there is a $1-1$ continuous map $f : P\restriction_{\preccurlyeq n} \to \mathbb{Y}$ (where $P\restriction_{\preccurlyeq n} = \{x\restriction_{\preccurlyeq n} : x \in P\}$) such that $\vartheta(x) = f(x\restriction_{\preccurlyeq n})$ for all $x \in P$. Now, let $x = \{x_i\}_{i\in\mathbb{N}} \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$. Define $\zeta(x) = z = \{z_i\}_{i\in\mathbb{N}}$ so that $z_i = \emptyset$ for $i < n$ and $z_{n+i} = x_i$ for all i. Finally set $\vartheta'(x) = f(\eta(\zeta(x))\restriction_{\preccurlyeq n})$ for all $x \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$: this is a continuous embedding E_1 in F.

Now we prove the second claim. We can assume that $\mathbb{Y} = \mathcal{P}(\mathbb{N})$ and that $\vartheta : \mathcal{P}(\mathbb{N})^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ is already a continuous embedding E_1 into F. Let $Y = \mathtt{ran}\,\vartheta$

and $Z = [Y]_{\mathsf{F}}$. Normally Y, Z are analytic, but in this case they are even Borel. Indeed Z is the projection of $P = \{\langle z, x\rangle : z\,\mathsf{F}\,\vartheta(x)\}$, a Borel subset of $\mathcal{P}(\mathbb{N})\times\mathcal{P}(\mathbb{N})^{\mathbb{N}}$ whose all cross-sections are E_1-equivalence classes, *i.e.*, σ-compact sets. It is known that in this case Z is Borel and, moreover, there is a Borel map *reference*
$f : Z \to \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ such that $f(z)\,\mathsf{E}_1\,x$ whenever $z\,\mathsf{F}\,\vartheta(x)$.

We can convert f to a $1-1$ map $g : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ with the same properties: namely, let $g(z)_n = f(z)_n$ for $n \ge 1$, but $g(z)_0 = z$. To conclude, $f : \mathcal{P}(\mathbb{N})^{\mathbb{N}} \to Z \subseteq \mathcal{P}(\mathbb{N})$ and $g : Z \to \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ are Borel $1-1$ maps (ϑ is even continuous, but this does not matter now), and, for any $x \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$, ϑ maps $[x]_{\mathsf{E}_1}$ into $[\vartheta(x)]_{\mathsf{F}} \subseteq Z$, and g maps $[\vartheta(x)]_{\mathsf{F}}$ back into $[x]_{\mathsf{E}_1}$. It remains to apply the construction from the Cantor – Bendixson theorem, to get a Borel embedding, say, F of E_1 into F with $\mathtt{ran}\,F = Z$, that is, an invariant embedding. $\square$

12.b Polish group action ERs incompatible with E_1

Suppose that a group $\mathbb{G}$ acts on a space $\mathbb{X}$, meaning that there is a map

$$\langle g, x\rangle \mapsto g\cdot x \quad : \quad \mathbb{G}\times\mathbb{X} \to \mathbb{X} \qquad (*)$$

such that $g\cdot(h\cdot x) = (gh)\cdot x$. Then $\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}$ denotes the induced equivalence relation, *or* $=$ *$(hg)\cdot x$? Nobody cares to give a complete def*
i.e., $x\,\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}\,y$ iff $\exists\, g \in \mathbb{G}\,(y = g\cdot x)$. The practically interesting case is when $\mathbb{G}$ is a Polish group (that is, the underlying set is a Polish space and the group operation is continuous) and the action $(*)$ is continuous, or, more generally, Borel. Then they say that $\mathbb{X}$ is a *Polish*, or *Borel*, $\mathbb{G}$*-space*. In this case $\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}$ is, generally, an analytic ER on $\mathbb{X}$, but sometimes it is Borel (for instance if $\mathbb{G}$ is countable) or at least with Borel classes.

Example 37 Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be a Polishable Borel ideal (see Subsection **??**), so that we can consider $\mathcal{I}$ as a Polish group, with Δ as the operation, continuously acting on $\mathcal{P}(\mathbb{N})$, a Polish space, by $g\cdot x = g\,\Delta\,X$ for all $g \in \mathcal{I}$ and $X \in \mathcal{P}(\mathbb{N})$. Then $\mathcal{P}(\mathbb{N})$ is a Polish $\mathcal{I}$-space, and $\mathsf{E}^{\mathcal{I}}_{\mathcal{P}(\mathbb{N})}$ coincides with the relation: $X\,\mathsf{E}_{\mathcal{I}}\,Y$ iff $X\,\Delta\,Y \in \mathcal{I}$. $\square$

Theorem 38 (Kechris and Louveau [32]) *Suppose that* $\mathbb{G}$ *is a Polish group and* $\mathbb{X}$ *is a Borel* $\mathbb{G}$*-space. Then* E_1 *is <u>not</u> Borel reducible to* $\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}$.

Proof Towards the contrary, let $\vartheta : \mathcal{P}(\mathbb{N})^{\mathbb{N}} \to \mathbb{X}$ be a Borel reduction of E_1 to E. We can assume, by Theorem 36, that ϑ is in fact an invariant embedding, *i.e.*, $1-1$ and $Y = \mathtt{ran}\,\vartheta$ is an E-invariant set. Define, for $g \in \mathbb{G}$ and $x \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$, $g\cdot x = \vartheta^{-1}(g\cdot\vartheta(x))$. Then this is a Borel action of $\mathbb{G}$ on $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$ such that the induced relation $\mathsf{E}^{\mathbb{G}}_{\mathcal{P}(\mathbb{N})^{\mathbb{N}}}$ coincides with E_1.

Let us fix $x \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$.

Consider any $y = \{y_n\}_n \in [x]_{\mathsf{E}_1}$. Then $[x]_{\mathsf{E}_1} = \bigcup_n C_n(y)$, where each set $C_n(y) = \{y' \in \mathcal{P}(\mathbb{N})^{\mathbb{N}} : \forall\, m \ge n\ (y_n = y'_n)\}$ is Borel (even compact). It follows that $\mathbb{G} = \bigcup_n G_n(y)$, where each $G_n(y) = \{g \in \mathbb{G} : g(x) \in C_n(y)\}$ is Borel. Thus, as $\mathbb{G}$ is Polish, there is a number n such that $G_n(y)$ is not meager in $\mathbb{G}$ (then this will hold for all $n' \geq n$, of course). Let $n(y)$ be the least such an n.

We assert that for any n the set $Y_n(x) = \{y \restriction [n, \infty) : y \in [x]_{\mathsf{E}_1} \wedge n(x) = n\}$ is at most countable. Indeed suppose that $Y_n(x)$ is not countable. Note that if y_1 and y_2 in $[x]_{\mathsf{E}_1}$ have different restrictions $y_i \restriction [n, \infty)$ then the sets $C_n(y_1)$ and $C_n(y_2)$ are disjoint, therefore, the sets $G_n(y_1)$ and $G_n(y_2)$ are disjoint, so we would have uncountably many pairwise disjoint non-meager sets in $\mathbb{G}$, contradiction. Thus all sets $Y_n(x)$ are countable.

It is most important that $Y_n(x)$ depends on $[x]_{\mathsf{E}_1}$ rather than x itself, more exactly, if $x' \in [x]_{\mathsf{E}_1}$ then $Y_n(x) = Y_n(x')$: this is because any set $G_n(y)$ in the sense of x' is just a shift, within $\mathbb{G}$, of $G_n(y)$ in the sense of x. Therefore, putting $Y(x) = \bigcup_n \{\bar{u} : u \in Y_n(x)\}$, where, for $u \in \mathcal{P}(\mathbb{N})^{[n,\infty)}$, $\bar{u} \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ is defined by $\bar{u} \restriction [n, \infty) = u$ and $\bar{u}(k) = \emptyset$ for $k < n$, we have the set $Y = \bigcup_{x \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}} Y(x)$ with the property that $Y \cap [x]_{\mathsf{E}_1}$ is non-empty and at most countable for any $x \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$.

The other important fact is that the relation $y \in Y(x)$ is Borel: this is because it is assembled from Borel relations via the Vaught quantifier "there exists nonmeager-many", known to preserve the Borelness. It follows that *reference*

$$Y = \{y : \exists x\, (y \in Y_x)\} = \{y : \forall x\, (x \in [y]_{\mathsf{E}_1} \Longrightarrow y \in Y(x)\}$$

is a Borel subset of $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$. However $\mathsf{E}_1 \restriction Y$ has countable equivalence classes by the above, hence, is both countable and hypersmooth, so that $\mathsf{E}_1 \restriction Y \leq_{\mathrm{B}} \mathsf{E}_0$ by a result of [4]. On the other hand, by the uniformization theorem for Borel sets with countable sections, there is a Borel map f defined on $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$ so that $f(x) \in Y(x)$ for any x, which implies $\mathsf{E}_1 \leq_{\mathrm{B}} \mathsf{E}_1 \restriction Y$, and finally $\mathsf{E}_1 \leq_{\mathrm{B}} \mathsf{E}_0$, which cannot be by Lemma 22. □ *reference*

12.c ERs below E_1

Theorem 39 (Kechris and Louveau [32]) *Suppose that* E *is a Borel ER on some Polish space, and* $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_1$. *Then either* $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_0$ *or* $\mathsf{E}_1 \leq_{\mathrm{B}} \mathsf{E}$.

Proof Starting the proof, we may assume that E is a Δ^1_1 ER on $\mathcal{P}(\mathbb{N})$, and that there is a reduction ρ of E to E_1, of class Δ^1_1. Then $R = \mathtt{ran}\,\rho$ is a Σ^1_1 subset of $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$. The idea behind the proof is to show that the set R is either small enough for $\mathsf{E}_1 \restriction R$ to be Borel reducible to E_0, or otherwise it is big enough to contain a closed subset X such that $\mathsf{E}_1 \restriction X$ is Borel isomorphic to E_1.

Relations $\prec$ and $\preccurlyeq$ will denote the inverse order relations on $\mathbb{N}$, *i.e.*, $m \preccurlyeq n$ iff $n \leq m$, and $m \prec n$ iff $n < m$. If $x \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ then $x \restriction_{\preccurlyeq n}$ denotes the restriction of x (a function defined on $\mathbb{N}$) on the domain $\preccurlyeq n$, *i.e.*, $[n, \infty)$. If $X \subseteq \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ then let $X \restriction_{\preccurlyeq n} = \{x \restriction_{\preccurlyeq n} : x \in X\}$. Define $x \restriction_{\prec n}$ and $X \restriction_{\prec n}$ similarly. In particular, $\mathcal{P}(\mathbb{N})^{\mathbb{N}} \restriction_{\preccurlyeq n} = \mathcal{P}(\mathbb{N})^{\preccurlyeq n} = \mathcal{P}(\mathbb{N})^{[n,\infty)}$.

For a sequence $x \in \mathcal{P}(\mathbb{N})^{\preccurlyeq n}$, let $\mathtt{dep}\,x$ (the *depth* of x) be the number (finite or ∞) of elements of the set $\nabla(x) = \{j \preccurlyeq n : x(j) \notin \Delta^1_1(x \restriction_{\prec j})\}$. The formula $\mathtt{dep}\,x \geq d$ (of two variables, d running over $\mathbb{N} \cup \{\infty\}$) is obviously Σ^1_1.

We have two cases:

***Case 1*:** all $x \in R = \mathtt{ran}\,\rho$ satisfy $\mathtt{dep}\,x < \infty$.

***Case 2*:** there exist $x \in R$ with $\mathtt{dep}\,x = \infty$.

12.d Case 1

This is the easier case. First of all we observe that R, a Σ^1_1 set, is a subset of the Π^1_1 set $Z = \{x : \mathtt{dep}\,x < \infty\}$, hence, there is a Δ^1_1 set Y with $\mathtt{ran}\,\rho \subseteq Y \subseteq Z$. The following lemma ends the argument.

Lemma 39.1 *Suppose that* $X \subseteq \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ *is a* Δ^1_1 *set and any* $x \in X$ *satisfies* $\mathtt{dep}\,x < \infty$. *Then* $\mathsf{E}_1 \restriction X \leq_{\mathrm{B}} \mathsf{E}_0$.

Proof By the choice of X for any $x \in X$ there is a number n such that $\forall m \preccurlyeq n\, (x(m) \in \Delta^1_1(x \restriction_{\prec m}))$. As the relation between x and n here is clearly Π^1_1, the "Kreisel selection" theorem yields a Δ^1_1 map $\nu : X \to \mathbb{N}$ such that $x(m) \in \Delta^1_1(x \restriction_{\prec n})$ holds whenever $x \in X$ and $m \preccurlyeq \nu(x)$. Now define, for each $x \in X$, $\vartheta(x) \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ as follows: $\vartheta(x) \restriction_{\preccurlyeq \nu(x)} = x \restriction_{\preccurlyeq \nu(x)}$, but $\vartheta(x)(j) = \emptyset$ for all $j < \nu(x)$. Note that $x \mathrel{\mathsf{E}_1} \vartheta(x)$ for any $x \in X$.

The other important thing is that $\mathtt{ran}\,\vartheta \subseteq Z = \{x \in \mathcal{P}(\mathbb{N})^{\mathbb{N}} : \mathtt{dep}\,x = 0\}$, where Z is a Π^1_1 set, hence, there is a Δ^1_1 set Y with $\mathtt{ran}\,\vartheta \subseteq Y \subseteq Z$. In particular ϑ reduces $\mathsf{E}_1 \restriction X$ to $\mathsf{E}_1 \restriction Y$. We observe that $\mathsf{E}_1 \restriction Y$ is a countable ER: any E_1-class in $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$ intersects Y by an at most countable set (as so is the property of Z, a bigger set). Thus $\mathsf{E}_1 \restriction Y$ is both countable and smooth. It follows, by a theorem of [4], that $\mathsf{E}_1 \restriction Y \leq_{\mathrm{B}} \mathsf{E}_0$, as required. □ *check [4] !!!*

12.e Case 2

Since $\mathtt{dep}\,x = \infty$ is a Σ^1_1 formula, it suffices to show that for any non-empty Σ^1_1 set $R \subseteq \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ with $\forall x \in R\, (\mathtt{dep}\,x = \infty)$ we have a Δ^1_1 subset $X \subseteq R$ with $\mathsf{E}_1 \leq_{\mathrm{B}} \mathsf{E}_1 \restriction X$. Fix a set R, as indicated, for the course of the proof. The subset X of R will be defined with the help of a splitting construction developed in [26] for the study of "ill"founded Sacks iterations.

We shall define a map $\varphi : \mathbb{N} \to \mathbb{N}$, which assumes infinitely many values and assumes each its value infinitely many times (but $\mathtt{ran}\,\varphi$ may be a proper subset of $\mathbb{N}$), and, for each $u \in 2^{<\omega}$, a non-empty Σ^1_1 subset $X_u \subseteq R$, which satisfy a quite long list of properties. First of all, if φ is already defined at least on $[0, n)$ and $u \neq v \in 2^{<\omega}$ then let $\nu_\varphi[u, v] = \mathtt{min}_{\preccurlyeq}\{\varphi(k) : k < n \wedge u(k) \neq v(k)\}$. (Note that the minimum is taken in the sense of $\preccurlyeq$, hence, it is $\mathtt{max}$ in the sense of $\leq$, the usual order). Separately, put $\varphi[u, u] = -1$ for any u.

Now we give the list of requirements.

(i) if $\varphi(n) \notin \{\varphi(k) : k < n\}$ then $\varphi(n) \prec \varphi(k)$ for any $k < n$;

(ii) every X_u is a non-empty Σ^1_1 subset of R;

(iii) if $u \in 2^n$, $x \in X_u$, and $k < n$, then $\varphi(k) \in \nabla(x)$;

(iv) if $u, v \in 2^n$ then $X_u \restriction_{\prec \nu_\varphi[u,v]} = X_v \restriction_{\prec \nu_\varphi[u,v]}$;

(v) if $u, v \in 2^n$ then $X_u \restriction_{\preccurlyeq \nu_\varphi[u,v]} \cap X_v \restriction_{\preccurlyeq \nu_\varphi[u,v]} = \emptyset$;

(vi) $X_{u^\wedge i} \subseteq X_u$ for all $u \in 2^{<\omega}$ and $i = 0, 1$;

(vii) $\mathtt{max}_{u \in 2^n} \mathtt{diam}\, X_u \to 0$ as $n \to \infty$ (a reasonable Polish metric on $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$ is assumed to be fixed);

(viii) a certain condition, in terms of the Choquet game, which connects each $X_{u^\wedge i}$ with X_u so that, as a consequence, $\bigcap_n X_{a \restriction n} \neq \emptyset$ for any $a \in 2^{\mathbb{N}}$.

Let us demonstrate how such a system of sets and a function φ accomplish Case 2. According to (vii) and (viii), for any $a \in 2^{\mathbb{N}}$ the intersection $\bigcap_n X_{a \restriction n}$ contains a single point, let it be $F(a)$, and F is continuous and $1-1$.

More definitions is necessary. Put $J = \mathtt{ran}\, \varphi$, so that J is an infinite subset of $\mathbb{N}$. Let $J = \{j_m : m \in \mathbb{N}\}$, in the $<$-increasing order.

Let $n \in \mathbb{N}$. Then $\varphi(n) = j_m$ for some (unique) m: we put $\psi(n) = m$. Thus $\psi : \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$ and the preimage $\psi^{-1}(m) = \varphi^{-1}(j_m)$ is an infinite subset of $\mathbb{N}$ for any m. This allows us to define a parallel system of sets Y_u, $u \in 2^{<\omega}$, as follows. Put $Y_\Lambda = \mathcal{P}(\mathbb{N})^{\mathbb{N}}$. Suppose that Y_u has been defined, $u \in 2^n$. Then $j = \varphi(n) = j_{\psi(n)}$. Let N be the number of all indices $k < n$ still satisfying $\varphi(k) = j$, perhaps $N = 0$. Put $Y_{u^\wedge i} = \{x \in Y_u : x(j)(N) = i\}$ for $i = 0, 1$.

