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Two cardinal combinatorics
and
higher-dimensional forcing

Notes from a lecture course given at the University of Bonn during the
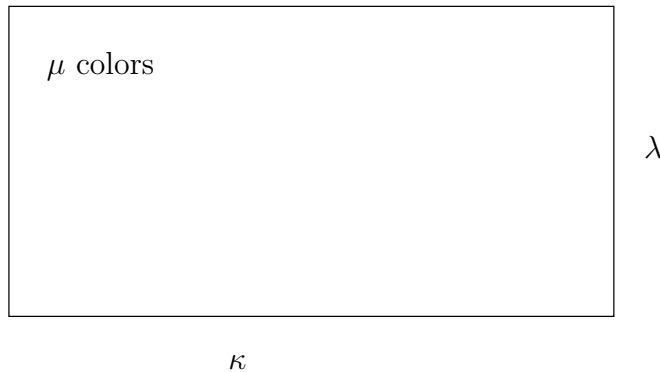Winter Semester 2008/2009.

September 17, 2009

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Introduction

Most, if not all, questions in infinite combinatorics can be posed in form of questions about colorings like $f : \kappa \times \lambda \to \mu$.

It makes sense to ask such questions for cardinals κ, λ, μ as well as for ordinals.

The most famous question about colorings is if they have homogeneous sets of a certain size. This can be denoted in the famous arrow notation. We write
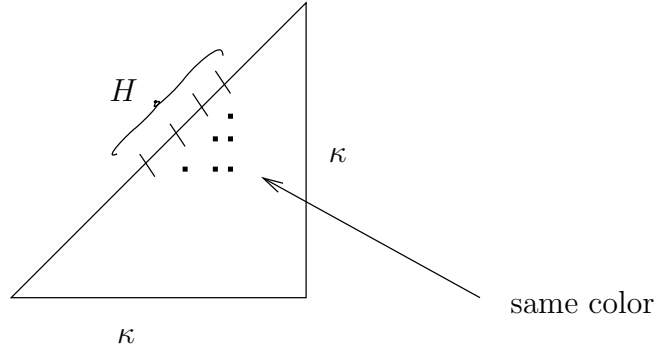
$$\kappa \to (\lambda)^n_\mu$$

for the following statement: Every partition $f : [\kappa]^n \to \mu$ of $[\kappa]^n$ into μ pieces has a homogeneous set h of size λ, i.e. $f : [\kappa]^n \to \mu$ is constant on $[H]^n$ for some $H \subseteq \kappa$ with $otp(h) = \lambda$.

We write $\kappa \to (\lambda)^n_\mu$ for the negation of this statement.

The relation $\kappa \to (\lambda)^n_\mu$ remains true if κ is made larger or if λ, n, μ are made

smaller.

If $n = 2$, then $f : [\kappa]^2 \to \mu$ is a coloring of a triangle with sides of length κ and μ colors.

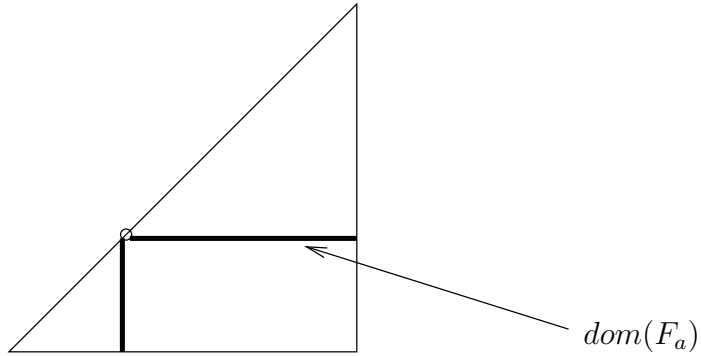**Theoerem 1.1** (Erdös-Rado)

