Forcings constructed along morasses

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A theorem of Todorcevic's

Theorem (Todorcevic)

If $\square_{\omega_1}$ holds, then there exists a ccc forcing that adds a function $f : \omega_2 \times \omega_2 \to \omega$ which is not constant on any $A \times B \subseteq \omega_2 \times \omega_2$ with $otp(A) = otp(B) = \omega$.

Question (Todorcevic)

Is it consistent that there exists a function $f : \omega_3 \times \omega_3 \to \omega$ which is not constant on any $A \times B \subseteq \omega_3 \times \omega_3$ with $otp(A) = otp(B) = \omega$?

The naive idea

First step:

Replace $\Box_{\omega_1}$ by the existence of a simplified $(\omega_1, 1)$-morass. Note that the existence of a simplified $(\omega_1, 1)$-morass implies $\Box_{\omega_1}$.

Second step:

Reformulate the construction of the forcing as a typical morass construction. That is, use the morass as an index set for a recursive construction of a system of embeddings between forcings and take its direct limit.

Third step:

Carry out the same construction along a simplified $(\omega_1, 2)$-morass. This yields a forcing of size ω_3 which might do the right thing.

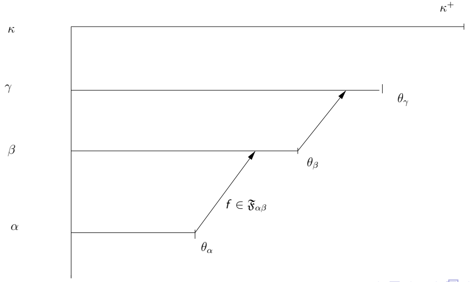
Gap-1 morasses

A simplified $(\kappa, 1)$-morass is a structure $\mathfrak{M} = \langle\langle\theta_\alpha \mid \alpha \leq \kappa\rangle, \langle\mathfrak{F}_{\alpha\beta} \mid \alpha < \beta \leq \kappa\rangle\rangle$ satisfying the following conditions:

(P0) (a) $\theta_0 = 1$, $\theta_\kappa = \kappa^+$, $\forall\alpha < \kappa \;\; 0 < \theta_\alpha < \kappa$.

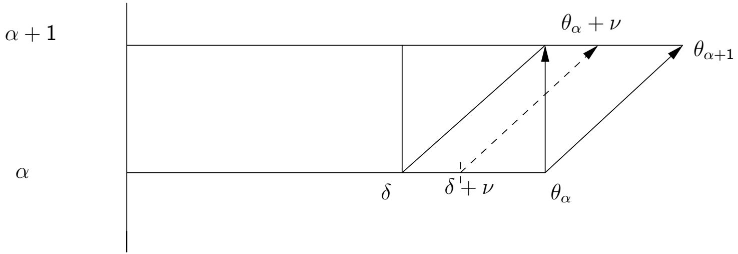
(b) $\mathfrak{F}_{\alpha\beta}$ is a set of order-preserving functions $f : \theta_\alpha \longrightarrow \theta_\beta$.

(P1) - (P5)



Gap-1 morasses

(P3) If $\alpha < \kappa$, then $\mathfrak{F}_{\alpha,\alpha+1} = \{id \upharpoonright \theta_\alpha, f_\alpha\}$ where f_α is such that $f_\alpha \upharpoonright \delta = id \upharpoonright \delta$ and $f_\alpha(\delta + \nu) = \theta_\alpha + \nu$ for some $\delta < \theta_\alpha$.