Each of Y_u is clearly a basic clopen set in $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$, and one easily verifies that conditions (i) – (vii), except for (iii), are satisfied for the sets Y_u (instead of X_u) and the map ψ (instead of φ), in particular, for any $a \in 2^{\mathbb{N}}$, $\bigcap_n Y_{a \restriction n} = \{G(a)\}$ is a singleton, and the map G is continuous and $1-1$. (We can, of course, define G explicitly: $G(a)(m)(l) = a(n)$, where $n \in \mathbb{N}$ is chosen so that $\psi(n) = m$ and there is exactly l numbers $k < n$ with $\psi(k) = m$.) Note finally that $\{G(a) : a \in 2^{\mathbb{N}}\} = \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ since by definition $Y_{u^\wedge 1} \cup Y_{u^\wedge 0} = Y_u$ for all u.

We conclude that the map $\vartheta(x) = F(G^{-1}(x))$ is a continuous bijection (hence, in this case, a homeomorphism by compactness) $\mathcal{P}(\mathbb{N})^{\mathbb{N}} \xrightarrow{\text{onto}} X$. We further assert that ϑ satisfying the following: for each $y, y' \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ and m,

$$y \restriction_{\preccurlyeq m} = y' \restriction_{\preccurlyeq m} \quad \text{iff} \quad \vartheta(y) \restriction_{\preccurlyeq j_m} = \vartheta(y') \restriction_{\preccurlyeq j_m}. \tag{$*$}$$

Indeed, let $y = G(a)$ and $x = F(a) = \vartheta(y)$, and similarly $y' = G(a')$ and $x' = F(a') = \vartheta(y')$, where $a, a' \in 2^{\mathbb{N}}$. Suppose that $y \restriction_{\preccurlyeq m} = y' \restriction_{\preccurlyeq m}$. According to (v) for ψ and the sets Y_u, we then have $m \prec \nu_\psi[a \restriction n, a' \restriction n]$ for any n. It follows, by the definition of ψ, that $j_m \prec \nu_\varphi[a \restriction n, a' \restriction n]$ for any n, hence, $X_{a \restriction n} \restriction_{\preccurlyeq j_m} = X_{a \restriction n} \restriction_{\preccurlyeq j_m}$ for any n by (iv). Assuming now that Polish metrics on all spaces $\mathcal{P}(\mathbb{N})^{\preccurlyeq j}$ are chosen so that $\mathtt{diam}\, Z \geq \mathtt{diam}\, (Z \restriction_{\preccurlyeq j})$ for all $Z \subseteq \mathcal{P}(\mathbb{N})$ and j, we easily obtain that $x \restriction_{\preccurlyeq j_m} = x' \restriction_{\preccurlyeq j_m}$, *i.e.*, the right-hand side of $(*)$. The inverse implication in $(*)$ is proved similarly.

Thus we have $(*)$, but this means that ϑ is a continuous reduction E_1 to $\mathsf{E}_1 \restriction X$, thus, $\mathsf{E}_1 \leq_{\mathrm{B}} \mathsf{E}_1 \restriction X$, as required.

$\square$ (*Theorem 39 modulo the construction* (i) – (viii))

12.f The construction

Recall that $R \subseteq \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ is a fixed non-empty Σ^1_1 set such that $\mathtt{dep}\, x = \infty$ for each $x \in R$. Set $X_\Lambda = R$.

Now suppose that the sets $X_u \subseteq R$ with $u \in 2^n$ have been defined and satisfy the applicable part of (i) – (viii).

Step 1. Our 1st task is to choose $\varphi(n)$. Let $\{j_1 < ... < j_m\} = \{\varphi(k) : k < n\}$. For any $1 \leq p \leq m$, let N_p be the number of all $k < n$ with $\varphi(k) = j_p$.

Case 1a. If some numbers N_p are $< m$ then choose $\varphi(n)$ among j_p with the least N_p, and among them the least one.

Case 1b: $N_p \geq m$ (then actually $N_p = m$) for all $p \leq m$. It follows from our assumptions, in particular (iv), that $X_u \restriction_{\prec j_m} = X_v \restriction_{\prec j_m}$ for all $u, v \in 2^n$. Let $Y = X_u \restriction_{\prec j_m}$ for any such u. Take any $y \in Y$. Then $\nabla(y)$ is infinite, hence, there is some $j \in \nabla(y)$ with $j \prec j_m$. Put $\varphi(n) = j$.

We have something else to do in this case. Let $X'_u = \{x \in X_u : j \in \nabla(y)\}$ for any $u \in 2^m$. Then we easily have $X'_u = \{x \in X_u : x \restriction_{\prec j_m} \in Y'\}$, where $Y' = \{y \in Y : j \in \nabla(y)\}$ is a non-empty Σ^1_1 set, so that the sets $X'_u \subseteq X_u$ are non-empty Σ^1_1. Moreover, as j_m is the $\preccurlyeq$-least in $\{\varphi(k) : k < n\}$, we can easily show that the system of sets X'_u still satisfies (iv). This allows us to assume, without any loss of generality, that, in Case 1b, $X'_u = X_u$ for all u, or, in other words, that any $x \in X_u$ for any $u \in 2^n$ satisfies $j = \varphi(n) \in \nabla(x)$. (This is true in Case 1a, of course, because then $\varphi(n) = \varphi(k)$ for some $k < n$.)

Note that this manner to choose $\varphi(n)$ implies (i) and also implies that φ takes infinitely many values and takes each its value infinitely many times.

The continuation of the construction requires the following

Lemma 39.2 *If $u_0 \in 2^n$ and $X' \subseteq X_{u_0}$ is a non-empty Σ^1_1 set then there is a system of Σ^1_1 sets $\emptyset \neq X'_u \subseteq X_u$ with $X'_{u_0} = X'$, which still satisfies* (iv).

Proof For any $u \in 2^n$, let $X'_u = \{x \in X_u : x \restriction_{\prec n(u)} \in X' \restriction_{\prec n(u)}\}$, where $n(u) = \nu_\varphi[u, u_0]$. In particular, this gives $X'_{u_0} = X'$, because $\nu_\varphi[u_0, u_0] = -1$. The sets X'_u are as required, via a routine verification. $\square$ (*Lemma*)

Step 2. First of all put $j = \varphi(n)$ and $Y_u = X_u \restriction_{\prec j}$. (All Y_u are equal to Y in Case 1b, but the argument pretends to make no difference between 1a and 1b). Take any $u_1 \in 2^n$. By the construction any element $x \in X_{u_1}$

satisfies $j \in \nabla(x)$, so that $x(j) \notin \Delta^1_1(x\restriction_{\prec j})$. As X_{u_1} is a Σ^1_1 set, it follows that $\{x'(j) : x' \in X_{u_1} \wedge x'\restriction_{\prec j} = x\restriction_{\prec j}\}$ is not a singleton, in fact is uncountable. It follows that there is a number l_{u_1} having the property that the Σ^1_1 set

$$Y'_{u_1} \;=\; \{y \in Y_{u_1} : \exists\, x, x' \in X_{u_1}\; (x'\restriction_{\prec j} = x\restriction_{\prec j} = y \wedge l_{u_1} \in x(j) \wedge l_{u_1} \notin x'(j))\}$$

is non-empty. We now put $X' = \{x \in X_{u_1} : x\restriction_{\prec j} \in Y'_{u_1}\}$ and define Σ^1_1 sets $\emptyset \neq X'_u \subseteq X_u$ as in the lemma, in particular, $X'_{u_1} = X'$, $X'_{u_1}\restriction_{\prec j} = Y'_{u_1}$, still (iv) is satisfied, and in addition

$$\forall\, y \in X'_{u_1}\restriction_{\prec j}\; \exists\, x, x' \in X'_{u_1}\; (x'\restriction_{\prec j} = x\restriction_{\prec j} = y \wedge l_{u_1} \in x(j) \wedge l_{u_1} \notin x'(j)) \qquad (1)$$

Now take some other $u_2 \in 2^n$. Let $\nu = \nu_\varphi[u_1, u_2]$. If $j \prec \nu$ then $X_{u_1}\restriction_{\prec j} = X_{u_2}\restriction_{\prec j}$, so that we already have, for $l_{u_2} = l_{u_1}$, that

$$\forall\, y \in X'_{u_2}\restriction_{\prec j}\; \exists\, x, x' \in X'_{u_2}\; (x'\restriction_{\prec j} = x\restriction_{\prec j} = y \wedge l_{u_2} \in x(j) \wedge l_{u_2} \notin x'(j))\,, \qquad (2)$$

and can pass to some $u_3 \in 2^n$. Suppose that $\nu \preccurlyeq j$. Now things are somewhat nastier. As above there is a number l_{u_2} such that

$$Y'_{u_2} \;=\; \{y \in Y_{u_2} : \exists\, x, x' \in X_{u_2}\; (x'\restriction_{\prec j} = x\restriction_{\prec j} = y \wedge l_{u_2} \in x(j) \wedge l_{u_2} \notin x'(j))\}$$

is a non-empty Σ^1_1 set, thus, we can define $X'' = \{x \in X_{u_1} : x\restriction_{\prec j} \in Y'_{u_1}\}$ and maintain the construction of Lemma 39.2, getting non-empty Σ^1_1 sets $X''_u \subseteq X'_u$ still satisfying (iv) and $X''_{u_2} = X''$, therefore, we still have (2) for the set X''_{u_2}.

Yet it is most important in this case that (1) is preserved, *i.e.*, it still holds for the set X''_{u_1} instead of X'_{u_1}! Why is this ? Indeed, according to the construction in the proof of Lemma 39.2, we have $X''_{u_1} = \{x \in X'_{u_1} : x\restriction_{\prec\nu} \in X''\restriction_{\prec\nu}\}$. Thus, although, in principle, X''_{u_1} is smaller than X'_{u_1}, for any $y \in X''_{u_1}\restriction_{\prec j}$ we have

$$\{x \in X''_{u_1} : x\restriction_{\prec j} = y\} \;=\; \{x \in X'_{u_1} : x\restriction_{\prec j} = y\}\,,$$

simply because now we assume that $\nu \preccurlyeq j$. This implies that (1) still holds.

Iterating this construction so that each $u \in 2^n$ is eventually encountered, we obtain, in the end, a system of non-empty Σ^1_1 sets, let us call them "new" X_u, but they are subsets of the "original" X_u, still satisfying (iv), still satisfying that $\varphi(n) \in \nabla(x)$ for each $x \in \bigcap_{u\in 2^n} X_u$, and, in addition, for any $u \in 2^n$ there is a number l_u such that $j \prec \nu_\varphi[u,v] \implies l_u = l_v$ and

$$\forall\, y \in X_u\restriction_{\prec j}\; \exists\, x, x' \in X_u\; (x'\restriction_{\prec j} = x\restriction_{\prec j} = y \wedge l_u \in x(j) \wedge l_u \notin x'(j))\,. \qquad (*)$$

Step 3. We define the $(n+1)$-th level of sets by $X_{u^\wedge 0} = \{x \in X_u : l_u \in x(j)\}$ and $X_{u^\wedge 1} = \{x \in X_u : l_u \notin x(j)\}$ for all $u \in 2^n$, where still $j = \varphi(n)$. It follows from $(*)$ that all these Σ^1_1 sets are non-empty.

Lemma 39.3 *The just defined system of sets* X_s, $s \in 2^{n+1}$, *satisfies* (iv), (v).

Proof Let $s = u^\wedge i$ and $t = v^\wedge i'$ belong to 2^{n+1}, so that $u, v \in 2^n$ and $i, i' \in \{0,1\}$. Let $\nu = \nu_\varphi[u,v]$ and $\nu' = \nu_\varphi[s,t]$.

Case 3a: $\nu \preccurlyeq j = \varphi(n)$. Then easily $\nu = \nu'$, so that (v) immediately follows from (v) at level n for X_u and X_v. As for (iv), we have $X_s\restriction_{\prec\nu} = X_u\restriction_{\prec\nu}$ (because by definition $X_s\restriction_{\prec j} = X_u\restriction_{\prec j}$), and similarly $X_t\restriction_{\prec\nu} = X_v\restriction_{\prec\nu}$, therefore, $X_t\restriction_{\prec\nu'} = X_s\restriction_{\prec\nu'}$ since $X_u\restriction_{\prec\nu} = X_v\restriction_{\prec\nu}$ by (iv) at level n.

Case 3b: $j \prec \nu$ and $i = i'$. Then still $\nu = \nu'$, thus we have (v). Further, $X_u\restriction_{\prec\nu} = X_v\restriction_{\prec\nu}$ by (iv) at level n, hence, $X_u\restriction_{\preccurlyeq j} = X_v\restriction_{\preccurlyeq j}$, hence, $l_u = l_v$ (see above). Now, assuming that, say, $i = i' = 1$ and $l_u = l_v = l$, we conclude that

$$X_s\restriction_{\prec\nu'} = \{y \in X_u\restriction_{\prec\nu} : l \in y(j)\} = \{y \in X_v\restriction_{\prec\nu} : l \in y(j)\} = X_t\restriction_{\prec\nu'}\,.$$

Case 3c: $j \prec \nu$ and $i \neq i'$, say, $i = 0$ and $i' = 1$. Now $\nu' = j$. Yet by definition $X_s\restriction_{\prec j} = X_u\restriction_{\prec j}$ and $X_t\restriction_{\prec j} = X_v\restriction_{\prec j}$, so it remains to apply (iv) for level n. As for (v), note that by definition $l \notin x(j)$ for any $x \in X_s = X_{u^\wedge 0}$ while $l \in x(j)$ for any $x \in X_t = X_{v^\wedge 1}$, where $l = l_u = l_v$. $\square$ (*Lemma*)

Step 4. In addition to (iv) and (v), we already have (i), (ii), (iii), (vi) at level $n+1$. To achieve the remaining properties (vii) and (viii), it suffices to consider, one by one, all elements $s \in 2^{n+1}$, finding, at each such a substep, a non-empty Σ^1_1 subset of X_s which is consistent with the requirements of (vii) and (viii) (for instance, for (vii), just take it so the diameter is $\le 2^{-n}$), and then reducing all other sets X_t by Lemma 39.2 at level $n+1$.

$\square$ (*Construction and Theorem 39*)

13 Summable ideals

Farah [5, § 1.12] gives the following classification of summable ideals $\mathcal{S}_{\{r_n\}}$, based on the distribution of numbers r_n:

(S1) *Atomic* ideals: there is $\varepsilon > 0$ such that the set $A_\varepsilon = \{n : r_n \ge \varepsilon\}$ is infinite and satisfies $\mu_{\{r_n\}}(\complement A_\varepsilon) < +\infty$. In this case $\mathcal{S}_{\{r_n\}} = \{X : X \cap A_\varepsilon \in \mathtt{Fin}\}$; Kechris [30] called such ideals *trivial variations of* $\mathtt{Fin}$.

(S2) *Dense* (summable) ideals: $r_n \to 0$.

(S3) There is a decreasing sequence of positive reals $\varepsilon_n \to 0$ sich that all sets $D_n = A_{\varepsilon_{n+1}} \setminus A_{\varepsilon_n}$ are infinite.

(S4) Ideals of the form $\mathtt{Fin} \oplus \text{dense}$: there is a real $\varepsilon > 0$ such that the set A_ε is infinite, $\mu_{\{r_n\}}(\complement A_\varepsilon) = +\infty$, and $\mathtt{lim}_{n\to\infty,\, n \in \complement A_\varepsilon} r_n = 0$.

define $\oplus$ somewhere

In the sense of $\le_{\mathrm{B}}$, all ideals of types (S2), (S3), (S4) are equivalent to each other, and all ideals of type (S1) are equivalent to each other, so that we have just 2 summable ideals modulo $\sim_{\mathrm{B}}$, namely, $\mathtt{Fin}$ and $\mathcal{S}_{\{1/n\}}$. The structure under $\le_{\mathrm{RB}}$ or $\le_{\mathrm{BE}}$ is much more complicated (Farah ?).

13.a Some simple results

Lemma 40 (Folklore ?) $0 \times \mathtt{Fin} \le^{+}_{\mathrm{RB}} \mathcal{Z}_0$.

Proof The sets $w_{kn} = \{2^k(2j+1) : 2^n \le j < 2^{n+1}\}$ are finite and pairwise disjoint. Clearly any $Z_k = \bigcup_n w_{kn}$ is a set of density 2^{-k-1}, moreover, if N is infinite then $\bigcup_{n\in N} w_{kn}$ still has upper density at least 2^{-k-2}, hence, if $X \subseteq \mathbb{N}^2$ does not belong to $0 \times \mathtt{Fin}$ then $w_X = \bigcup_{\langle n,k\rangle \in X} w_{kn} \notin \mathcal{Z}_0$. On the other hand, for any k, the union $Z_{\ge k} = \bigcup_{k' \ge k} Z_{k'}$ has density 2^{-k}, which yields the opposite implication. □

Lemma 41 (Folklore ?) $\mathcal{S}_{\{1/n\}} \subsetneqq \mathcal{Z}_0$.

Proof Profe first that $\mathcal{S}_{\{1/n\}} \subseteq \mathcal{Z}_0$. Let $X \notin \mathcal{Z}_0$, so that there is $\varepsilon > 0$ such that $\#(X \cap [0,n)) > 2n\varepsilon$ for infinitely many numbers n. Take any such n, big enough w.r.t. ε^{-1}. Then clearly $\#(X \cap [n\varepsilon, n)) > n\varepsilon$. It follows that the sum $\sum_{k\in X,\, k>n\varepsilon} k^{-1}$ is not smaller than $\frac{n\varepsilon}{n} = \varepsilon$. As this holds for infinitely many n, the set X cannot belong to $\mathcal{S}_{\{1/n\}}$.

