Infinitary Combinatorics without the Axiom of Choice
Consistency Strengths of Choiceless Failures of SCH

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Outline

1. ICWAC
2. ¬SCH
3. Parallel Prikry forcing
4. The lower bound
5. Further results and questions
The ICWAC Project

- Study strong combinatorial principles like Chang’s Conjecture, Rowbottom Cardinals, ¬SCH, ... without assuming AC
- Consistency strengths go down without AC and become amenable to forcing and inner model arguments for relatively small large cardinals
- Equiconsistencies are possible in several cases
- (Also combinatorics under AD)
- Joint DFG-NWO project with Benedikt Löwe and Arthur Apter
Cardinals without AC

- $\kappa = \lambda^+$
  - $\leftrightarrow \forall \gamma < \kappa \exists f : \gamma \to \lambda$ injective
  - $\leftrightarrow (AC!!) \exists F : \kappa \times \kappa \to \lambda \forall \gamma < \kappa F(\gamma, \nu) : \gamma \to \lambda$ injective

- Under AC, $\kappa = \lambda^+$ is not Ramsey:
  - define a partition $P : \kappa^3 \to 2$ by $P(\alpha, \beta, \gamma) = 1$ iff $F(\alpha, \gamma) < F(\beta, \gamma)$, for $\alpha < \beta < \gamma < \kappa$
Cardinals without \( AC \)

To get strong combinatorial properties at accessible cardinals:

- arrange \( \forall \gamma < \kappa \exists f : \gamma \rightarrow \lambda \) injective
  
  without \( \exists F : \kappa \times \kappa \rightarrow \lambda \ \forall \gamma < \kappa \ F(\ast, \nu) : \gamma \rightarrow \lambda \) injective

- use symmetric submodels \( N \) of forcing extensions

- make \( N \) a limit of models \( M_i \vDash ZFC \):
  
  \[
  N \cap \mathcal{P}(\text{Ord}) = \bigcup_i (M_i \cap \mathcal{P}(\text{Ord}))
  \]

- let every \( M_i \) be a “small” forcing extension of the ground model \( V \)
Example: Chang’s Conjecture

- Let $\kappa \rightarrow (\omega_2)_2^{<\omega}$
- Levy collapse $\kappa$ to $\omega_3$: $V[G] \models \kappa = \omega_3$
- Let $N$ be a submodel of $V[G]$ spanned by $V[G \upharpoonright i]$ for $i < \kappa$
- $N \cap \mathcal{P}(\text{Ord}) = \bigcup_i (V[G \upharpoonright i] \cap \mathcal{P}(\text{Ord}))$
- $V[G \upharpoonright i]$ is a small forcing extension relative to $\kappa$
- $V[G \upharpoonright i] \models \kappa \rightarrow (\omega_2)_2^{<\omega}$
- $N \models \kappa \rightarrow (\omega_2)_2^{<\omega}$
- $N \models$ Chang’s Conjecture for $(\omega_3, \omega_2)$
- Chang’s Conjecture for $(\omega_3, \omega_2)$ is equiconsistent with $\exists \kappa \ \kappa \rightarrow (\omega_2)_2^{<\omega}$
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For a fixed $\alpha \geq 2$, the following theories are equiconsistent:

$$ZFC + \exists \kappa [\kappa \text{ is measurable}]$$

and

$$ZF + \neg AC + GCH \text{ holds below } \aleph_\omega +$$

There is a surjective $f : [\aleph_\omega]^\omega \to \aleph_\omega+\alpha$. 
Theorem

For a fixed $n < \omega$, $n \geq 1$, the following theories are equiconsistent:

$$ZFC + \exists \kappa[(\text{cof}(\kappa) = \omega)]$$

$$\wedge (\forall i < \omega)(\forall \lambda < \kappa)(\exists \delta < \kappa) [(\delta > \lambda) \wedge (o(\delta) \geq \delta^+ i)]$$

and

$$ZF + \neg AC + GCH \text{ holds below } \aleph_\omega +$$

There is an injective $f : \aleph_{\omega n} \rightarrow [\aleph_\omega]^\omega$. 
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Parallel Prikry forcing

Fix a normal measure $\mathcal{U}$ on $\kappa$ and a set $Z \subseteq \text{Ord}$.

$p = (s_\alpha, A_\alpha)_{\alpha \in Z}$ is a condition in $\mathbb{P}$ iff

1. $\forall \alpha \in Z[(s_\alpha \in [\kappa]^\omega) \land (A_\alpha \in \mathcal{U}) \land (\max(s_\alpha) < \min(A_\alpha))]$

2. $\text{dom}(p) := \{\alpha \in Z \mid A_\alpha \neq \kappa\}$ is finite.