The morass tree

A simplified morass defines a tree $\langle T, \prec\rangle$:

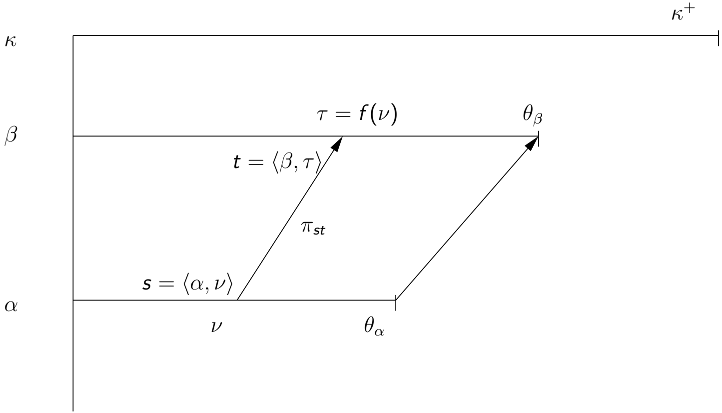
Let $T = \{\langle\alpha,\gamma\rangle \mid \alpha \leq \kappa, \gamma < \theta_\alpha\}$.

For $t = \langle\alpha,\nu\rangle \in T$ set $\alpha(t) = \alpha$ and $\nu(t) = \nu$.

Let $\langle\alpha,\nu\rangle \prec \langle\beta,\tau\rangle$ iff $\alpha < \beta$ and $f(\nu) = \tau$ for some $f \in \mathfrak{F}_{\alpha\beta}$.

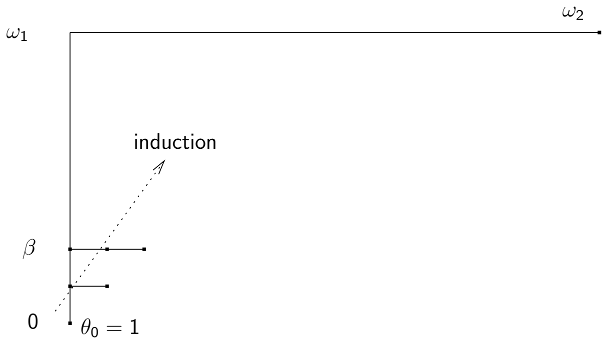
If $s := \langle\alpha,\nu\rangle \prec \langle\beta,\tau\rangle =: t$, $f \in \mathfrak{F}_{\alpha\beta}$ and $f(\nu) = \tau$, then $f \upharpoonright (\nu(s)+1)$ does not depend on f. So we may define $\pi_{st} := f \upharpoonright (\nu(s)+1)$.

The morass tree



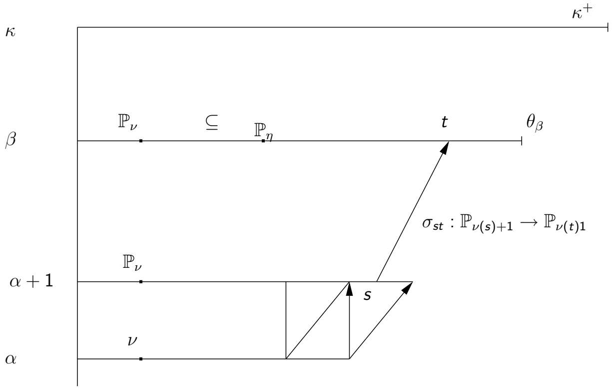
Morass constructions

Morass constructions proceed by induction.



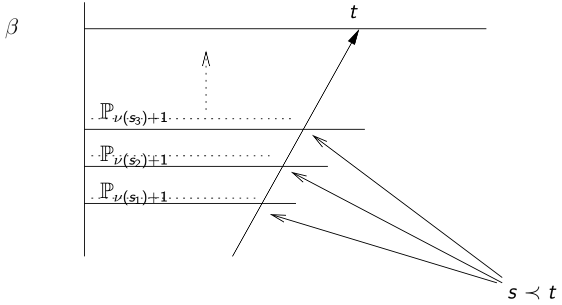
Morass constructions

For every $\nu \leq \omega_2$ a forcing $\mathbb{P}_\nu$ is constructed. Whenever $s \prec t$ an embedding $\sigma_{st} : \mathbb{P}_{\nu(s)+1} \longrightarrow \mathbb{P}_{\nu(t)+1}$ is definied.