Let us now produce a set $X \in \mathcal{Z}_0 \setminus \mathcal{S}_{\{1/n\}}$. Let m_j be the entire part of $j \log j$. Then $X = \{m_j : j \ge 2\} \notin \mathcal{S}_{\{1/n\}}$ because the integral $\int_2^\infty (x \log x)^{-1} dx$ diverges. On the other hand, easily $X \in \mathcal{Z}_0$. □

A simple modification of the argument proves the following:

Lemma 42 *Suppose that* $r_n \ge 0$ *and* $r_n \to 0$. *Then* $\mathcal{Z}_0 \not\subseteq \mathcal{S}_{\{r_n\}}$.

Question 43 What about $\boldsymbol{c}_0 / \mathcal{S}_{\{1/n\}}$? □

Lemma 44 (Folklore ?) *Suppose that* $r_n \ge 0$, $r_n \to 0$, *and* $\sum_n r_n = +\infty$. *Then any summable ideal* $\mathcal{I}$ *satisfies* $\mathcal{I} \le^{++}_{\mathrm{RB}} \mathcal{S}_{\{r_n\}}$.

Proof Let $I = \mathcal{S}_{\{p_n\}}$, where $p_n \ge 0$ (no other requirements !). Under the assumptions of the lemma we can associate a finite set $w_n \subseteq \mathbb{N}$ to any n so that $\mathtt{max}\, w_n < \mathtt{min}\, w_{n+1}$ and $|r_n - \sum_{j\in w_n} r_i| < 2^{-n}$. □

Extension: full picture of summables.

13.b Above the summable ideals

Is there any example of Borel ideals $\mathcal{I} \le_{\mathrm{B}} \mathcal{J}$ which do not satisfy $\mathcal{I} \le_\Delta \mathcal{J}$? Typically the reductions found to witness $\mathcal{I} \le_{\mathrm{B}} \mathcal{J}$ are Δ-homomorphisms, and even better maps. The following lemma proves that Borel reduction yields $\le^{++}_{\mathrm{RB}}$-reduction in quite a representative case. Let us say that $\mathcal{I} \le^{++}_{\mathrm{RB}} \mathcal{J}$ *holds exponentially* if there is a map $i \mapsto w_i$ withessing $\mathcal{I} \le^{++}_{\mathrm{RB}} \mathcal{J}$ and in addition a sequence of natural numbers k_i with $w_i \subseteq [k_i, k_{i+1})$ and $k_{i+1} \ge 2k_i$.

Lemma 45 (Farah [6]) *Let* $\mathcal{J}$ *be a Borel P-ideal. Suppose that* $r_n \ge 0$, $r_n \to 0$, *and* $\sum_n r_n = +\infty$. *Then* $\mathcal{S}_{\{r_n\}} \le_{\mathrm{B}} \mathcal{J}$ *implies* $\mathcal{S}_{\{r_n\}} \le^{++}_{\mathrm{RB}} \mathcal{J}$ *exponentially.*

Proof Let a Borel $\vartheta : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ witness $\mathcal{S}_{\{r_n\}} \le_{\mathrm{B}} \mathcal{J}$. Let, according to Theorem 25, φ be a l.s.c. submeasure on $\mathbb{N}$ with $\mathcal{J} = \mathtt{Exh}_\varphi$. The construction will make use of stabilizers. Below, "generic" will mean: Cohen-generic over a sufficiently large countable model of a big enough fragment of **ZFC**.

Suppose that $n \in \mathbb{N}$. If $u, v \subseteq [0,n)$ then $(u \cup X) \,\Delta\, (v \cup X) \in \mathcal{S}_{\{r_n\}}$ for any $X \subseteq [n, +\infty)$, hence, $\vartheta(u \cup X) \,\Delta\, \vartheta(v \cup X) \in \mathcal{J}$. It follows, by the choice of φ, that for any $\varepsilon > 0$ there are numbers $n < k < n'$ and a set $s \subseteq [n, n')$ with

$$\varphi((\vartheta(u \cup s \cup X) \,\Delta\, \vartheta(v \cup s \cup X)) \cap [k, \infty)) < \varepsilon$$

being true for all $u, v \subseteq [0,n)$ and all generic $X \subseteq [n', \infty)$.

This observation allows us to define an increasing sequence of natural numbers $0 = k_0 = a_0 < b_0 < k_1 < a_1 < b_1 < k_2 < \dots$ and, for any i, a set $s_i \subseteq [b_i, a_{i+1})$ such that, for all generic $X, Y \subseteq [b_{i+1}, \infty)$ and all $u, v \subseteq [0, a_i)$, we have

(i) $\varphi((\vartheta(u \cup s_i \cup X) \,\Delta\, \vartheta(v \cup s_i \cup X)) \cap [k_{i+1}, \infty)) < 2^{-i}$;

(ii) $(\vartheta(u \cup s_i \cup X) \,\Delta\, \vartheta(u \cup s_i \cup Y)) \cap [0, k_{i+1}) = \emptyset$;

(iii) any $Z \subseteq \mathbb{N}$, satisfying $X \cap [b_i, a_{i+1}) = s_i$ for infinitely many i, is generic;

(iv) $k_{i+1} \ge 2k_i$ for all i;

and in addition, under the assumptions on $\{r_n\}$,

(v) there is $g_i \subseteq [a_i, b_i)$ such that $|r_i - \sum_{n \in g_i} r_n| < 2^{-i}$.

It follows from (v) that $A \mapsto g_A = \bigcup_{i \in A} g_i$ is a reduction $\mathcal{S}_{\{r_n\}} \to \mathcal{S}_{\{r_n\}} \restriction N$, where $N = \bigcup_i [a_i, b_i)$.

Let $\xi(Z) = \vartheta(Z \cup S) \,\Delta\, \vartheta(S)$ for any $Z \subseteq N$, where $S = \bigcup_i s_i$. (Note that $S \cap N = \emptyset$.) Then, for $X, Y \subseteq N$, $X \Delta Y \in \mathcal{S}_{\{r_n\}}$ iff $\vartheta(X \cup S) \Delta \vartheta(Y \cup S) \in \mathcal{J}$ iff $\xi(X) \,\Delta\, \xi(Y) \in \mathcal{J}$, so ξ reduces $\mathcal{S}_{\{r_n\}} \restriction N$ to $\mathcal{J}$. Now put $w_i = \xi(g_i) \cap [k_i, k_{i+1})$ and $w_A = \bigcup_{i \in A} w_i$. We assert that the map $i \mapsto w_i$ proves $\mathcal{S}_{\{r_n\}} \le^{++}_{\text{RB}} \mathcal{J}$. Ii view of the above, it remains to show that $\xi(g_A) \,\Delta\, w_A \in \mathcal{J}$ for any $A \in \mathcal{P}(\mathbb{N})$.

pachimu dobavleno $\Delta\vartheta(S)$? razo-brat'sya

As $\mathcal{J} = \texttt{Exh}_\varphi$, it suffices to demonstrate that $\varphi(w_i \Delta(\xi(g_A) \cap [k_i, k_{i+1}))) < 2^{-i}$ for all $i \in A$ while $\varphi(\xi(g_A) \cap [k_i, k_{i+1})) < 2^{-i}$ for $i \notin A$. Getting rid of the common term $\vartheta(S)$, we see that it suffices to check that

(1) $\varphi((\vartheta(g_i \cup S) \,\Delta\, \vartheta(g_A \cup S)) \cap [k_i, k_{i+1})) < 2^{-i}$ for all $i \in A$ while

(2) $\varphi((\vartheta(S) \,\Delta\, \vartheta(g_A \cup S)) \cap [k_i, k_{i+1})) < 2^{-i}$ for $i \notin A$.

Note that, as any set of the form $X \cup S$, where $S \subseteq N$, is generic by (iii). It follows, by (ii), that we can assume, in (1) and (2), that $A \subseteq [0, i]$, *i.e.*, resp. $\max A = i$ and $\max A < i$. We can finally apply (i), with $u = A \cup \bigcup_{j<i} s_j$, $X = \bigcup_{j>i} s_j$, and $v = u_i \cup \bigcup_{j<i} s_j$ if $i \in A$ while $v = \bigcup_{j<i} s_j$ if $i \notin A$. □

Question 46 (Farah [6]) For which ideals other than summables the lemma is true ? (Farah points out that it also holds for $0 \times \mathtt{Fin}$.) □

Corollary 47 (Hjorth ?) *If r_n are as in the lemma then $\mathcal{S}_{\{r_n\}}$ is not Borel-reducible to $\mathcal{Z}_0$.*

Proof Otherwise, by the lemma, we have $\mathcal{S}_{\{r_n\}} \le^{++}_{\text{RB}} \mathcal{Z}_0$ exponentially. Let this be witnessed by $i \mapsto w_i$ and a sequence of numbers k_i, so that $k_{i+1} \ge 2k_i$ and $w_i \subseteq [k_i, k_{i+1})$. If $d_i = \frac{\#(w_i)}{k_{i+1}} \to 0$ then easily $\bigcup_i w_i \in \mathcal{Z}_0$ by the choice of $\{k_i\}$. Otherwise there is a set $A \in \mathcal{S}_{\{r_n\}}$ such that $d_i > \varepsilon$ for all $i \in A$ and one and the same $\varepsilon > 0$, so that $w_A = \bigcup_{i \in A} w_i \notin \mathcal{Z}_0$. In both cases we have a contradiction with the assumption that the map $i \mapsto w_i$ witnesses $\mathcal{S}_{\{r_n\}} \le^{++}_{\text{RB}} \mathcal{Z}_0$. □

13.c A useful lemma

Lemma 48 (Attributed to Kechris in [17]) *Suppose that A, X are Borel sets, E is a Borel ER on A, and $\rho : A \to X$ is a Borel map satisfying the following: first, the ρ-image of any E-class is at most countable, secong, ρ-images of any different E-classes are disjoint. Then E is an essentially countable ER.*

Proof The relation: $x \,\mathsf{R}\, y$ iff $x, y \in Y$ belong to the ρ-image of one and the same E-class in A, is a $\mathbf{\Sigma}^1_1$ ER on the set $Y = \mathtt{ran}\,\vartheta$, moreover,

$$\mathsf{R} \subseteq P = \{\langle x, y\rangle : \neg\, \exists\, a, b \in A\ (a \not\mathsf{E}\, b \wedge x = \rho(a) \wedge y = \rho(b))\},$$

where P is $\mathbf{\Pi}^1_1$, hence, there is a Borel set U with $\mathsf{R} \subseteq U \subseteq P$, in particular, $U \cap (Y \times Y) = \mathsf{R}$. As all R-equivalence classes are at most countable, we can assume that all cross-sections of U are at most countable, too.

Now it suffices to find a Borel ER F with $\mathsf{R} \subseteq \mathsf{F} \subseteq U$. Say that a set $Z \subseteq X$ is "stable" if $U \cap (Z \times Z)$ is a ER, for example, Y is "stable". We observe that the set $D_0 = \{y : Y \cup \{y\}$ is "stable"$\}$ is $\mathbf{\Pi}^1_1$ and satisfies $Y \subseteq D_0$, hence, there is a Borel set Z_1 with $Y \subseteq Z_1 \subseteq D_0$. Similarly,

$$D_1 = \{y' \in Z_1 : Y \cup \{y, y'\} \text{ is "stable" for any } y \in Z_1\}$$

is $\mathbf{\Pi}^1_1$ and satisfies $Y \subseteq D_1$ by the definition of Z_1, so that there is a Borel set Z_2 with $Y \subseteq Z_2 \subseteq D_1$. Generally, we define

$$D_n = \{y' \in Z_n : Y \cup \{y_1, ..., y_n, y'\} \text{ is "stable" for all } y_1, ..., y_n \in Z_n\}$$

find that $Y \subseteq D_n$, and choose a Borel set Z_n with $Y \subseteq Z_n \subseteq D_n$. Then, by the construction, $Y \subseteq Z = \bigcap_n Z_n$, and, for any finite $Z' \subseteq Z$, the set $Y \cup Z'$ is "stable", so that Z itself is "stable", and we can take $\mathsf{F} = U \cap (Z \times Z)$. □

13.d Under the summable ideal

Subsets of $\mathbb{N}$ will be systematically identified with their characteristic functions.

For $a, b \in 2^{\mathbb{N}}$ put $a \,\Delta\, b = \{n : a(n) \ne b(n)\}$ (identified with the function $c(n) = 1$ iff $a(n) \ne b(n)$) and $\Sigma(a, b) = \sum_{n \in a \Delta b} \frac{1}{n+1}$. (This can be a nonnegative real or $+\infty$.) Generally, we define $\Sigma^m_k(a, b) = \sum_{n \in a\Delta b,\, k \le n \le m} \frac{1}{n+1}$, where m can be ∞. We also define $\Sigma(a) = \sum_{\{n : a(n)=1\}} \frac{1}{n+1}$ and similarly $\Sigma^k_m(a)$.

Recall that the *summable ideal* is defined as

$$\mathcal{S}_{\{1/n\}} = \{a \in 2^{\mathbb{N}} : \Sigma(a) < +\infty\}.$$

(The notation $\mathcal{I}_2$ and $\mathcal{I}_0$ is also used.) $\mathsf{E}_{\{1/n\}}$ will denote the associated Borel ER on $2^{\mathbb{N}}$, *i.e.*, $a \,\mathsf{E}_{\{1/n\}}\, b$ iff $\Sigma(a, b) < +\infty$.

Theorem 49 *Let E be a Borel ER on a Polish space $\mathbb{X}$, and $\mathsf{E} \le_{\text{B}} \mathsf{E}_{\{1/n\}}$. Then either $\mathsf{E} \sim_{\text{B}} \mathsf{E}_{\{1/n\}}$ or E is essentially countable.*

Proof This is a long proof. Let $\vartheta : \mathbb{X} \to 2^{\mathbb{N}}$ be a Borel reduction E to $\mathsf{E}_{\{1/n\}}$. We can assume that ϑ is in fact continuous: indeed it is known that there is a stronger Polish topology on $\mathbb{X}$ which makes ϑ continuous but does not add new Borel subsets of $\mathbb{X}$. Now, as any Polish $\mathbb{X}$ is a $1-1$ continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$, we can assume that $\mathbb{X} = \mathbb{N}^{\mathbb{N}}$.

Finally, we can assume that ϑ is Δ^1_1, not merely Borel.

If $a \in A \subseteq 2^{\mathbb{N}}$ and $q \in \mathbb{Q}^+$ then let $\mathtt{Gal}^q_A(a)$ be the set of all $b \in A$ such that there is a finite chain $a = a_0, a_1, ..., a_n = b$ of reals $a_i \in A$ such that $\Sigma(a_i, a_{i+1}) < q$ for all i, the *q-galaxy of a in A*.

Definition 49.1 A set $A \subseteq 2^{\mathbb{N}}$ is q-*"grainy"*, where $q \in \mathbb{Q}^+$, iff $\Sigma(a,b) < 1$ for all $a \in A$ and $b \in \mathtt{Gal}^q_A(a)$. A set A is *"grainy"* if it is q-"grainy" for some $q \in \mathbb{Q}^+$. (In other words it is required that the galaxies are rather small.) □

Claim 49.2 *Any q-"grainy" Σ^1_1 set $A \subseteq 2^{\mathbb{N}}$ is covered by a q-"grainy" Δ^1_1 set.*

Proof [12] The set $D_0 = \{b \in 2^{\mathbb{N}} : A \cup \{b\}$ is q-"grainy"$\}$ is Π^1_1 and $A \subseteq D_0$, hence, there is a Δ^1_1 set B_1 with $A \subseteq B_1 \subseteq D_0$. Note that $A \cup \{a\}$ is q-"grainy" for any $a \in B_1$. It follows that the Π^1_1 set

$$D_1 = \{b \in B_1 : A \cup \{a,b\} \text{ is } q\text{-"grainy" for any } a \in B_1\}$$

still contains A, hence, there is a Δ^1_1 set B_2 with $A \subseteq B_2 \subseteq D_1 \subseteq B_1$. Note that $A \cup \{a_1, a_2\}$ is q-"grainy" for any $a_1, a_2 \in B_2$. In general, as soon as we have got a Δ^1_1 set B_n with $A \subseteq B_n$ and such that $A \cup \{a_1,...,a_n\}$ is q-"grainy" for any $a_1,...,a_n \in B_n$, then the Π^1_1 set

$$D_n = \{b \in B_n : A \cup \{a_1,...,a_n,b\} \text{ is } q\text{-"grainy" for any } a_1,...,a_n \in B_n\}$$

contains A, hence, there is a Δ^1_1 set B_{n+1} with $A \subseteq B_{n+1} \subseteq D_n \subseteq B_n$.

As usual in similar cases, the choice of the sets B_n can be made effective enough for the set $B = \bigcap_n B_n$ to be still Δ^1_1, not merely Borel. On the other hand, $A \subseteq B$ and B is q-"grainy". □ (*Claim*)

Coming back to the proof of the theorem, let C be the union of all "grainy" Δ^1_1 sets. An ordinary computation shows that C is Π^1_1. We have two cases.

***Case 1*:** $\mathtt{ran}\,\vartheta \subseteq C$. ***Case 2*:** otherwise.

13.e Case 1

We are going to prove that, in this case, E is essentially countable. First note that, by Separation, there is a Δ^1_1 set $H^* \subseteq 2^{\mathbb{N}}$ with $\mathtt{ran}\,\vartheta \subseteq H^* \subseteq C$.