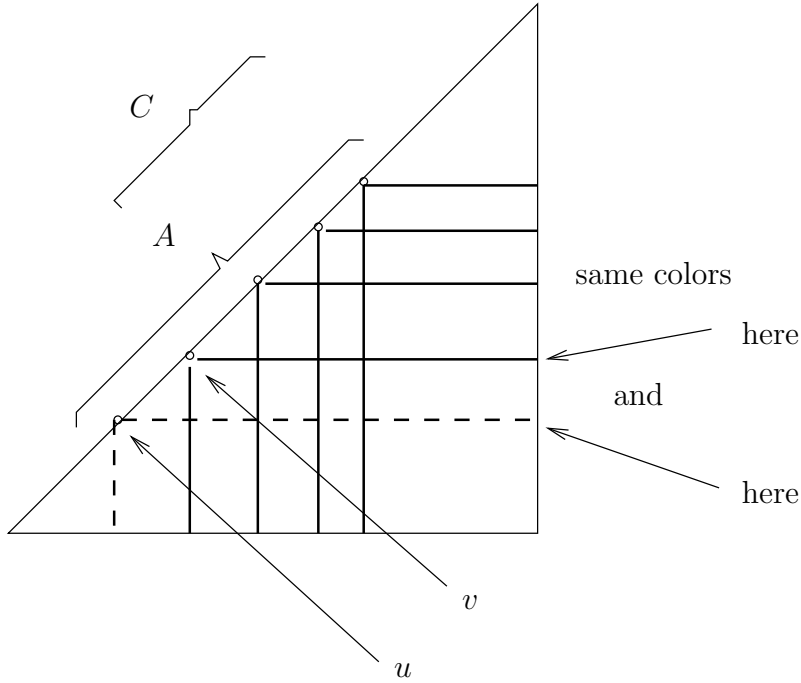$$(2^{<\kappa})^+ \to (\kappa)^2_\gamma$$

for all regular $\kappa \geq \omega$ and all $\gamma < \kappa$.

Proof: Let $\lambda = (2^{<\kappa})^+$ and let $f : [\lambda]^2 \to \gamma$ be a partition of $[\lambda]^2$ into γ pieces. For $a \in \kappa$, let $F_a : \lambda - \{a\} \to \gamma$ be defined by $F_a(\beta) = F(\{a, \beta\})$.

We claim that there exists a set $a \subseteq \lambda$ such that $|A| = 2^{<\kappa}$ and such that for every $C \subseteq A$ of $otp(C) < \kappa$ and every $u \in \lambda - C$ there exists $\nu \in A - C$ such that F_ν agrees with F_u on C.

To prove the claim, we construct a sequence $\langle A_i \mid i < \kappa \rangle$ such that $a_i \subseteq A_j$ for all $i \leq j < \kappa$, $A_i \subseteq \lambda$ and $card(A_i) = 2^{<\kappa}$ for all $i < \kappa$. Let $A_0 = \bigcup \{ A_j \mid j < i \}$ for $i \in Lim$. For the successor step assume that A_i is given. Then choose $A_i \subseteq A_{i+1}$ such that $card(A_{i+1}) = 2^{<\kappa}$ and for every $C \subseteq A_i$ with $otp(C) < \kappa$ and every $u \in \lambda - C$ there exists $v \in A_{i+1} - C$ such that F_v and F_u agree on C. This is possible since there are at most $(2^{<\kappa})^{<\kappa} = 2^{<\kappa}$ many subsetes $C \subseteq A_i$ and $(2^{<\kappa})^{<\kappa} = 2^{<\kappa}$ many functions $g : C \to \gamma$. If we set $A = \bigcup \{ A_i \mid i < \kappa \}$, then A is as wanted.