Morass constructions

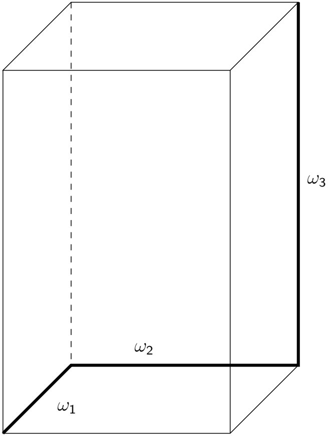
If $\beta \in Lim$ and $\alpha(t) = \beta$, then $\mathbb{P}_{\nu(t)+1}$ is obtained as a direct limit.



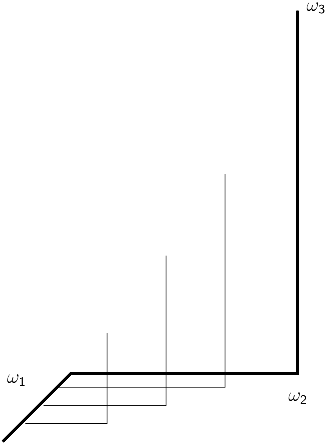
Gap-2 morasses

In a simplified $(\omega_1, 2)$-morass, a simplified $(\omega_2, 1)$-morass is approximated by small pieces.

Gap-2 morasses



Gap-2 morasses



A Suslin tree

As an example, we construct along an $(\omega_1, 1)$-morass a ccc forcing $\mathbb{P}$ of size ω_1 which adds an ω_2-Suslin tree. An ω_2-Suslin tree is a tree of size (or equivalently height) ω_2 which has neither a chain nor an antichain of size ω_2.

Let $P(\theta)$ be the set of all finite trees $p = \langle x_p, <_p \rangle$, $x_p \subseteq \theta$, such that $\alpha < \beta$ if $\alpha <_p \beta$.

Set $p \leq q$ iff $x_p \supset x_q$ and $<_q = <_p \cap x_q^2$.

For $\theta = \omega_1$, $P(\theta)$ is Tennenbaum's forcing to add an ω_1-Suslin tree which satisfies ccc. However, if $\theta > \omega_1 + 1$, then $P(\theta)$ does not satisfy ccc.

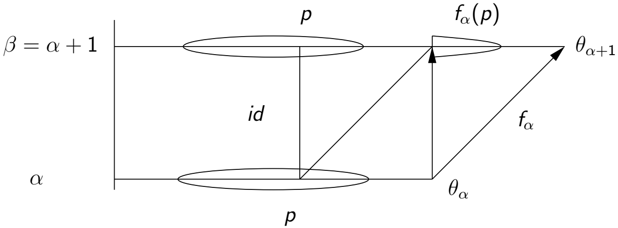
A Suslin tree

We can get a ccc forcing by thinning out $P(\omega_2)$ along the morass. At limit levels, we take the direct limit. So we can concentrate on the successor levels.

Let $\beta = \alpha + 1$. Let the forcing on level α be already defined. Then the forcing on level β consists just of the conditions of the following form:

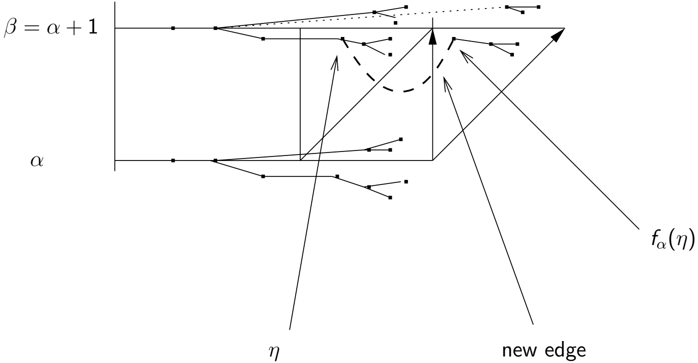
A Suslin tree

`Type 1:` A condition of type 1 is just the union of the two possible copies of some p from level α to the next level.