Fix a standard enumeration $\{W_e\}_{e \in E}$ of all Δ^1_1 subsets of $2^{\mathbb{N}}$, where, as usual, $E \subseteq \mathbb{N}$ is a Π^1_1 set. By Kreisel Selection, there exist Δ^1_1 functions $a \longmapsto e(a)$ and $a \longmapsto q(a)$, defined on H^*, such that for any $a \in H^*$ the Δ^1_1 set $W(a) = W_{e(a)}$ contains a and is $q(a)$-"grainy". The final point of our argument will be an application of Lemma 48, where ρ will be a derivate of the function $G(a) = \mathtt{Gal}^{q(a)}_{W(a)}(a)$. We prove

Claim 49.3 *If $a \in H^*$ then $\gamma_a = \{G(b) : b \in [a]_{\mathsf{E}_{\{1/n\}}} \cap H^*\}$ is at most countable.*

[12] The result can be achieved as a routine application of a reflection principle, yet we would like to show how it works with a low level technique.

Proof Otherwise there is a pair of $e \in E$ and $q \in \mathbb{Q}^+$ and an uncountable set $B \subseteq [a]_{\mathsf{E}_{\{1/n\}}} \cap H^*$ such that $q(b) = q$ and $e(b) = e$ for any $b \in B$ and $G(b') \neq G(b)$ for any two different $b, b' \in B$. Note that any $G(b)$, $b \in B$, is a q-galaxy in one and the same set $W(a) = W(b) = W_e$, therefore, if $b \neq b' \in B$ then $b' \notin G(b)$ and $\Sigma(b,b') \geq q$. On the other hand, as $B \subseteq [a]_{\mathsf{E}_{\{1/n\}}}$, we have $\Sigma(a,b) < +\infty$ for all $b \in B$, hence, there is m and a still uncountable set $B' \subseteq B$ such that $\Sigma^{\infty-1}_m(a,b) < q/2$ for all $b \in B'$. Now take a pair of $b \neq b' \in B'$ with $b \restriction [0,m) = b' \restriction [0,m)$: then $\Sigma(b,b') < q$, contradiction. □ (*Claim*)

It follows that $x \mapsto G(\vartheta(x))$ maps any E-class into a countable set of galaxies $G(a)$. To code the galaxies by single points, let $S(a) = \{b \restriction m : b \in G(a) \wedge m \in \mathbb{N}\}$, so that $S(a) \subseteq 2^{<\omega}$ codes the topological closure of the galaxy $G(a)$.

Claim 49.4 *If $a, b \in H^*$ and $a \not\mathsf{E}_{\{1/n\}} b$ then b does not belong to the (topological) closure of $G(a)$, in particular, $b \restriction m \notin S(a)$ for some m.*

Proof Take m big enough for $\Sigma^{m-1}_0(a,b) \geq 2$. Then $s = b \restriction m$ does not belong to $S(a)$ because any $a' \in G(a)$ satisfies $\Sigma(a,a') < 1$. □ (*Claim*)

Elementary computation shows that the sets

$$\mathbf{G} = \{\langle a,b\rangle : a \in H^* \wedge b \in G(a)\} \quad \text{and} \quad \mathbf{S} = \{\langle a,s\rangle : a \in H^* \wedge s \in S(a)\}.$$

belong to Σ^1_1, but this is not enough to claim that $a \mapsto S(a)$ is a Borel map. Yet we can change it appropriately to get a Borel map with similar properties. First of all define the following Σ^1_1 ER on H^*:

$$a \,\mathsf{F}\, b \quad \text{iff} \quad e(a) = e(b) \wedge q(a) = q(b) \wedge G(a) = G(b).$$

(To see that F is Σ^1_1 note that here $G(a) = G(b)$ is equivalent to $b \in G(a)$, and that $\mathbf{G}$ is Σ^1_1.) It follows from Claim 49.4 and Kreisel Selection that there is a Δ^1_1 function $\mu : H^* \times H^* \to \mathbb{N}$ such that for any pair of $a, b \in H^*$ with $a \not\mathsf{E}_{\{1/n\}} b$ we have $b \restriction \mu(a,b) \notin S(a)$. Then the set

$$R(a) = \{b \restriction \mu(a',b) : a', b \in H^* \wedge a \,\mathsf{F}\, a' \wedge a' \not\mathsf{E}_{\{1/n\}} b)\} \subseteq 2^{<\omega}$$

does not intersect $S(a)$, for any $a \in H^*$, hence, the Σ^1_1 set

$$\mathbf{R} = \{\langle a,s\rangle : a \in H^* \wedge s \in R(a)\}$$

does not intersect $\mathbf{S}$. Note that by definition $\mathbf{R}$ is F-invariant w.r.t. the 1st argument, *i.e.*, if $a, a' \in H^*$ satisfy $a \,\mathsf{F}\, a'$ then $R(a) = R(a')$. It follows from Lemma 17.1 that there is a Δ^1_1 set $\mathbf{Q} \subseteq H^* \times 2^{<\omega}$ with $\mathbf{S} \subseteq \mathbf{Q}$ but $\mathbf{R} \cap \mathbf{Q} = \emptyset$, F-invariant in the same sense. Then the map $a \mapsto Q(a) = \{s : \mathbf{Q}(a,s)\}$ is Δ^1_1.

Claim 49.5 *Suppose that $a, b \in H^*$. Then: $a \,\mathsf{F}\, b$ implies $Q(a) = Q(b)$ and $a \not\mathsf{E}_{\{1/n\}} b$ implies $Q(a) \neq Q(b)$.*

Proof The first statement holds just because Q is F-invariant. Now suppose that $a \not\mathsf{E}_{\{1/n\}} b$. Then by definition $s = b \restriction \mu(a,b) \in R(a)$, hence, $s \notin Q(a)$. On the other hand, $s \in S(b) \subseteq Q(b)$. □ (*Claim*)

Define $\tau(x) = Q(\vartheta(x))$ for $x \in \mathbb{N}^{\mathbb{N}}$, so that τ is a Δ^1_1 map $\mathbb{N}^{\mathbb{N}} \to \mathcal{P}(2^{<\omega})$.

Claim 49.6 *If* $x \in \mathbb{N}^{\mathbb{N}}$ *then* $T_a = \{\tau(y) : y \in [x]_{\mathsf{E}}\}$ *is at most countable.*

Proof Suppose that $y, z \in [x]_{\mathsf{E}}$. Then $a = \vartheta(x)$, $b = \vartheta(y)$, and $c = \vartheta(z)$ belong to H^*, and $b, c \in [a]_{\mathsf{E}_{\{1/n\}}}$. It follows from Claim 49.5 that if $G(b) = G(c)$, $e(b) = e(c)$, and $q(b) = q(c)$, then $Q(b) = Q(c)$. It remains to note that G takes only countably many values on $H^* \cap [a]_{\mathsf{E}_{\{1/n\}}}$ by Claim 49.3. □ (*Claim*)

Finally note that, if $x \not\mathsf{E}\, y \in \mathbb{N}^{\mathbb{N}}$ then $\vartheta(x)$, $\vartheta(y)$ belong to H^* and satisfy $\vartheta(x) \not\mathsf{E}_{\{1/n\}} \vartheta(y)$, hence, $\tau(x) \neq \tau(y)$ by Claim 49.5. Thus, the Borel map τ witnesses that the given ER E is essentially countable by Lemma 48.

13.f Case 2

Thus we suppose that the Σ^1_1 set $B^* = \mathtt{ran}\,\vartheta \setminus C$ is non-empty. Note that, by Claim 49.2, there is no non-empty Σ^1_1 "grainy" set $A \subseteq B^*$.

To prove that in this case $\mathsf{E}_{\{1/n\}} \le_{\mathrm{B}} \mathsf{E}$, we shall define an increasing sequence of natural numbers $0 = k_0 < k_1 < k_2 < \ldots$, and also objects A_s, g_s, v_s for any $s \in 2^{<\omega}$, which satisfy the following list of requirements (i) – (vii).

0^m will denote the sequence of m zeros.

Let $\mathcal{B}_s = \{a \in 2^{\mathbb{N}} : s \subset a\}$ for $s \in 2^{<\omega}$ and $\mathcal{N}_u = \{x \in \mathbb{N}^{\mathbb{N}} : u \subset x\}$ for $u \in \mathbb{N}^{<\omega}$ (basic open nbhds in $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$).

If $A, B \subseteq 2^{\mathbb{N}}$ and $m, k \in \mathbb{N}$, then $A\, \mathsf{R}^m_{\ge k}\, B$ will mean that for any $a \in A$ there is $b \in B$ with $\Sigma^\infty_k(a,b) < 2^{-m}$, and conversely, for any $b \in B$ there is $a \in A$ with $\Sigma^\infty_k(a,b) < 2^{-m}$. This is not a ER, of course, yet we can assert that if $A\, \mathsf{R}^m_{\ge k}\, B$ and $B\, \mathsf{R}^m_{\ge k}\, C$ then $A\, \mathsf{R}^{m-1}_{\ge k}\, C$.

(i) if $s \in 2^m$ then $g_s \in 2^{k_m}$, and $s \subset t \Longrightarrow g_s \subset g_t$;

(ii) $\emptyset \neq A_s \subseteq B^* \cap \mathcal{B}_{g_s}$, A_s is Σ^1_1, and $s \subset t \Longrightarrow A_t \subseteq A_s$.

(iii) if $s \in 2^n$ then $A_{0^n}\, \mathsf{R}^{n+2}_{\ge k_n}\, A_s$;

(iv) if $s \in 2^n$, $m < n$, $s(m) = 0$, then $\Sigma^{k_{m+1}-1}_{k_m}(g_s, g_{0^m}) < 2^{-m-1}$;

(v) if $s \in 2^n$, $m < n$, $s(m) = 1$, then $|\Sigma^{k_{m+1}-1}_{k_m}(g_s, g_{0^m}) - \frac{1}{m+1}| < 2^{-m-1}$;

(vi) if $s \in 2^n$ then $v_s \in \mathbb{N}^n$, and $s \subset t \Longrightarrow v_s \subset v_t$;

(vii) $A_s \subseteq \{a \in B^* : \vartheta^{-1}(a) \cap \mathcal{N}_{v_s} \neq \emptyset\}$.

We can now accomplish Case 2 as follows. For any $a \in 2^{\mathbb{N}}$ define $F(a) = \bigcup_n g_{a \restriction n} \in 2^{\mathbb{N}}$ (the only element satisfying $g_{a\restriction n} \subset F(a)$ for all n) and $\rho(a) = \bigcup_n v_{a\restriction n} \in \mathbb{N}^{\mathbb{N}}$. It follows, by (vii) and the continuity of ϑ, that $F(a) = \vartheta(\rho(a))$ for any $a \in 2^{\mathbb{N}}$. Thus the next claim proves that ρ is a Borel (in fact, here continuous) reduction $\mathsf{E}_{\{1/n\}}$ to E and ends Case 2.

Claim 49.7 *The map* F *reduces* $\mathsf{E}_{\{1/n\}}$ *to* $\mathsf{E}_{\{1/n\}}$*, that is, the equivalence* $a\, \mathsf{E}_{\{1/n\}}\, b \Longleftrightarrow F(a)\, \mathsf{E}_{\{1/n\}}\, F(b)$ *holds for all* $a, b \in 2^{\mathbb{N}}$.

Proof By definition $\Sigma(F(a), F(b)) = \mathtt{lim}_{n\to\infty} \Sigma^{k_n-1}_0(g_{a\restriction n}, g_{b\restriction n})$. However it follows from (iv) and (v) that

$$|\Sigma^{k_n-1}_0(g_{a\restriction n}, g_{b\restriction n}) - \Sigma^{n-1}_0(a \restriction n, b \restriction n)| \le \textstyle\sum_{m<n} 2^{-m} < 2.$$

We conclude that $|\Sigma(F(a), F(b)) - \Sigma(a,b)| \le 2$, as required. □ (*Claim*)

13.g Construction

The construction goes on by induction. To begin with we set $k_0 = 0$, $g_\Lambda = \Lambda$ and $A_\Lambda = B^*$. Suppose that, for some n, we have the objects as required for all $n' \le n$, and extend the construction on the level $n+1$.

As A_{0^n} is not "grainy" (see above), there is a pair of elements $a^0, a^1 \in A_{0^n}$ such that $|\Sigma(a^0,a^1) - \frac{1}{n+1}| < 2^{-n-2}$. Note that $a^0 \restriction k_n = a^1 \restriction k_n$ by (i) and (ii), hence, there is $k_{n+1} > k_n$ such that $|\Sigma^{k_{n+1}-1}_{k_n}(a^0,a^1) - \frac{1}{n+1}| < 2^{-n-2}$. According to (iii), for any $s \in 2^n$ there exist $b^0_s, b^1_s \in A_s$ such that and $\Sigma^\infty_{k_n}(a^i, b^i_s) < 2^{-n-2}$ for $i = 0,1$; we can, of course, assume that $b^i_{0^m} = a^i$. Moreover, the number k_{n+1} can be chosen big enough for the following to hold:

$$\Sigma^\infty_{k_{n+1}}(b^i_s, a^0) < 2^{-n-3} \quad \text{— for all} \quad s \in 2^n \quad \text{and} \quad i = 0,1. \tag{1}$$

We let $g_{s^\wedge i} = b^i_s \restriction k_{n+1}$ for all $s^\wedge i \in 2^{n+1}$. This definition preserves (i). To check (iv) for $s' = s^\wedge 0 \in 2^{n+1}$ and $m = n$, note that

$$\Sigma^{k_{n+1}-1}_{k_n}(g_{s'}, g_{0^{n+1}}) = \Sigma^{k_{n+1}-1}_{k_n}(b^0_s, a^0) < 2^{-n-2}.$$

To check (v) for $s' = s^\wedge 1 \in 2^{n+1}$ and $m = n$, note that

$$|\Sigma^{k_{n+1}-1}_{k_n}(g_{s'}, g_{0^{n+1}}) - \tfrac{1}{n+1}| \le \Sigma^{k_{n+1}-1}_{k_n}(b^1_s, a^1) + |\Sigma^{k_{n+1}-1}_{k_n}(a^0,a^1) - \tfrac{1}{n+1}| < 2^{-n-1}.$$

To fulfill (vi), choose, for any $s^\wedge i \in 2^{n+1}$, a sequence $v_{s^\wedge i} \in \mathbb{N}^{n+1}$ so that $v_s \subset v_{s^\wedge i}$ and there is $\mathcal{N}_{v_{s^\wedge i}} \cap \vartheta^{-1}(b^i_s) \neq \emptyset$.

Let us finally define the sets $A_{s'} \subseteq A_s$, for all $s' = s^\wedge i \in 2^{n+1}$ (so that $s \in 2^n$ and $i = 0,1$). To fulfill (ii) and (vii), we begin with

$$A'_{s^\wedge i} = \{a \in A_s \cap \mathcal{B}_{g_{s^\wedge i}} : \vartheta^{-1}(a) \cap \mathcal{N}_{v_{s^\wedge i}} \neq \emptyset\}.$$

This is a Σ^1_1 subset of A_s, containing b^i_s. To fulfill (iii), we define $A_{0^{n+1}}$ to be the set of all $a \in A'_{0^{n+1}}$ such that

$$\forall\, s' = s^\wedge i \in 2^{n+1}\ \exists\, b \in A'_{s'}\ (\Sigma^\infty_{k_{n+1}}(a,b) < 2^{-n-3})\ ;$$

this is still a Σ^1_1 set containing $b^0_{0^n} = a^0$ by (1). It remains to define, for any $s^\wedge i \neq 0^{n+1}$, $A_{s^\wedge i}$ to be the set of all $b \in A'_{s^\wedge i}$ such that

$$\exists\, b \in A_{0^{n+1}}\ (\Sigma^\infty_{k_{n+1}}(a,b) < 2^{-n-3})\ .$$

This ends the definition for the level $n+1$.

□ (*Construction and Theorem 49*)

14 Connections with Banach spaces

14.a Some classical Banach spaces

Recall the following definitions:

$$\begin{array}{lcll} \boldsymbol{\ell}^p & = & \{x \in \mathbb{R}^{\mathbb{N}} : \sum_n |x(n)|^p < +\infty\} & ; \\ \boldsymbol{\ell}^\infty & = & \{x \in \mathbb{R}^{\mathbb{N}} : \exists\, C\ \forall\, k\ (|x(k)| < C)\} & ; \\ \boldsymbol{c}_0 & = & \{x \in \mathbb{R}^{\mathbb{N}} : \mathtt{lim}_{n\to\infty}\, x(n) = 0\} & ; \end{array}$$

These symbols will also denote the induced ERs on $\mathbb{R}^{\mathbb{N}}$, for instance, $x\ \boldsymbol{\ell}^p\ y$ means that $x - y \in \boldsymbol{\ell}^p$, where $x - y$ is the coordinate-wise subtraction.

Lemma 50 (Oliver ?) $\mathsf{E}_{\mathcal{Z}_0} \sim_{\mathrm{B}} \boldsymbol{c}_0$.

Proof Prove direction $\boldsymbol{c}_0 \le_{\mathrm{B}} \mathsf{E}_{\mathcal{Z}_0}$. This consists of two parts. First of all let us define a Borel reduction $\mathbb{R}^{\mathbb{N}}/\ \boldsymbol{c}_0 \to [0,1]^{\mathbb{N}}/\ \boldsymbol{c}_0$. We have to define a Borel map $\xi : \mathbb{R}^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ such that $x\ \boldsymbol{c}_0\ y \iff \xi(x)\ \boldsymbol{c}_0\ \xi(y)$. Let $\pi : \mathbb{N} \times \mathbb{Z} \xrightarrow{\text{onto}} \mathbb{N}$ be any bijection. For $x \in \mathbb{R}^{\mathbb{N}}$, let $\xi(x) \in [0,1]^{\mathbb{N}}$ be defined as follows. Take any $n = \pi(k,a) \in \mathbb{N}$. If $a \le x(k) < a+1$ put $\xi(x)(n) = x(k) - a$. If $x(k) \ge a+1$ put $\xi(x)(n) = 1$. If $x(k) < a$ put $\xi(x)(n) = 0$. Easily ξ is as required.