Write $(s_\alpha, A_\alpha)$ instead of $(s_\alpha, A_\alpha)_{\alpha \in Z}$. 
Conditions $p' = (s'_\alpha, A'_\alpha)$ and $p = (s_\alpha, A_\alpha)$ in $\mathbb{P}$ are partially ordered by $p' \leq p$ iff there is an integer $n < \omega$ such that

1. $\forall \alpha \in \text{dom}(p)[(\text{otp}(s'_\alpha \setminus s_\alpha) = n) \land (s'_\alpha \setminus s_\alpha \subseteq A_\alpha)]$.

2. $(\forall \alpha, \beta \in \text{dom}(p))(\forall \xi \in s'_\alpha \setminus s_\alpha)(\forall \zeta \in s_\beta)[\xi > \zeta]$.

3. $(\forall \alpha < \beta \in \text{dom}(p))(\forall i < n)[(s'_\alpha \setminus s_\alpha)[i] < (s'_\beta \setminus s_\beta)[i]]$ (s[i] is the $i$-th element of the monotone enumeration of the set s)

4. $(\forall \alpha, \beta \in \text{dom}(p))(\forall i < n)[(i + 1 < n) \implies ((s'_\alpha \setminus s_\alpha)[i] < (s'_\beta \setminus s_\beta)[i + 1])]$.

5. $\forall \alpha \in \text{dom}(p)[A'_\alpha \subseteq A_\alpha]$. 
The partial order on $\mathcal{P}$

1. The stems $s_\alpha$ are extended into the corresponding reservoir sets $A_\alpha$ in a systematic fashion.
2. The extension points are chosen greater than all of the previous stem points.
3. There are the same number of new points at all indices in $\text{dom}(\rho)$, and these are chosen in layers which are strictly ascending.
4. Reservoirs may be thinned out, and new stems outside the old domain may be grown.
Properties of $\mathbb{P}$

Let $G$ be $\mathbb{P}$-generic over $V$. $G$ adjoins a system $(C_\alpha \mid \alpha \in Z)$

$$C_\alpha = \bigcup \{s_\alpha \mid (s_\beta, A_\beta)_{\beta \in Z} \in G\}.$$

**Lemma**

a) Let $\gamma \in Z$. Then $C_\gamma$ is a Prikry sequence for $\mathcal{U}$, i.e.,

$$\forall X \in \wp(\kappa) \cap V[(X \in \mathcal{U}) \iff (C_\gamma \setminus X \text{ is finite})].$$

b) Let $\gamma, \delta \in Z$, $\gamma < \delta$. Then $C_\gamma \cap C_\delta$ is finite, and $C_\gamma \Delta C_\delta$ is infinite.
Lemma

$(\mathbb{P}, \leq)$ satisfies the $\kappa^+$-chain condition.
Define

\[ N = \text{HOD}^{V[G]} \left( \bigcup_{\alpha \in \mathbb{Z}} \tilde{C}_\alpha \cup \{(\tilde{C}_\alpha \mid \alpha \in \mathbb{Z})\} \right), \]

where \( \tilde{C}_\alpha = \{ C \in \mathcal{P}(\kappa) \mid C \Delta C_\alpha \text{ is finite} \} \). \( N \) is the class of sets which are hereditarily definable in the generic extension from finitely many parameters from the class \( \text{Ord} \cup \{ C_\alpha \mid \alpha \in \mathbb{Z} \} \cup \{(\tilde{C}_\alpha \mid \alpha \in \mathbb{Z})\} \).
The powerset of $\kappa$ is large

**Lemma**

In $N$, there is a surjection $f : [\kappa]^\omega \to Z$.

**Proof.**

Define $f$ using the parameter $(\tilde{\mathcal{C}}_\alpha \mid \alpha \in Z)$ by

$$X \mapsto \begin{cases} \text{The unique } \alpha \in Z \text{ such that } X \in \tilde{\mathcal{C}}_\alpha, & \text{if that exists,} \\ 0, & \text{otherwise.} \end{cases}$$
Finite support approximations

**Lemma**

Let $G$ be $\mathbb{P}_Z$-generic for $V$, where $\text{card}(Z) < \omega$. Then $V[G]$ is an extension of $V$ by Prikry forcing $\mathbb{P}_1$. Therefore, by the properties of standard Prikry forcing, $V[G]$ has the same bounded subsets as $V$. 
Lemma

Let $G$ be $\mathbb{P}$-generic, with $C_\alpha = (\dot{C}_\alpha)^G$ for $\alpha \in \mathbb{Z}$ and $D = \dot{D}^G$. Let $X \in V[G]$ be defined by

$$X = \{\zeta \in \text{Ord} \mid V[G] \models \varphi(\zeta, \bar{\xi}, C_{\alpha_0}, \ldots, C_{\alpha_{n-1}}, D)\}$$

where $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{Z}$. Then $X \in V[G \upharpoonright \{\alpha_0, \ldots, \alpha_{n-1}\}]$. 
We may assume that $V \models \text{GCH}$.
Define $(\mathbb{P}, \leq) = (\mathbb{P}_Z, \leq)$ with $Z = \kappa^{+\beta}$. Let $V[G]$ be a generic extension of $V$ by $\mathbb{P}$ with Prikry sequences $(C_\alpha)_{\alpha < \kappa^{+\beta}}$.