A Suslin tree

Type 2: A condition of type 2 is just the union of the two possible copies of p to the next level plus one additional edge.



A Suslin tree

Let $\mathbb{P}$ be the forcing which we obtain by thinning-out $P(\omega_2)$. As it turns out, there is a dense embedding from $\mathbb{P}$ to some forcing $\mathbb{Q}$ of size ω_1.

Theorem

If there is a simplified $(\omega_1, 1)$-morass, then there exists a ccc forcing of size ω_1 that adds an ω_2-Suslin tree.

The coloring

The method we used to construct the forcing $\mathbb{P}$ which add an ω_2-Suslin tree always produces forcings that can be densely embedded in a ccc forcing $\mathbb{Q}$ of size ω_1. Hence we cannot use it to construct forcings that destroy CH.

But if we allow in the successor step to amalgamate the copies of two different conditions, then this is possible.

In this way it is possible to prove the following known results.

The coloring

Theorem (Galvin)

It is consistent that there exists a function $f : [\omega_2]^2 \to \omega$ such that $\{\xi < \alpha \mid f(\xi, \alpha) = f(\xi, \beta)\}$ is finite for all $\alpha < \beta < \omega_2$.

Theorem (Todorcevic)

It is consistent that there exists a function $f : \omega_2 \times \omega_2 \to \omega$ which is not constant on any $A \times B \subseteq \omega_2 \times \omega_2$ with $otp(A) = otp(B) = \omega$.

Similar applications

Theorem (Koszmider)

It is consistent that there exists a sequence $\langle X_\alpha \mid \alpha < \omega_2 \rangle$ such that $X_\alpha \subseteq \omega_1$, $X_\beta - X_\alpha$ is finite and $X_\alpha - X_\beta$ is uncountable for all $\beta < \alpha < \omega_2$.

Theorem (Baumgartner, Shelah)

It is consistent that there exists an (ω, ω_2)-superatomic Boolean algebra.

A topological space

Let X be a topological space. Its spread is defined by

$$spread(X) = sup\{card(D) \mid D \text{ discrete subspace of } X\}.$$

Theorem (Hajnal,Juhasz - 1967)
If X is a Hausdorff space, then $card(X) \leq 2^{2^{spread(X)}}$.

In his book "Cardinal functions in topology" (1971), Juhasz explicitly asks if the second exponentiation is really necessary. This was answered by Fedorcuk (1975).

Theorem (Fedorcuk)

In L, there exists a 0-dimensional Hausdorff (and hence regular) space with spread ω of size $\omega_2 = 2^{2^{spread(X)}}$.

This is a consequence of $\diamondsuit$ (and GCH).

There was no such example for the case $spread(X) = \omega_1$. By thinning-out Cohen forcing along a simplified gap-2 morass, one obtains:

Theorem

If there is a simplified $(\omega_1, 2)$-morass, then there exists a ccc forcing of size ω_1 which adds a 0-dimensional Hausdorff space X of size ω_3 with spread ω_1.

Hence there exists such a forcing in L. By the usual argument for Cohen forcing, it preserves GCH. So the existence of a 0-dimensional Hausdorff space with spread ω_1 and size $2^{2^{spread(X)}}$ is consistent.

Summary

	two-dimensional	three-dimensional
preserving GCH	Suslin tree	topological space
not preserving GCH	coloring and similar examples	???

Morasses and finite support iterations,
Proc. Amer. Math. Soc. 137 (2009), 1103-1113

Forcings constructed along morasses,
to appear in the JSL

Two cardinal combinatorics and higher-dimensional forcing,
available under `www.bernhard-irrgang.de`