It remains to define a Borel reduction $[0,1]^{\mathbb{N}}/\ \boldsymbol{c}_0 \to \mathcal{P}(\mathbb{N})/\mathcal{Z}_0$, *i.e.*, a Borel map $\vartheta : [0,1]^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ such that $x\ \boldsymbol{c}_0\ y \iff \vartheta(x)\,\Delta\,\vartheta(y) \in \mathcal{Z}_0$ for all $x, y \in \mathbb{R}^{\mathbb{N}}$. Let $x \in [0,1]^{\mathbb{N}}$. Then $0 \le x(n) \le 1$ for any n. The value of $x(n)$ will determine the intersection $\vartheta(x) \cap [2^n, 2^{n+1})$, namely,

$$2^n + j \in \vartheta(x) \quad \text{iff} \quad j\,2^{-n} < x(n)\ : \qquad \text{for all} \quad j < 2^n.$$

Then we have $|x(n) - \frac{\#(\vartheta(x)\cap[2^n,2^{n+1}))}{2^n}| < 2^{-n}$ for any n. It follows that $0.4 < \frac{\#(\vartheta(x)\cap[0,2^n))}{2^n x(n)} < 2.1$ for all $n \ge 4$. This easily implies that ϑ is as required.

Prove direction $\mathsf{E}_{\mathcal{Z}_0} \le_{\mathrm{B}} \boldsymbol{c}_0$. We have to define a Borel map $\vartheta : \mathcal{P}(\mathbb{N}) \to \mathbb{R}^{\mathbb{N}}$ such that $X\,\Delta\,Y \in \mathcal{Z}_0 \iff \vartheta(X)\ \boldsymbol{c}_0\ \vartheta(Y)$. Very simple ideas like $\vartheta(X)(n) = \frac{\#(X\cap[0,n))}{n}$ do not work, the right way is based on the following observation: for sets $s, t \subseteq [0,n)$ to satisfy $\#(s\,\Delta\,t) \le k$ it is necessary and sufficient that $|\#(s\,\Delta\,z) - \#(t\,\Delta\,z)| \le k$ for any $z \subseteq [0,n)$. To make use of this fact, let us fix an enumeration (with repetitions) $\{z_j\}_{j\in\mathbb{N}}$ of all finite subsets of $\mathbb{N}$ such that

$$\{z_j : 2^n \le j < 2^{n+1}\} \quad = \quad \text{all subsets of } [0,n)$$

for every n. Define, for any $X \in \mathcal{P}(\mathbb{N})$ and $2^n \le j < 2^{n+1}$, $\vartheta(X)(j) = \frac{\#(X\cap z_j)}{n}$. Then $\vartheta : \mathcal{P}(\mathbb{N}) \to [0,1]^{\mathbb{N}}$ is a required reduction. □

Lemma 51 (Who ?) $\mathsf{E}_{\mathcal{S}_{\{1/n\}}} \sim_{\mathrm{B}} \boldsymbol{\ell}^1$.

Proof To prove direction $\mathsf{E}_{\mathcal{S}_{\{1/n\}}} \le_{\mathrm{B}} \boldsymbol{\ell}^1$, simply define $\vartheta(X) \in \mathbb{R}^{\mathbb{N}}$ for any $X \in \mathcal{P}(\mathbb{N})$ as follows: $\vartheta(X)(n) = n^{-1}$ for any $n \ge 1$, $n \in X$, and $\vartheta(X)(n) = 0$ for any other n. Then $X \,\Delta\, Y \in \mathcal{S}_{\{1/n\}} \iff \vartheta(X)\ \boldsymbol{\ell}^1\ \vartheta(Y)$, as required.

Let us prove the other direction. The map ξ as in the proof of Lemma 50 reduces $\boldsymbol{\ell}^1$ to $\boldsymbol{\ell}^1 \restriction [0,1]^{\mathbb{N}}$. It remains to define a reduction $\boldsymbol{\ell}^1 \restriction [0,1]^{\mathbb{N}} \to \mathsf{E}_{\mathcal{S}_{\{1/n\}}}$, *i.e.*, a Borel map $\vartheta : T \to \mathcal{P}(\mathbb{N})$ such that $x\ \boldsymbol{\ell}^1\ y \iff \vartheta(x) \,\Delta\, \vartheta(y) \in \mathcal{S}_{\{1/n\}}$. We can associate, with any pair of n and $k < 2^n$, a set $s_{nk} \subseteq \mathbb{N}$ so that all sets s_{nk} are pairwise disjoint and $\sum_{j \in s_{nk}} j^{-1} = 2^{-n}$. Now define $\vartheta(x) = \bigcup_n \bigcup_{k < x(n)} s_{nk}$ for $x \in T$. □

14.b Mutual reducibility of $\boldsymbol{\ell}^p$

Theorem 52 (Dougherty and Hjorth [3]) *If* $1 \le p < q < \infty$ *then* $\boldsymbol{\ell}^p \le_{\mathrm{B}} \boldsymbol{\ell}^q$.

Proof To simplify the task note that, as above, $\boldsymbol{\ell}^p \le_{\mathrm{B}} \boldsymbol{\ell}^p \restriction [0,1]^{\mathbb{N}}$, so that it suffices to prove that $\boldsymbol{\ell}^p \restriction [0,1]^{\mathbb{N}} \le_{\mathrm{B}} \boldsymbol{\ell}^q$. We shall assume that $q < 2p$: this does not reduce the generality because any bigger q cal be approached in several steps. For $\vec{x} = \langle x, y \rangle \in \mathbb{R}^2$, let $||\vec{x}||_h = (x^h + y^h)^{1/h}$. The proof is based on

Lemma 53 *For any* $\frac{1}{2} < \alpha < 1$ *there is a continuous map* $K_\alpha : [0,1] \to [0,1]^2$ *and positive real numbers* $m < M$ *such that, for all* $x, y \in [0,1]$, *we have* $m(x-y)^\alpha \le ||K_r(x) - K_r(y)||_2 \le M(x-y)^\alpha$.

Taking the lemma on trust, let us proceed with the theorem. Let $\alpha = p/q$, and let K_α be as guaranteed by the lemma, too. Let $x = \langle x_0, x_1, x_2, \dots \rangle \in [0,1]^{\mathbb{N}}$. Then $K_\alpha(x_i) = \langle x_i', x_i'' \rangle \in [0,1]^2$. We put $\vartheta(x) = \langle x_0', x_0'', x_1', x_1'', x_2', x_2'', \dots \rangle$. Prove that ϑ reduces $\boldsymbol{\ell}^p \restriction [0,1]^{\mathbb{N}}$ to $\boldsymbol{\ell}^q$.

Let $x = \{x_i\}_{i \in \mathbb{N}}$ and $y = \{y_i\}_{i \in \mathbb{N}}$ belong to $[0,1]^{\mathbb{N}}$; we have to prove that $x\ \boldsymbol{\ell}^p\ y$ iff $\vartheta(x)\ \boldsymbol{\ell}^q\ \vartheta(y)$. To simplify the picture note the following:

$$2^{-1/2}||w||_2 \le \mathtt{max}\{w', w''\} \le ||w||_q \le ||w||_1 \le 2||w||_2$$

for any $w = \langle w', w'' \rangle \in \mathbb{R}^2$. The task takes the following form:

$$\sum_i (x_i - y_i)^p < \infty \iff \sum_i ||K_\alpha(x_i) - K_\alpha(y_i)||_2^q < \infty .$$

Furthermore, by the choice of K_α, this converts to

$$\sum_i (x_i - y_i)^p < \infty \iff \sum_i (x_i - y_i)^{\alpha q} < \infty ,$$

which holds because $\alpha q = p$. □ (*Theorem 52 modulo Lemma 53*)

Proof of the lemma. Let $r = 4^{-\alpha}$, so that $\frac{1}{4} < r < \frac{1}{2}$ and $\alpha = -\log_4 r$. (Dougherty–Hjorth construction follows.) □

14.c Mutual irreducibility of $\boldsymbol{\ell}^p$

Theorem 54 (Dougherty and Hjorth [3]) *If* $1 \le p < q < \infty$ *then* $\boldsymbol{\ell}^q \not\le_{\mathrm{B}} \boldsymbol{\ell}^p$.

Proof It suffices to prove that $\boldsymbol{\ell}^q \restriction [0,1]^{\mathbb{N}} \not\le_{\mathrm{B}} \boldsymbol{\ell}^p$. Now let B_i be the (finite) set of multiples of 2^{-i} that belong to $[0,1]$, and B_i^* be the set of all integer multiples of 2^{-i}, positive and negative. Define

$$B = \prod_{i \in \mathbb{N}} B_i = \{x = \{x_i\}_{i \in \mathbb{N}} \in [0,1]^{\mathbb{N}} : \forall\, i\ (x_i \in B_i)\} ,$$

and similarly $B^* = \prod_{i \in \mathbb{N}} B_i^*$. Note that $\boldsymbol{\ell}^p \sim_{\mathrm{B}} \boldsymbol{\ell}^p \restriction B^*$: the direction $\le_{\mathrm{B}}$ follows by the identity embedding, the other direction holds by the approximation of real terms x_i by multiples of 2^{-i}. Similarly, $\boldsymbol{\ell}^q \restriction [0,1]^{\mathbb{N}} \sim_{\mathrm{B}} \boldsymbol{\ell}^q \restriction B$.

Exercise 55 (Not used in the proof of the theorem.) Prove, using the same transformation as in the beginning of the proof of Lemma 50, that moreover $\boldsymbol{\ell}^p \sim_{\mathrm{B}} \boldsymbol{\ell}^p \restriction B$. Here this is not so transparent. □

Thus it suffices to show that $\boldsymbol{\ell}^q \restriction B \not\le_{\mathrm{B}} \boldsymbol{\ell}^p \restriction B^*$. Suppose, on the contrary, that $\boldsymbol{\ell}^q \restriction B \le_{\mathrm{B}} \boldsymbol{\ell}^p \restriction B^*$. First we prove that there is a reduction of a special sort.

Let, for any $I \subseteq \mathbb{N}$, $B \restriction I$ be the set of all maps x defined on I so that $x(i) \in B_i$ for any $i \in I$: for instance, $B = B \restriction \mathbb{N}$. Define $B^* \restriction I$ accordingly. Suppose that $0 = k_0 < k_1 < k_2 < \dots$ and, for any i, f_i is a map $B_i \to B^* \restriction [k_i, k_{i+1})$. Define then, for each $x \in B$, $\rho_{\{f_i\}}(x) = \bigcup_{i \in \mathbb{N}} f_i(x_i) \in B^*$, so that $y \restriction [k_i, k_{i+1}) = f_i(x_i)$ for all i.

Lemma 56 (If $\boldsymbol{\ell}^q \restriction B \le_{\mathrm{B}} \boldsymbol{\ell}^p \restriction B^*$ then) *there is a reduction of the form* $\rho_{\{f_i\}}$.

Proof of the lemma. Let $\vartheta : B \to B$ be any Borel reduction $\boldsymbol{\ell}^q \restriction B$ to $\boldsymbol{\ell}^p \restriction B^*$. Let $\vartheta_i(x) = (\rho(x))_i$, so that $\vartheta(x) = \{\rho_i(x)\}_{i \in \mathbb{N}}$. Arguing as in the proof of Lemma 45, we obtain a sequence $0 = k_0 = a_0 < b_0 < k_1 < a_1 < b_1 < k_2 < \dots$ of natural numbers and, for any i, a map $s_i \in B \restriction [b_i, a_{i+1})$, such that

(i) if $x \in B$ is "generic" then $\sum_{j \ge k_{i+1}} |\vartheta_j(u \cup s_i \cup x') - \vartheta_j(v \cup s_i \cup x')|^p < 2^{-i}$ for all $u, v \in B \restriction [0, b_i)$, where $x' = x \restriction [a_{i+1}, \infty)$;

(ii) if $u \in B \restriction [0, b_i)$ then $u \cup s_i$ "forces" the value of $\vartheta(x) \restriction [0, k_{i+1})$ in the sense that there is $g_i(u) \in B^* \restriction [0, k_{i+1})$ such that $\vartheta(x) \restriction [0, k_{i+1}) = g_i(u)$ whenever $x \in B$ is "generic" with $x \restriction [0, b_i) = u$ and $x \restriction [b_i, a_{i+1}) = s_i$;

(iii) any $x \in B$ with $x \restriction [b_i, a_{i+1}) = s_i$ for infinitely many i is "generic".

Now define $u_i(r) \in B \restriction [a_i, b_i)$ for any $r \in B_i$ as follows: $u_i(r)_{a_i} = r$ while $u_i(r)_j = 0$ for all $a_i < j < b_i$. In particular, put $\mathbf{0}_i = u_i(0)$ (the constant-0 member of $B \restriction [a_i, b_i)$). Define, for each $u \in B \restriction [a_i, b_i)$,

$$\overline{u} = \mathbf{0}_0 \cup s_0 \cup \mathbf{0}_1 \cup s_i \cup \dots \cup \mathbf{0}_{i-1} \cup s_{i-1} \cup u \cup s_i \quad \text{(belongs to } B \restriction [0, b_{i+1})) .$$

Put $h_i(u) = g_i(\overline{u}) \restriction [k_i, k_{i+1})$, then $h_i(u) \in B^* \restriction [k_i, k_{i+1})$. Finally let

$$\eta(x) = h_0(x \restriction [a_0, b_0)) \cup h_1(x \restriction [a_1, b_1)) \cup h_2(x \restriction [a_2, b_2)) \cup \ldots$$

for any $x \in B$ with $x \restriction [b_i, a_{i+1}) = s_i$ for all i, so that $\eta(x) \in B^*$. Then $||\eta(x) - \vartheta(x)||_p < +\infty$ for each $x \in B$ with $x \restriction [b_i, a_{i+1}) = s_i$ for all i. (Indeed, it suffices to show that $\sum_{k_i \le j < k_{i+1}} |\eta_j(x) - \vartheta_j(x)|^p < 2^{-i}$ for all i, in other words, by (ii), that $\sum_{k_i \le j < k_{i+1}} |g_i(\overline{u})_j - g_i(v)_j|^p < 2^{-i}$, where $v = x \restriction [0, b_i)$, $u = x \restriction [a_i, b_i)$, and $\overline{u}$ is defined as above. But this easily follows from (i).)

Let, for any $z = \{z_i\}_{i \in \mathbb{N}} \in B$, $\varphi(z) \in B$ be defined so that [13] we have $\varphi(z) \restriction [b_i, a_{i+1}) = s_i$ and $\varphi(z) \restriction [a_i, b_i) = u_i(z_i)$ for all i. Then clearly φ is a Borel reduction $\boldsymbol{\ell}^q \restriction B$ to itself, hence, by the above, $\rho(z) = \eta(\varphi(z))$ still reduces $\boldsymbol{\ell}^q \restriction B$ to $\boldsymbol{\ell}^p \restriction B^*$. On the other hand, we obviously have $\rho = \rho_{\{f_i\}}$, where $f_i(r) = h_i(u_i(r))$ for $r \in B_i$. □ (*Lemma*)

Coming back to the theorem, let $\rho = \rho_{\{f_i\}} : B \to B^*$ be a reduction given by the lemma. Let, as usual, $||\sigma - \tau||_p = \sum_{k_i \le k < k_{i+1}} |\sigma_k - \tau_k|^p$ for any pair of sequences $\sigma, \tau \in B^* \restriction [k_i, k_{i+1})$.

Lemma 57 *There are $c > 0$ and a number N such that $||f_i(1) - f_i(0)||_p \ge c$ for all $i \ge N$.*

Proof of the lemma. Otherwise we have an increasing sequence $\{i_m\} \to \infty$ with $||f_{i_m}(1) - f_{i_m}(0)||_p \le 2^{-m}$ for all m. Let now $x \in B$ be the constant 0 while $y \in B$ be defined by $y_{i_m} = 1$ for all m and $y_i = 0$ for all other i. Then $x \;\boldsymbol{\ell}^q\; y$ fails but $\rho(x) \;\boldsymbol{\ell}^p\; \rho(y)$ holds, contradiction. □ (*Lemma*)

We can assume now that $||f_i(1) - f_i(0)||_p \ge c$ for all natural i.

Coming back to the proof of the theorem, we observe that, as $p < q$, there is a non-decreasing sequence of natural numbers $\nu(i) \le i$, $i = 0, 1, 2, \ldots$, such that $\sum_i 2^{-p\,\nu(i)}$ diverges but $\sum_i 2^{-q\,\nu(i)}$ converges. (*Hint*. For any n, the number of indices i with $\nu(i) = n$ must be approximately equal to 2^{pn}.)