Let

$$N = \text{HOD}^{V[G]}(\{ C_\alpha \mid \alpha < \kappa^{+\beta} \} \cup \{(\tilde{C}_\alpha \mid \alpha < \kappa^{+\beta})\}).$$

Every set of ordinals in $N$ is of the form

$$X = \{ \zeta \in \text{Ord} \mid V[G] \models \varphi(\zeta, \tilde{\xi}, C_{\alpha_0}, \ldots, C_{\alpha_{n-1}}, (\tilde{C}_\alpha \mid \alpha < \kappa^{+\beta})) \}$$

Then

$$X \in V[G \upharpoonright \{\alpha_0, \ldots, \alpha_{n-1}\}].$$

Finite support parallel Prikry forcing does not add bounded subsets of $\kappa$. So $\kappa$ is a singular cardinal in $N$, and $N \models \text{“GCH holds below } \kappa \text{”}.$

There is a surjection $f : [\kappa]^\omega \to (\kappa^{+\beta})^V$ in $N$.

By the $\kappa^+$-cc, $(\kappa^{+\beta})^V = (\kappa^{+\beta})^N$. So $f$ yields a choiceless, surjective failure of SCH.
Let $\kappa_0, \kappa_1, \ldots$ be a Prikry sequence in $N$ for the cardinal $\kappa$. Extend $N$ generically by collapsing each $\kappa_{n+1}$ to $\kappa_n^{++}$. Then $\kappa$ becomes $\aleph_\omega$ without destroying GCH below $\kappa$. So SCH can fail at $\aleph_\omega$. 
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The lower bound

**Theorem**

Assume that SCH fails in a surjective way in a model V of ZF. Then there is an inner model of ZFC with a measurable cardinal.
Using the Dodd-Jensen Core Model $K$

Let $\kappa$ be a singular cardinal such that $(\forall \nu < \kappa)[2^\nu < \kappa]$, and let $f : [\kappa]^{\text{cof}(\kappa)} \to \kappa^{++}$ be a surjection. Let $\lambda = \text{cof}(\kappa) + \aleph_2$. Assume that there were no inner model of ZFC with a measurable cardinal. For $Y \subseteq \text{Ord}$, take $g_Y : \text{otp}(Y) \leftrightarrow Y$ to be the uniquely defined order preserving map.

Consider $X \in [\kappa]^{\text{cof}(\kappa)}$. By the Dodd-Jensen covering theorem (in $\text{HOD}[X]$), there is $Y \in K$, $X \subseteq Y \subseteq \kappa$, $\text{otp}(Y) < \lambda$. Let $Z = g_Y^{-1}[X] \in \wp(\lambda)$. Then

$$X = g_Y[Z] \text{ for some } Y \in \wp(\kappa) \cap K \text{ and } Z \in \wp(\lambda).$$

Since GCH holds in $K$, take a surjective $k : \kappa^+ \to \wp(\kappa) \cap K$. Since $2^\lambda < \kappa$, take a surjective $h : \kappa \to \wp(\lambda)$. By (4), the map

$$(\gamma, \eta) \mapsto f(g_k(\gamma)[h(\eta)])$$

is a surjection from $\kappa^+ \times \kappa$ onto $\kappa^{++}$. 
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Injective failures

Theorem

The following theories are equiconsistent:

\[ ZFC + \exists \kappa [o(\kappa) = \kappa^{++} + \omega_2] \]

and

\[ ZF + \neg AC + GCH \text{ holds below } \aleph_{\omega_2} \]

+ There is an injective \( f : \aleph_{\omega_2 + 2} \rightarrow [\aleph_{\omega_2}]^{\omega_2} \).
Injective failures

**Theorem**

a) *If the theory*

\[ ZFC + \exists \kappa [o(\kappa) = \kappa^{++} + \omega_1] \]

*is consistent, then so is the theory*

\[ ZF + \neg AC + GCH \text{ below } \aleph_1 + \text{ there is injective } f : \aleph_{1+2} \to [\aleph_1]^{\omega_1}. \]

b) *If the theory*

\[ ZF + \neg AC + GCH \text{ below } \aleph_1 + \text{ there is injective } f : \aleph_{1+2} \to [\aleph_1]^{\omega_1} \]

*is consistent, then so is the theory*

\[ ZFC + \exists \kappa [o(\kappa) = \kappa^{++}]. \]
Questions

- Can one achieve equiconsistencies in all cases?
- Can one lift the equiconsistency for the surjective failure to uncountable cofinalities?