Now consider any i. As $||f_i(1) - f_i(0)||_p \ge c$ and because $||\ldots||_p$ is a norm, there exists a pair of reals $z_i < z_i'$ in B_i with $z_i' - z_i = 2^{-\nu(i)}$ and $||f_i(z_i') - f_i(z_i)||_p \ge c\,2^{-\nu(i)}$. (For instance z_i and z_i' may be adjacent multiples of $2^{-\nu(i)}$ in B_i.) Then the $\boldsymbol{\ell}^q$-distance between $z = \{z_i\}_{i \in \mathbb{N}}$ and $z' = \{z_i'\}_{i \in \mathbb{N}}$ is $\sum_i 2^{-q\,\nu(i)} < +\infty$, while the $\boldsymbol{\ell}^p$-distance between $\rho(z)$ and $\rho(z')$ is

$$\textstyle\sum_i ||f_i(z_i') - f_i(z_i)||_p^p \;\ge\; \sum_i c^p \cdot 2^{-p\,\nu(i)} \;=\; +\infty\,,$$

which contradicts the assumption that ρ is a reduction. □ (*Theorem 54*)

[13] φ belongs to the innumerable class of useful auxiliary embeddings "into own generic part". This step was implicit in the proof of Lemma 45.

15 c_0-equalities

Suppose that $\langle X_k\,; d_k\rangle$ is a finite metric space for each $k \in \mathbb{N}$. Farah [8] defines an equivalence relation $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$ on $\mathbb{X} = \prod_{k \in \mathbb{N}} X_k$ as follows:

$$x \,\mathsf{D}\, y \quad \text{iff} \quad \lim_{k \to \infty} d_k(x_k, y_k) = 0\,.$$

ERs of this form are called c_0-*equalities*. In addition, $\mathsf{D}(\langle X_k\,; d_k\rangle)$ is *nontrivial* if $\limsup_{k\to\infty} \mathtt{diam}(X_k) > 0$ (otherwise $\mathsf{D}(\langle X_k\,; d_k\rangle)$ makes everything equivalent). Every c_0-equality is easily a Borel ER, more exactly, of class $\boldsymbol{\Pi}^0_3$.

15.a Some examples and simple results

Example 58 (1) Let $X_k = \{0,1\}$ with $d_k(0,1) = 1$ for all k. Then clearly the relation $\mathsf{D}(\langle X_k\,; d_k\rangle)$ on $2^{\mathbb{N}} = \prod_k \{0,1\}$ is just E_0.

(2) Let $X_{kl} = \{0,1\}$ with $d_{kl}(0,1) = k^{-1}$ for all $k, l \in \mathbb{N}$. Then the relation $\mathsf{D}(\langle X_{kl}\,; d_{kl}\rangle)$ on $2^{\mathbb{N}\times\mathbb{N}} = \prod_{k,l}\{0,1\}$ is just $\mathsf{E}_3 = \mathsf{E}_{0\times\mathtt{Fin}}$.

(3) Generally, if $0 = n_0 < n_1 < n_2 < \ldots$ and φ_i is a submeasure on $[n_i, n_{i+1})$, then let $X_i = \mathcal{P}([n_i, n_{i+1}))$ and $d_i(u,v) = \varphi_i(u \,\Delta\, v)$ for $u, v \subseteq [n_i, n_{i+1})$. Then $\mathsf{D}(\langle X_i\,; d_i\rangle)$ is clearly isomorphic to $\mathsf{E}_{\mathcal{I}}$, where

$$\mathcal{I} = \mathtt{Exh}(\varphi) = \{x \subseteq \mathbb{N} : \lim_{n\to\infty} \varphi(x \cap [n, \infty)) = 0\}$$

and $\varphi(x) = \mathtt{sup}_i\, \varphi_i(x \cap [n_i, n_{i+1}))$.

(4) Let $\mathsf{D}_{\mathtt{max}} = \mathsf{D}(\langle X_k\,; d_k\rangle)$, where $X_k = \{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$ and d_k is the distance on X_k inherited from $\mathbb{R}$. □

Proposition 59 (Farah [8] with a reference to Oliver) *$\mathsf{D}_{\mathtt{max}} \sim_{\mathrm{B}} \mathsf{E}_{\mathcal{Z}_0}$. If D is a c_0-equality then $\mathsf{D} \le_{\mathrm{B}} \mathsf{D}_{\mathtt{max}}$, moreover, $\mathsf{D} \le_{\mathrm{A}} \mathsf{D}_{\mathtt{max}}$.*

Thus $\mathsf{D}_{\mathtt{max}}$ is a maximal, in a sense, among c_0-equalities.

Proof Is there a half-page proof ? □

The structure of c_0-equalities tend to be connected more with the additive reducibility $\le_{\mathrm{A}}$ (see § 2.e on $\le_{\mathrm{A}}$ and the associated relations $<_{\mathrm{A}}$ and $\sim_{\mathrm{A}}$) than with the general Borel reducibility. In particular, we have

Lemma 60 *For any c_0-equality $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$, if D' is a Borel ER on a set $\prod_k X_k'$ (with finite nonempty X_k') and $\mathsf{D}' \le_{\mathrm{A}} \mathsf{D}$ then D' is a c_0-equality.*

Proof Let a sequence $0 = n_0 < n_1 < n_2 < \ldots$ and a collection of maps $H_i : X_i' \to \prod_{n_i \le k < n_{i+1}} X_k$ witness $\mathsf{D}' \le_{\mathrm{A}} \mathsf{D}$. For $x', y' \in X_i'$ put

$$d_i'(x', y') = \max_{n_i \le k < n_{i+1}} d_k(H_i(x')_k, H_i(y')_k)\,.$$

Then easily $\mathsf{D}' = \mathsf{D}(\langle X_k'\,; d_k'\rangle)$. □

Lemma 61 (Farah [8] with a reference to Hjorth) *Every c_0-equality $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$ is induced by a continuous action of a Polish group.*

(The domain $\mathbb{X} = \prod_k X_k$ of D is considered with the product topology.)

Proof (sketch) For any k let S_k be the (finite) group of all permutations of X_k, with the distance $\rho_k(s,t) = \max_{x\in X_k} d_k(s(x),t(x))$. Then

$$\mathbb{G} = \{g \in \textstyle\prod_k S_k : \lim\limits_{k\to\infty} \rho_k(g_k, e_k) = 0\}, \quad \text{where } e_k \in S_k \text{ is the identity},$$

is easily a subgroup of $\prod_k S_k$, moreover, the distance $d(g,h) = \sup_k \rho_k(g_k,h_k)$ converts $\mathbb{G}$ into a Polish group, the natural action of which on $\mathbb{X}$ (*i.e.*, $(g\cdot x)_k = g_k(x_k)$, $\forall\, k$) is continuous and induces D. $\square$

15.b Classification

Recall that for a metric space $\langle A; d\rangle$, a rational $q > 0$, and $a \in A$, $\mathtt{Gal}^q_A(a)$ is the set of all $b \in A$ which can be connected with a by a finite chain $a = a_0, a_1, ..., a_n = b$ with $d(a_i, a_{i+1}) < q$ for all i. Farah defines, for $r > 0$,

$$\delta(r,A) = \inf\{q \in \mathbb{Q}^+ : \exists\, a \in A\ (\mathtt{diam}(\mathtt{Gal}^q_A(a)) \ge r)\}$$

(with the understanding that here $\inf\emptyset = +\infty$), and

$$\Delta(A) = \{d(a,b) : a \ne b \in A\}, \quad \text{so that} \quad \mathtt{diam}\, A = \sup(\Delta(A) \cup \{0\}).$$

Now let $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$ be a c_0-equality on $\mathbb{X} = \prod_{k\in\mathbb{N}} X_k$. The basic properties of D are determined by the following two conditions:

(co1) $\mathtt{liminf}_{k\to\infty}\, \delta(r, X_k) = 0$ for some $r > 0$.

(co2) $\forall\, \varepsilon > 0\ \exists\, \varepsilon' \in (0,\varepsilon)\ \exists^\infty k\ (\Delta(X_k) \cap [\varepsilon', \varepsilon) \ne \emptyset)$.

Easily (co1) implies both the nontriviality of $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$ and (co2).

Theorem 62 (Farah [8]) *Let $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$ be a nontrivial c_0-equality. Then*

(i) *If* (co2), *hence,* (co1) *fail then $\mathsf{D} \sim_{\rm A} \mathsf{E}_0$, hence, $\mathsf{D} \sim_{\rm B} \mathsf{E}_0$;*

(ii) *If* (co1) *fails but* (co2) *holds then $\mathsf{D} \sim_{\rm A} \mathsf{E}_3$, hence, $\mathsf{D} \sim_{\rm B} \mathsf{E}_3$;*

(iii) *If* (co1), *hence,* (co2) *hold then $\mathsf{E}_0 <_{\rm A} \mathsf{D}$ and $\mathsf{D}_1 \le_{\rm A} \mathsf{D}$ for a turbulent c_0-equality D_1 satisfying $\mathsf{E}_3 \le_{\rm A} \mathsf{D}_1$.*

Comment upon turbulent.

Proof (i) To show that $\mathsf{E}_0 \le_{\rm A} \mathsf{D}$ note that, by the nontriviality of D, there exist: a number $p > 0$, an increasing sequence $0 = n_0 < n_1 < n_2 < ...$, and, for any i, a pair of points $x_{n_i}, y_{n_i} \in X_{n_i}$ with $d_{n_i}(x_{n_i}, y_{n_i}) \ge p$. For n not of the form n_i fix an arbitrary $x_n \in X_n$. Now, if $a \in 2^{\mathbb{N}}$, then define $\vartheta(a) \in \prod_k X_k$ so that $\vartheta(a)_n = z_n$ for n not of the form n_i, while $\vartheta(a)_{n_i} = x_{n_i}$ or $= y_{n_i}$ if resp. $a_i = 0$ or $= 1$. This map ϑ witnesses $\mathsf{E}_0 \le_{\rm A} \mathsf{D}$.

Now prove that $\mathsf{D} \le_{\rm A} \mathsf{E}_0$. As (co2) fails, there is $\varepsilon > 0$ such that for each ε' with $0 < \varepsilon' < \varepsilon$ we have only finitely many k with the propery that $\varepsilon' \le d_k(\xi,\eta) < \varepsilon$ for some $\xi, \eta \in X_k$. Let G_k be the (finite) set of all $\frac{\varepsilon}{2}$-galaxies in X_k, and let $\vartheta : \mathbb{X} = \prod_k X_k \to G = \prod_k G_k$ be defined as follows: $\vartheta(x)_k$ is that galaxy in G_k to which x_k belongs. Let E be the G-version of E_0, *i.e.*, if $g, h \in G$ then $g \mathrel{\mathsf{E}} h$ iff $g_k = h_k$ for all but finite k. As easily $\mathsf{E} \le_{\rm A} \mathsf{E}_0$, it suffices to demonstrate that $\mathsf{D} \le_{\rm A} \mathsf{E}$ via ϑ. Suppose that $x, y \in \mathbb{X}$ and $\vartheta(x) \mathrel{\mathsf{E}} \vartheta(y)$ and prove $x \mathrel{\mathsf{D}} y$ (the nontrivial direction). Let, on the contrary, $x \mathrel{\not\!\mathsf{D}} y$, so that there is a number $p > 0$ with $d_k(x_k, y_k) > p$ for infinitely many k. We can assume that $p < \frac{\varepsilon}{2}$. On the other hand, as $\vartheta(x) \mathrel{\mathsf{E}} \vartheta(y)$, there is k_0 such that x_k and y_k belong to one and the same $\frac{\varepsilon}{2}$-galaxy in X_k for all $k > k_0$. Then, for any $k > k_0$ with $d_k(x_k, y_k) > p$ (*i.e.*, for infinitely many values of k) there exists an element $z_k \in X_k$ in the same galaxy such that $p < d_k(x_k, z_k) < \varepsilon$, but this is a contradiction to the choice of ε (indeed, take $\varepsilon' = p$).

(ii) Let us show first that if (co2) holds then $\mathsf{E}_3 \le_{\rm A} \mathsf{D}$ (independently of (co1)). It follows from (co2) that there exist: an infinite sequence $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > ... > 0$, for any i an infinite set J_i, and for any $j \in J_i$ a pair of elements $x_{ij}, y_{ij} \in X_j$ with $d_j(x_{ij}, y_{ij}) \in [\varepsilon_{i+1}, \varepsilon_i)$. We may assume that the sets J_i are pairwise disjoint. Then the c_0-equality $\mathsf{D}' = \mathsf{D}(\langle\{x_{ij}, y_{ij}\}\,; d_j\rangle_{i\in\mathbb{N},\, j\in J_i})$ satisfies both $\mathsf{D}' \le_{\rm A} \mathsf{D}$ and $\mathsf{D}' \cong \mathsf{E}_3$ (via a bijection between the underlying sets).

Now, assuming that, in addition, (co1) fails, we show that $\mathsf{D} \le_{\rm A} \mathsf{E}_3$. For all $k, n \in \mathbb{N}$ let G_{kn} be the (finite) set of all $\frac{1}{n}$-galaxies in X_k. For any $x \in \mathbb{X} = \prod_i X_i$ define $\vartheta(x) \in G = \prod_{k,n} G_{kn}$ so that $\vartheta(x)_{kn}$ is that $\frac{1}{n}$-galaxy in G_{kn} to which x_k belongs (for all k, n). The ER E on G, defined so that $g \mathrel{\mathsf{E}} h$ iff $\forall\, n\ \forall^\infty k\ (g_{kn} = h_{kn})$ ($g, h \in G$) is easily $\le_{\rm A} \mathsf{E}_3$, so it suffices to show that $\mathsf{D} \le_{\rm A} \mathsf{E}$ via ϑ. Suppose that $x, y \in \mathbb{X}$ and $\vartheta(x) \mathrel{\mathsf{E}} \vartheta(y)$ and prove $x \mathrel{\mathsf{D}} y$ (the nontrivial direction). Otherwise there is some $r > 0$ with $d_k(x_k, y_k) > r$ for infinitely many k. As (co1) fails for this r, there is n big enough for $\delta(r, X_k) > \frac{1}{n}$ to hold for almost all k. Then, by the choice of r, we have $\vartheta(x)_{kn} \ne \vartheta(y)_{kn}$ for infinitely many k, hence, $\vartheta(x) \mathrel{\not\!\mathsf{E}} \vartheta(y)$, contradiction.

(iii) Fix $r > 0$ with $\mathtt{liminf}_{k\to\infty}\, \delta(r, X_k) = 0$. As for any increasing sequence $n_0 < n_1 < n_2 < ...$ we have $\mathsf{D}(\langle X_{n_i}\,; d_{n_i}\rangle) \le_{\rm A} \mathsf{D}$, it can be assumed that $\mathtt{lim}_k\, \delta(r, X_k) = 0$, and further that $\delta(r, X_k) < \frac{1}{k}$ for all k. Then every X_k contains a $\frac{1}{k}$-galaxy $Y_k \subseteq X_k$ of $\mathtt{diam}\, Y_k \ge r$. As easily $\mathsf{D}(\langle Y_k\,; d_k\rangle) \le_{\rm A} \mathsf{D}$, the following lemma suffices to prove (iii).

Lemma 62.1 *Suppose that $r > 0$ and each X_k is a single $\frac{1}{k}$-galaxy in itself with $\mathtt{diam}(X_k) \ge r$. Then $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$ is turbulent and $\mathsf{E}_3 \le_{\rm A} \mathsf{D}$.*

Proof We know from the proof of (iii) above that $\mathsf{E}_3 \le_{\rm A} \mathsf{D}$. Now prove that the natural action of the Polish group $\mathbb{G}$ defined as in the proof of Lemma 61 is turbulent under the assumptions of the lemma.

That every D-class is dense in $\mathbb{X} = \prod_k X_k$ (with the product topology on $\mathbb{X}$) is an easy exercise. To see that every D-class $[x]_{\mathsf{D}}$ also is meager in $\mathbb{X}$, note that by the assumptions of the lemma any X_k contains a pair of elements x'_k, x''_k with $d_k(x'_k, x''_k) \ge r$. Let y_k be one of x'_k, x''_k which is d_k-fahrer than $\frac{r}{2}$ from x_k. Now the set $Z = \{z \in \mathbb{X} : \exists^\infty k\ (z_k = y_k)\}$ is comeager in $\mathbb{X}$ and disjoint from $[x]_{\mathsf{D}}$. It remains to prove that local orbits are somewhere dense.

Let G be an open nbhd of the identity in $\mathbb{G}$ and $\emptyset \ne X \subseteq \mathbb{X}$ be open in $\mathbb{X}$. We can assume that, for some n, G is the $\frac{1}{n}$-ball around the identity in $\mathbb{G}$ while $X = \{x \in \mathbb{X} : \forall k < n\ (x_k = \xi_k)\}$, where elements $\xi_k \in X_k$, $k < n$, are fixed. It is enough to prove that all classes of the local orbit relation $\sim^G_X$ are dense in X. Consider an open set $Y = \{y \in \mathbb{X} : \forall k < m\ (y_k = \xi_k)\} \subseteq X$, where $m > n$ and elements $\xi_k \in X_k$, $n \le k < m$, are fixed in addition to the above.

Let $x \in X$. Then $x_k = \xi_k$ for $k < n$. Let $n \le k < m$. The elements ξ_k and x_k belong to X_k, which is a $\frac{1}{k}$-galaxy, therefore, there is a chain, of a length $\ell(k)$, of elements of X_k, which connects x_k and ξ_k so that every step within the chain has d_k-length $< \frac{1}{k}$. Then there is a permutation g_k of X_k such that $g_k^{\ell(k)}(x_k) = \xi_k$, $g_k(\xi_k) = x_k$, and $d_k(\xi, g_k(\xi)) < \frac{1}{k}$ for all $\xi \in X_k$. Let g_k be the identity on X_k whenever $k < n$ or $k \ge m$. This defines an element $g \in \mathbb{G}$ which obviously belongs to G, moreover, X is g-invariant and $g^\ell(x) \in U$, where $\ell = \prod_{n \le k < m} \ell(k)$, hence, $x \sim^G_X g(x)$, as required. □ *(Lemma)*

□ *(Theorem 62)*

Remark 63 Theorem 62 shows that any nontrivial c_0-equality D $\le_{\rm A}$-contains a turbulent c_0-equality D' with $\mathsf{E}_3 \le_{\rm A} \mathsf{D}'$ (and the turbulence of D' holds, in particular, via the natural action defined in the proof of Lemma 61), unless D is $\sim_{\rm A}$ to E_0 or E_3, and that (co1) is necessary for the turbulence of D itself and sufficient for a turbulent c_0-equality $\mathsf{D}' \le_{\rm A} \mathsf{D}$ to exist. □

15.c LV-equalities

By Farah, an LV-*equality* is a c_0-equality $\mathsf{D} = \mathsf{D}(\langle X_k\,;d_k\rangle)$ satisfying

$$\text{(LV1)}\quad \forall m\,\forall\varepsilon > 0\,\forall^\infty k\,\forall x_0,...,x_m \in X_k\ \big(d_k(x_0,x_m) \le \mathtt{max}_{j<m}\, d_k(x_j,x_{j+1}) + \varepsilon\big)\,.$$

In other words, the metrics involved are postulated to be "asymptotically close" to ultrametrics. This sort of c_0-equalities was first considered by Louveau and Velickovic [33]. The following simple fact is analogous to Lemma 60.

Lemma 64 *For any* LV-*equality* D, *if* D' *is a Borel ER on a set* $\prod_k X'_k$ *(with finite nonempty* X'_k*) and* $\mathsf{D}' \le_{\rm A} \mathsf{D}$ *then* D' *is an* LV-*equality.* □

Example 65 (Louveau and Velickovic [33]) We define $X_k = \{1,2,...,2^k\}$ and $d_k(m,n) = \log(|m-n|+1)/k$ for $1 \le m,n \le 2^k$. □

Theorem 66 (Essentially, Louveau and Velickovic [33]) *Let* $\mathsf{D} = \mathsf{D}(\langle X_k\,;d_k\rangle)$ *be a turbulent* LV-*equality. Then we can associate, with each infinite* $A \subseteq \mathbb{N}$, *a* LV-*equality* $\mathsf{D}_A \le_{\rm A} \mathsf{D}$ *such that for all* $A, B \subseteq \mathbb{N}$ *the following are equivalent:*

(i) $A \subseteq^* B$ (*i.e.*, $A \setminus B$ is finite);

(ii) $\mathsf{D}_A \le_{\rm A} \mathsf{D}_B$;

(iii) $\mathsf{D}_A \le_{\rm BM} \mathsf{D}_B$ (*i.e.*, via a Baire measurable reduction).

This theorem was the first major application of c_0-equalities. One of its corollaries is that there exist big families of mutually irreducible Borel ERs !

Proof As D is turbulent, the necessary turbulence condition (co1) of §15.b holds, moreover, as in the proof of Theorem62 (case (iii)), we can assume that it takes the following special form for some $r > 0$:

(1) Each X_k is a single $\mathtt{min}\{\frac{r}{2}, \frac{1}{k+1}\}$-galaxy of $\mathtt{diam}(X_k) \ge 4r$.

The intended transformations (reduction to a certain infinite subsequence of spaces $\langle X_k\,;d_k\rangle$, and then each X_k to a suitable galaxy $Y_k \subseteq X_k$) preserve (LV1), of course, moreover, going to subsequences once again, we can assume that (LV1) holds in the following special form:

(2) $d_k(x_0, x_{m_k}) \le \mathtt{max}_{i<m_k}\, d_k(x_i,x_{i+1}) + \frac{1}{k+1}$ whenever $x_0,...,x_{m_k} \in X_k$, where $m_k = 2^{\prod_{j=0}^{k-1} \#(X_j)}$.

We can derive the following important consequence:

(3) For any k there is a set $Y_k \subseteq X_k$ of $\#(Y_k) = m_k$ such that we have $d_k(x,y) \ge r$ for all $x \ne y$ in Y_k .

To prove this note that by (1) there is a set $\{x_0,...,x_m\} \subseteq X_k$ such that $d_k(x_0,x_m) \ge 4r$ but $d_k(x_i,x_{i+1}) < r$ for all i. We may assume that m is the least possible length of such a sequence $\{x_i\}$. Now let us define a subsequence $\{y_0,y_1,...,y_n\}$ of $\{x_i\}$, the number $n \le m$ will be specified in the course of the construction. Put $y_0 = x_0$. If $y_j = x_{i(j)}$ has been defined, and there is $l > i(j)$, $l \le m$, such that $d_k(y_j,x_l) \ge r$, then let $y_{j+1} = x_l$ for the least such l, otherwise put $n = j$ and stop the construction.

By definition $d_k(y_j,y_{j+1}) \ge r$ for all $j < n$, moreover, $d_k(y_{j'},y_{j+1}) \ge r$ for any $j' < j$ by the minimality of m. Thus $Y_k = \{y_j : j \le n\}$ satisfies $d_k(x,y) \ge r$ for all $x \ne y$ in Y_k. It remains to prove that $n \ge m_k$. Indeed we have $d_k(y_j,y_{j+1}) < 2r$ by the construction, hence, if $n \le m_k$ then we would have $d_k(y_0,y_n) \le 3r$ by (2), which implies $d_k(y_n,x_m) \ge r$, a contradiction to the assumption that the construction stops with y_n ,

This said, we proceed to the proof of the theorem. First note that

Lemma 66.1 (iii) *implies that* (ii) *holds at least for some (infinite)* $A' \subseteq A$.

Is it true that for a pair of c_0-*equalities* D, D', *if* $\mathsf{D} \le_{\mathrm{B}} \mathsf{D}'$ *then* $\mathsf{D} \le_{\mathrm{A}} \mathsf{D}'$ *?*

Proof A Borel reduction can be extracted from a Baire measurable one by a version of the "stabilizers" construction (see proofs of ... .) $\square$ (*Lemma 66.1*)

Thus it remains only to show that (ii) implies (i), even simpler, that, for any disjoint infinite sets $A,\ B \subseteq \mathbb{N}$, $\mathsf{D}_A \le_{\mathrm{A}} \mathsf{D}_B$ fails. Suppose, towards the contrary, that $\mathsf{D}_A \le_{\mathrm{A}} \mathsf{D}_B$ holds, and let this be witnessed by a reduction Ψ defined (as in § 2.e) from an increasing sequence $\mathtt{min}\, B = n_0 < n_1 < n_2 < \dots$ of numbers $n_i \in B$ and a collection of maps $H_k : X_k \to \prod_{j \in [n_i, n_{i+1}) \cap B} X_j$, $k \in A$. Let

$$f_k(\delta) = \max_{\xi,\eta \in X_k,\ d_k(\xi,\eta) < \delta} \ \max_{j \in [n_i, n_{i+1}) \cap B} d_j(H_k(\xi)_j, H_k(\eta)_j),$$

for $k \in \mathbb{N}$ and $\delta > 0$ (with the understanding that $\mathtt{max}\, \emptyset = 0$ if applicable). Then $f(\delta) = \mathtt{sup}_{k \in A} f_k(\delta)$ is a nondecreasing map $\mathbb{R}^+ \to [0, \infty)$.

Lemma 66.2 $\mathtt{lim}_{\delta \to 0} f(\delta) = 0$.

Proof Otherwise there is $\varepsilon > 0$ such that $f(\delta) \ge \varepsilon$ for all δ. Then the numbers

$$\mu_k = \mathtt{min}_{\xi,\eta \in X_k,\ \xi \ne \eta}\ d_k(\xi,\eta) \quad \text{(all of them are } > 0\text{)}$$

must satisfy $\mathtt{inf}_{k \in A}\, \mu_k = 0$. This allows us to define a sequence $k_0 < k_1 < k_2 < \dots$ of numbers $k_i \in A$, and, for any k_i, a pair of $\xi_i,\ \eta_i \in X_{k_i}$ with $d_{k_i}(\xi_i, \eta_i) \to 0$, and also $j_i \in [n_{k_i}, n_{k_i+1}) \cap B$ such that $d_{j_i}(H_{k_i}(\xi_i)_{j_i}, H_{k_i}(\eta_i)_{j_i}) \ge \varepsilon$. Let $x,\ y \in \prod_{k \in A} X_k$ satisfy $x_{k_i} = \xi_i$ and $y_{k_i} = \eta_i$ for all i and $x_k = y_k$ for all $k \in A$ not of the form k_i. Then easily $x \,\mathsf{D}_A\, y$ holds but $\Psi(x) \,\mathsf{D}_B\, \Psi(y)$ fails, which is a contradiction. $\square$ (*Lemma 66.2*)

Let $k \in A$, and let $Y_k \subseteq X_k$ be as in (3). Then there exist elements $x_k \ne y_k$ in Y_k such that $H_k(x_k) \restriction k = H_k(y_k) \restriction k$. By (1) there is a chain $x_k = \xi_0, \xi_1, \dots, \xi_n = y_k$ of elements $\xi_i \in X_k$ with $d_k(z_i, z_{i+1}) \le \frac{1}{k+1}$ for all $i < n$. Now $H_k(\xi_i) \in \prod_{j \in [n_i, n_{i+1}) \cap B} X_j$ for each $i \le n$. Let $j \in [n_i, n_{i+1}) \cap B$. If $j > k$ then the elements $y_i^j = H_k(\xi_i)_j$, $i \le n$, satisfy $d_j(y_i^j, y_{i+1}^j) \le f_k(\frac{1}{k+1})$. As clearly $n < m_j$, we conclude that $d_j(H_k(x_k)_j, H_k(y_k)_j) \le f_k(\frac{1}{k+1}) + \frac{1}{j+1}$ by (2). If $j < k$ then simply $H_k(x_k)_j = H_k(y_k)_j$ by the choice of $x_k,\ y_k$. Thus totally

(4) $d_j(H_k(x_k)_j, H_k(y_k)_j) \le f(\frac{1}{k+1}) + \frac{1}{k+1}$ for all $j \in [n_i, n_{i+1}) \cap B$.

(as $k \notin B$). Let $x = \{x_k\}_{k \in A}$ and $y = \{y_k\}_{k \in A}$, both are elements of $\prod_{k \in A} X_k$, and $x \,\mathsf{D}_A\, y$ fails because $d_k(x_k, y_k) \ge r$ for all k. On the other hand, we have $\Psi(x) \,\mathsf{D}_B\, \Psi(y)$ by (4), because $f(\delta) \to 0$ with $\delta \to 0$ by Lemma 66.2. This is a contradiction to the assumption that Ψ reduces D_A to D_B.

$\square$ (*Theorem 66*)

16 T_2 is not reducible to ...

This section contains a theorem saying that the ER T_2 of equality of countable sets of the reals is not Borel reducible to ERs which belong to a family of *Borel-stable* ERs, including, for instance, continuous actions of CLI groups and some ideals, not only Polishable, and is closed under the Fubini product modulo Fin. But the *prima facie* definition of the family is based on a rather metamathematical property which we extracted from Hjorth [15].

Recall that T_2 is defined on $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ as follows: $x \,\mathsf{T}_2\, y$ iff $\mathtt{ran}\, x = \mathtt{ran}\, y$.

16.a Borel-stable ERs do not reduce T_2

By a *Borel code* for a recursivery presented Polish space $\mathbb{X}$ we shall understand a pair $\tau = \langle T, f \rangle$ of a wellfounded tree $\emptyset \ne T = T_\tau \subseteq \mathtt{Ord}^{<\omega}$ (then $\Lambda \in T$) and a map f defined on $\mathtt{Max}\, T$ (all $\subsetneq$-maximal elements of T) and taking values among indices of basic open sets of $\mathbb{X}$. We define $\mathrm{B}_\tau(t) \subseteq \mathbb{N}^{\mathbb{N}}$ for any $t \in T$ by induction on the rank of t in T, so that $\mathrm{B}_\tau(t) = B_{f(t)}$ for all $t \in \mathtt{Max}\, T$, where B_k is that basic set which has index k, and $\mathrm{B}_\tau(t) = \complement \bigcup_{t^\wedge \xi \in T} \mathrm{B}_\tau(t^\wedge \xi)$ for $t \in T \setminus \mathtt{Max}\, T$; finally, put $\mathrm{B}_\tau = \mathrm{B}_\tau(\Lambda)$.

A Borel cote τ is *countable* if its tree T_τ is countable or finite. In this case the coded set B_τ is a Borel subset of $\mathbb{N}^{\mathbb{N}}$.

Let us agree that if X is a $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$ in the universe $\mathbb{V}$, and we also consider an extension $\mathbb{V}^+$ of $\mathbb{V}$, then let $X^\#$ denote what results by the definition of X applied in $\mathbb{V}^+$. There is no ambiguity here by Shoenfield, and easily $X = X^\# \cap \mathbb{V}$.

Definition 67 A $\mathbf{\Sigma}^1_1$ ER E is *Borel-stable* if, for any (not necessarily countable) Borel code τ, if B_τ is pairwise $\mathsf{E}^\#$-equivalent in any generic extension of $\mathbb{V}$ then there is $x \in \mathbb{V}$ such that $\mathrm{B}_\tau \subseteq [x]_{\mathsf{E}^\#}$ in any generic extension of $\mathbb{V}$. $\square$

Lemma 68 (Hjorth [15]) *Borel-stable* $\mathbf{\Sigma}^1_1$ *ERs* E *do not satisfy* $\mathsf{T}_2 \le_{\mathrm{B}} \mathsf{E}$.

Proof To see that T_2 itself is not Borel-stable consider a Borel code $\tau \in \mathbb{V}$ for the set $\{x \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} : \mathtt{ran}\, x = \mathbb{N}^{\mathbb{N}} \cap \mathbb{V}\}$, so that $T_\tau \subseteq (\mathfrak{c}^{\mathbb{V}})^{<\omega}$ and $\mathbf{L}[\tau] = \mathbf{L}[\mathbb{N}^{\mathbb{N}} \cap \mathbb{V}]$. Then if of Definition 67 holds, actually, B_τ is a T_2-class in any universe where it is non-empty, but then fails.

Now suppose that $\vartheta : (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is, in $\mathbb{V}$, a Borel reduction of T_2 to a $\mathbf{\Sigma}^1_1$ ER E on $\mathbb{N}^{\mathbb{N}}$. There is perhaps no Borel code π such that $\mathrm{B}_\pi = \vartheta^\#\text{"}\mathrm{B}_\tau$ in any universe, but still there is a code π which witnesses the T_2-non-unlikeness of E. Consider the formula $A(\tau, y)$ saying:

- $y \in \mathbb{N}^{\mathbb{N}}$ and there is a forcing $P \in \mathbf{L}[\tau, y]$ which forces some $x \in \mathrm{B}_\tau$ such that $y = \vartheta^\#(x)$.

Formula $A(\tau, y)$ is relativized to $\mathbf{L}[\tau, y]$, hence, as it is known, there is a Borel code $\pi \in \mathbf{L}[\tau]$ such that $\mathrm{B}_\pi = \{y : \mathbf{L}[\tau, y] \models A(\tau, y)\}$ in any extension of *reference ?*

$\mathbb{V}$. Then easily $\mathsf{B}_\pi \subseteq \vartheta^{\#}{}^"\mathsf{B}_\tau$, hence, B_π is pairwise $\mathsf{E}^\#$-equivalent – in any universe, and B_π is non-empty somewhere, because if $x \in (\mathbb{N}^\mathbb{N})^\mathbb{N}$ is a generic, over $\mathbb{V}$, collapse function $\mathbb{N} \xrightarrow{\text{onto}} \mathbb{V} \cap \mathbb{N}^\mathbb{N}$ then $y = \vartheta^\#(x)$ satisfies $A(\tau, y)$, hence, belongs to B_π. Suppose, towards the contrary, that E is Borel-stable. Then there is $y \in \mathbb{N}^\mathbb{N} \cap \mathbb{V}$ such that $\mathsf{B}_\pi \subseteq [y]_{\mathsf{E}^\#}$ holds, in particular, in $\mathtt{Coll}(\mathbb{N}, \mathbb{V} \cap \mathbb{N}^\mathbb{N})$-generic extension $\mathbb{V}^+$ of $\mathbb{V}$, where $C_\pi \neq \emptyset$, hence, there is $x \in \mathsf{B}_\tau \cap \mathbb{V}^+$ with $y \,\mathsf{E}^\#\, \vartheta^\#(x)$. On the other hand, we have, by Shoenfield, $y \,\mathsf{E}\, \vartheta(x')$ holds for some $x' \in (\mathbb{N}^\mathbb{N})^\mathbb{N}$ in $\mathbb{V}$. Thus $x \,\mathsf{T}_2\, x'$, which is a contradiction because x codes countability of $\mathfrak{c}^\mathbb{V}$ while $x' \in \mathbb{V}$. $\square$

16.b Fubini product of Borel-stables is Borel-stable

Recall that the Fubini product $\mathsf{E} = \prod_{k\in\mathbb{N}} \mathsf{E}_k \,/\, \mathtt{Fin}$ of ERs E_k on $\mathbb{N}^\mathbb{N}$ modulo $\mathtt{Fin}$ is a ER on $(\mathbb{N}^\mathbb{N})^\mathbb{N}$ defined as follows: $x \mathsf{E} y$ if $x(k) \mathsf{E}_k y(k)$ for all but finite k.

Proposition 69 *The family of all Borel-stable $\mathbf{\Sigma}^1_1$ ERs is closed under Fubini products modulo* $\mathtt{Fin}$.

Proof Suppose that ERs E_k on $\mathbb{N}^\mathbb{N}$ are Borel-stable; prove that the Fubini product $\mathsf{E} = \prod_{k\in\mathbb{N}} \mathsf{E}_k \,/\, \mathtt{Fin}$ is Borel-stable. Define $x \,\mathsf{F}_k\, y$ iff $x(k) \,\mathsf{E}_k\, y(k)$: F_k are $\mathbf{\Sigma}^1_1$ ERs on $(\mathbb{N}^\mathbb{N})^\mathbb{N}$ and $x \,\mathsf{E}\, y$ iff $x \,\mathsf{F}_k\, y$ for almost all k.

Claim 69.1 *Each F_k is Borel-stable.*

Proof Consider a Borel code τ for a subset of $(\mathbb{N}^\mathbb{N})^\mathbb{N}$, satisfying <u>if</u> of Definition 67 w.r.t. F_k and such that $\mathsf{B}_\tau \neq \emptyset$ in some extension of $\mathbb{V}$. By the same argument as in the proof of Lemma 68, there is a Borel code π for a subset of $\mathbb{N}^\mathbb{N}$, such that $\mathsf{B}_\pi \neq \emptyset$ in some extension of $\mathbb{V}$ and $\mathsf{B}_\pi \subseteq \{x(k) : x \in \mathsf{B}_\tau\}$ in any extension of $\mathbb{V}$, hence, π satisfies <u>if</u> of Definition 67 w.r.t. E_k. As E_k is Borel-stable, there is $a \in \mathbb{N}^\mathbb{N}$ such that $\mathsf{B}_\pi \subseteq [a]_{\mathsf{E}_k^\#}$ in any extension, but then easily $\mathsf{B}_\tau \subseteq [x]_{\mathsf{F}_k^\#}$ in any extension, where $x \in (\mathbb{N}^\mathbb{N})^\mathbb{N} \cap \mathbb{V}$ has only to satisfy $x(k) = a$ for the given k. $\square$ *(Claim)*

In continuation of the proof of the proposition, consider a Borel code τ for a subset of $(\mathbb{N}^\mathbb{N})^\mathbb{N}$, satisfying <u>if</u> of Definition 67 w.r.t. E. Our plan is to find another Borel code φ with $\mathsf{B}_\varphi \subseteq \mathsf{B}_\tau$ everywhere, which satisfies <u>if</u> of Definition 67 for almost all E_k. This involves a forcing by Borel codes.

We have $T_\tau \subseteq \kappa^{<\omega}$ for a cardinal κ. Let Φ be the set of all Borel codes $\pi \in \mathbb{V}$ for subsets of $(\mathbb{N}^\mathbb{N})^\mathbb{N}$ such that $T_\pi \subseteq \lambda^{<\omega}$ for some $\lambda < \kappa^+$ and $\mathsf{B}_\pi \neq \emptyset$ in some generic extension of the universe $\mathbb{V}$. Φ is considered as a forcing, with the order $\pi \preccurlyeq \tau$ iff $\mathsf{B}_\pi \subseteq \mathsf{B}_\tau$ in all generic extensions of $\mathbb{V}$. It is known that Φ forces a point of $(\mathbb{N}^\mathbb{N})^\mathbb{N}$, so that $\bigcap_{\pi\in G} \mathsf{B}_\pi = \{x_G\}$ for any Φ-generic, over $\mathbb{V}$, set $G \subseteq \Phi$. Let $\dot{x}$ be the name of the generic element of $(\mathbb{N}^\mathbb{N})^\mathbb{N}$.

We can assume that B_τ is non-empty somewhere, as otherwise there is nothing to prove. Then $\langle\tau,\tau\rangle$ $\Phi\times\Phi$-forces $\dot{x}_{\mathtt{left}} \,\mathsf{E}^\#\, \dot{x}_{\mathtt{right}}$, hence, there are codes $\pi, \sigma \in \Phi$ and a number k_0 such that $\langle\pi,\sigma\rangle$ $\Phi\times\Phi$-forces $\dot{x}_{\mathtt{left}} \,\mathsf{F}_k^\#\, \dot{x}_{\mathtt{right}}$ for any $k \geq k_0$. Then, by a simple argument, we have $x \,\mathsf{F}_k{}^\#\, y$ in any extension of $\mathbb{V}$ for any two Φ-generic, over $\mathbb{V}$, elements $x, y \in \mathsf{B}_\pi$. We can straightforwardly define in $\mathbb{V}$ a Borel code φ (too big to be a member of Φ!) such that, in any extension of $\mathbb{V}$, B_φ is the set of all Φ-generic, over $\mathbb{V}$, elements of B_π. Then φ satisfies <u>if</u> of Definition 67 w.r.t. any F_k with $k \geq k_0$. Hence, by the claim, there is, in $\mathbb{V}$, a sequence of points $x_k \in (\mathbb{N}^\mathbb{N})^\mathbb{N}$ such that $\mathsf{B}_\varphi \subseteq [x_k]_{\mathsf{F}_k^\#}$ in any generic extension of $\mathbb{V}$, for any $k \geq k_0$. Define $x \in (\mathbb{N}^\mathbb{N})^\mathbb{N} \cap \mathbb{V}$ so that $x(k) = x_k(k)$ for any $k \geq k_0$, then, by the definition of F_k, we have $\mathsf{B}_\varphi \subseteq [x]_{\mathsf{F}_k^\#}$ for all $k \geq k_0$ in any extension of $\mathbb{V}$. Yet $\bigcap_{k\geq k_0} [x]_{\mathsf{F}_k^\#} \subseteq [x]_{\mathsf{E}^\#}$. $\square$ *(Proposition)*

16.c Borel stability of CLI actions

Recall that a Polish group $\mathbb{G}$ is *complete left-invariant*, CLI for brevity, if $\mathbb{G}$ admits a compatible left-invariant complete metric. Then easily $\mathbb{G}$ also admits a compatible <u>right</u>-invariant complete metric, which will be practically used.

Theorem 70 (Hjorth [15]) *Suppose that $\mathbb{G}$ is a Polish* CLI *group continuously acting on a Polish space $\mathbb{X}$. Then $\mathsf{E}^\mathbb{G}_\mathbb{X}$ is Borel-stable, hence, T_2 is not Borel reducible to $\mathsf{E}^\mathbb{G}_\mathbb{X}$.*

Proof Fix a Borel code $\hat{\tau}$ satisfying <u>if</u> of Definition 67. Let κ be a cardinal in $\mathbb{V}$ satisfying $T_{\hat{\tau}} \subseteq \kappa^{<\omega}$. Define Φ as above, so that now Φ forces an element of $\mathbb{X}$. Say that a set $X \subseteq \mathbb{X}$ is *forcible* if there is $\pi \in \Phi$ which forces $\dot{x} \in X^\#$. For instance, any B_π is forcible, which is witnessed by π itself.

Claim 70.1 *If $X = \bigcup_n X_n \subseteq \mathbb{X}$ is forcible then at least one of X_n is forcible.*

Proof By definition some $\pi \in \Phi$ forces $\dot{x} \in X^\#$. If none of X_n is forcible then π forces $\dot{x} \notin X_n{}^\#$ for each n, which is a contradiction. $\square$ *(Claim)*

Now, let ρ be a compatible <u>right</u>-invariant metric on $\mathbb{G}$. For any $r > 0$, let $G_r = \{g \in \mathbb{G} : \rho(g, 1_\mathbb{G}) < r\}$: this is an open nbhd of $1_\mathbb{G}$. Say that $\pi \in \Phi$ *is of size* $\leq r$ if $\langle\pi,\pi\rangle$ $(\Phi\times\Phi)$-forces that there is $g \in G_r{}^\#$ with $\dot{x}_{\mathtt{left}} = g \cdot \dot{x}_{\mathtt{right}}$. In this case, by the right-invariance of the metric, in any generic extension of the universe, for any two Φ-generic $x, y \in \mathsf{B}_\pi$ there is $g \in G_{2r}{}^\#$ with $y = g\cdot x$. (Indeed, take $z \in \mathsf{B}_\pi$ to be generic enough for both $\langle x,z\rangle$ and $\langle y,z\rangle$ to be $\Phi\times\Phi$-generic pairs. Then $x = f\cdot z$ and $y = h\cdot z$ for some $f, h \in G_r{}^\#$, so that $y = hf^{-1}\cdot x$. But $\rho(hf^{-1}, 1_\mathbb{G}) = \rho(h,f)$ by the right invariance, and this is $\leq \rho(h,1_\mathbb{G}) + \rho(f,1_\mathbb{G}) \leq 2r$.) By the same argument, if $\pi \in \Phi$ forces $\dot{x} \in (G_r \cdot y)^\#$ for some $y \in \mathbb{X}$ then π is of size $2r$.

Lemma 70.2 *If $\pi \in \Phi$, $\pi \preccurlyeq \widehat{\tau}$, and $r > 0$, then there exists a condition $\tau \in \Phi$, $\tau \preccurlyeq \pi$, of size $\le r$.*

Proof Take any $y \in \mathrm{B}_\pi$. Let Q be a countable dense subset of $\mathbb{G}$. For any $g \in Q$, the set $G_r g$ is the open r-ball in $\mathbb{G}$ by the choice of the metric, hence, $\mathbb{G} = \bigcup_{g\in Q} G_r g$. Then $\mathrm{B}_\pi \subseteq \bigcup_{g\in Q} G_r g\cdot y$, hence, by Lemma 70.1, one of the sets $(G_r g\cdot y) \cap \mathrm{B}_\pi = (G_r\cdot x) \cap \mathrm{B}_\pi$, where $x = g\cdot y$, is forcible, so that there is a "condition" $\tau \in \Phi$ which Φ-forces $\dot{x} \in ((G_r\cdot x)\cap \mathrm{B}_\pi)^{\#}$. We easily have $\tau \preccurlyeq \pi$. On the other hand, τ is of size $\le 2r$ by the remark above. □ (*Lemma*)

It follows that there is, in $\mathbb{V}$, a sequence of codes $\pi_n \in \Phi$ such that $\pi_0 \preccurlyeq \widehat{\tau}$, $\pi_{n+1} \preccurlyeq \pi_n$, π_n has size $\le 2^{-n-1}$, and B_{π_n} has $\mathbb{X}$-diameter $\le 2^{-n}$ for any n. The only limit point x of the sequence of sets B_{π_n} belongs to $\mathbb{V}$, so it remains to show that $\mathrm{B}_{\widehat{\tau}} \subseteq [x]_{(\mathsf{E}^{\mathbb{G}}_{\mathbb{X}})^{\#}}$ in any extension $\mathbb{V}^+$ of the universe $\mathbb{V}$.

We can assume that $\mathbb{V}^+$ is rich enough to contain a Φ-generic, over $\mathbb{V}$, element $x_n \in \mathrm{B}_{\pi_n}$ for any n, then $\mathtt{lim}_n\, x_n = x$. Moreover, for any n, both x_n and x_{n+1} belong to B_{π_n}, hence, as π_n has size $\le 2^{-n-1}$, there is $g_{n+1} \in \mathbb{G}^{\#}$ with $\rho(1,g) \le 2^{-n}$ such that $x_{n+1} = g_{n+1}\cdot x_n$. Thus, $x_n = h_n\cdot x_0$, where $h_n = g_n \dots g_1$. Note that $\rho(h_n, h_{n-1}) = \rho(g_n, 1_{\mathbb{G}}) \le 2^{-n+1}$ by the right-invariance of the metric, thus, $\{h_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{G}^{\#}$. Let $h = \mathtt{lim}_{n\to\infty} h_n \in \mathbb{G}^{\#}$ be its limit. As the action is continuous, we have $x = \mathtt{lim}_n\, x_n = h\cdot x_0$. It follows that $x \mathrel{\mathsf{E}^{\mathbb{G}}_{\mathbb{X}}} x_0$. However $x_0 \in \mathrm{B}_{\pi_0} \subseteq \mathrm{B}_{\widehat{\tau}}$, therefore, $\mathrm{B}_{\widehat{\tau}} \subseteq [x]_{(\mathsf{E}^{\mathbb{G}}_{\mathbb{X}})^{\#}}$, as required.

□ (*Theorem 70*)

16.d A family of Borel-stable ideals

Let us say that a Borel ideal $\mathcal{I}$ is *Borel-stable* if so is the induced ER $\mathsf{E}_{\mathcal{I}}$. It immediately follows from Theorem 70 that any polishable ideal is Borel-stable. Yet there are Borel-stable ideals among non-polishable ones. Suppose that $\{\varphi_i\}_{i\in\mathbb{N}}$ is a sequence of lower semicontinuous (l. s. c.) submeasures on $\mathbb{N}$. Define

$$\mathtt{Exh}_{\{\varphi_i\}} = \{X \subseteq \mathbb{N} : \varphi_\infty(X) = 0\}\,, \quad \text{where} \quad \varphi_\infty(X) = \mathtt{lim\,sup}_{i\to\infty} \varphi_i(X)\,.$$

the exhaustive ideal of the sequence of submeasures. By Solecki's Theorem 25 for any Borel P-ideal $\mathcal{I}$ there is a single l. s. c. submeasure φ such that $\mathcal{I} = \mathtt{Exh}_{\{\varphi_i\}} = \mathtt{Exh}_\varphi$, where $\varphi_i(X) = \varphi(X \cap [i,\infty))$, however, for example, the non-polishable ideal $\mathcal{I}_1 = \mathrm{Fin} \times 0$ also is of the form $\mathtt{Exh}_{\{\varphi_i\}}$, where for $X \subseteq \mathbb{N}^2$ we define $\varphi_i(X) = 0$ or 1 if resp. $X \subseteq$ or $\not\subseteq \{0,\dots,n-1\}\times\mathbb{N}$.

Theorem 71 *Any ideal of the form $\mathtt{Exh}_{\{\varphi_i\}}$ is Borel-stable.*

Proof Thus let $\mathcal{I} = \mathtt{Exh}_{\{\varphi_i\}}$, all φ_i being l. s. c. submeasures on $\mathbb{N}$. We can assume that the submeasures φ_i decrease, *i.e.*, $\varphi_{i+1}(X) \le \varphi_i(X)$ for any X,

for if not consider the l. s. c. submeasures $\varphi'_i(X) = \mathtt{sup}_{j\ge i}\, \varphi_j(X)$. Let $\widehat{\tau}$ be a Borel code, for a subset of $\mathcal{P}(\mathbb{N})$, satisfying if of Definition 67 w. r. t. the induced ER $\mathsf{E}_{\mathcal{I}}$ on $\mathcal{P}(\mathbb{N})$. We can assume that $\mathrm{B}_{\widehat{\tau}}$ is non-empty in some extension of $\mathbb{V}$, thus, $\widehat{\tau} \in \Phi$, where Φ is a forcing defined as in the proof of Proposition 69 (but now Φ forces a subset of $\mathcal{P}(\mathbb{N})$, of course).

Using the same arguments as above, we see that for any $\tau \in \Phi$, $\tau \preccurlyeq \widehat{\tau}$, and $i \in \mathbb{N}$, there are $j \ge i$ and codes $\pi, \sigma \in \Phi$ with $\pi, \sigma \preccurlyeq \tau$, such that $\langle \pi, \sigma\rangle$ $\Phi\times\Phi$-forces that $\varphi_j(\dot{x}_{\mathtt{left}} \,\Delta\, \dot{x}_{\mathtt{right}}) \le 2^{-i-1}$, hence, any two Φ-generic, over $\mathbb{V}$, elements $x, y \in \mathrm{B}_\pi$ satisfy $\varphi_j(x \,\Delta\, y) \le 2^{-i}$. It follows that, in $\mathbb{V}$, there is a sequence of numbers $i_0 < i_1 < i_2 < \dots$, a sequence $\widehat{\tau} \succcurlyeq \tau_0 \succcurlyeq \tau_1 \succcurlyeq \tau_2 \succcurlyeq \dots$ of codes in Φ, and, for any n, a set $s_n \subseteq [0,n)$, such that

(1) each τ_n Φ-forces $\dot{x} \cap [0,n) = s_n$;

(2) any Φ-generic, over $\mathbb{V}$, $x, y \in \mathrm{B}_{\tau_n}$ satisfy $\varphi_{i_n}(x \,\Delta\, y) \le 2^{-n}$.

Easily there is, in $\mathbb{V}$, a unique $a \in \mathcal{P}(\mathbb{N})$ such that $a \cap [0,n) = s_n$ for all n. Prove that $\mathrm{B}_{\widehat{\tau}} \subseteq [a]_{\mathsf{E}^{\#}_{\mathcal{I}}}$ in any extension of $\mathbb{V}$.

We can assume that, in the extension, for any n there is a Φ-generic, over $\mathbb{V}$, element $x_n \in \mathrm{B}_{\tau_n}$. Then we have, by (2), $\varphi_{i_n}(x_n \,\Delta\, x_m) \le 2^{-n}$ whenever $n \le m$. It follows that $\varphi_{i_n}(x_n \,\Delta\, x) \le 2^{-n}$, because obviously $x = \mathtt{lim}_m\, x_m$. However we assume that the submeasures φ_j decrease, hence, $\mathtt{lim\,sup}_j\, \varphi_j(x_n \,\Delta\, x) \le 2^{-n}$. On the other hand, $\mathtt{lim\,sup}_j\, \varphi_j(x_n \,\Delta\, x_0) = 0$ because all elements of B_{τ_0} are pairwise $\mathsf{E}^{\#}_{\mathcal{I}}$-equivalent. We conclude that $\mathtt{lim\,sup}_j\, \varphi_j(x_0 \,\Delta\, x) \le 2^{-n+1}$ for any n, in other words, $\mathtt{lim\,sup}_j\, \varphi_j(x_0 \,\Delta\, x) = 0$ and $x_0 \mathrel{\mathsf{E}^{\#}_{\mathcal{I}}} x$, as required. □

Question 1 Are all Borel ideals Borel-stable ? The expected answer "yes" would show that T_2 is not Borel reducible to any Borel ideal. □

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