Now we choose an arbitrary $a \in \lambda - A$. We construct a sequence $\langle x_\alpha \mid \alpha < \kappa \rangle$ in A as follows. Let x_0 be arbitrary. If $\langle x_\beta \mid \beta < \alpha \rangle$ is already defined, then set $C = \{ x_\beta \mid \beta < \alpha \}$ and let x_α be some $v \in A - C$ such that F_v agrees with F_a on C. Let $X = \{ x_\alpha \mid \alpha < \kappa \}$. Consider $G : F_\alpha \upharpoonright X : X \to \gamma$. Since $\gamma < \kappa$, there exists $X \subseteq H$ such that G is constant on H. On the other hand $F(\{x_\alpha, x_\beta\}) = F_{x_\beta}(x_\alpha) = F_a(x_\alpha) = G(x_\alpha)$ for all $\alpha < \beta < \kappa$. Hence F is constant on $[H]^2$. $\square$

For every cardinal κ define $exp_n(\kappa)$ by recursion:

$$exp_0(\kappa) = \kappa$$

$$exp_{n+1}(\kappa) = 2^{exp_n(\kappa)}.$$

Theorem 1.2 (Erdös-Rado)

$$(exp_n(2^{<\kappa}))^+ \rightarrow (\kappa)^{n+2}_\gamma$$

for each cardinal $\kappa \geq \omega$, for each ordinal $\gamma < cf(\kappa)$ and all $n \in \omega$.

Proof: By induction on n using the argument of the previous proof. See for example the book by Hajnal and Hamburger. $\square$

For colorings $f : \kappa \times \lambda \rightarrow \rho$ there is the so-called polarized partition relation

$$\begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \rightarrow \begin{pmatrix} \mu \\ \nu \end{pmatrix}_\rho$$

which holds iff for all $f : \kappa \times \lambda \rightarrow \rho$ there are $A \subseteq \kappa$ and $B \subseteq \lambda$ such that $otp(A) = \mu$ and $otp(B)\nu$ and f is constant on $A \times B$.

We write

$$\begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \nrightarrow \begin{pmatrix} \mu \\ \nu \end{pmatrix}_\rho$$

for the negation of this statement.

Theorem 1.3 (Baumgartner-Hajnal)

$$\begin{pmatrix} (2^{<\kappa})^{++} \\ (2^{<\kappa})^+ \end{pmatrix} \rightarrow \begin{pmatrix} \kappa \\ \kappa \end{pmatrix}_\gamma$$

for all infinite cardinals κ and all $\gamma < \kappa$.

Proof: See Baumgartner, Hajnal: "Polarized partition relations", JSL 66 (2001), 811 - 821. $\square$

Many questions can be understood as questions on colorings. An example is: Does there exist (at least consistently) a family of functions $f_\alpha : \omega_1 \rightarrow \omega$ such that $\{\xi < \omega_1 \mid f_\alpha(\xi) \neq f_\beta(\xi)\}$ is finite for all $\alpha \neq \beta \in \omega_2$? Obviously, the existence of such a family implies

$$\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \nrightarrow \begin{pmatrix} 2 \\ \omega \end{pmatrix}_\omega.$$

Another example is:

Is it consistent that there exists a chain $\langle X_\alpha \mid \alpha < \omega_2 \rangle$ such that $X_\alpha \subseteq \omega_1$, $X_\beta - X_\alpha$ is finite and $X_\alpha - X_\beta$ has size ω_1 for all $\beta < \alpha < \omega_2$?

As we will see, both consistency questions can be answered positively.

We will approach such questions by a method which we call "higher-dimensional forcing". The basic idea is to generalize iterated forcing.

Let us recall the following basic facts about iterated forcing (see e.g. Kunen's textbook):

Let $\mathbb{P}$ be a forcing notion and $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name such that $\mathbb{P} \Vdash (\dot{\mathbb{Q}}$ is a forcing).

Let $\mathbb{P} * \dot{\mathbb{Q}} = \{\langle p, \dot{q}\rangle \mid p \in \mathbb{P}$ and $\mathbb{P} \Vdash \dot{q} \in \dot{\mathbb{Q}}\}$. Define a partial order on $\mathbb{P} * \dot{\mathbb{Q}}$ by setting

$$\langle p_1, \dot{q}_1 \rangle \leq \langle p_2, \dot{q}_2 \rangle \text{ iff } p_1 \leq p_2 \text{ and } p_1 \Vdash \dot{q}_1 \leq \dot{q}_2.$$

Define $i : \mathbb{P} \rightarrow \mathbb{P} * \dot{\mathbb{Q}}$, $p \mapsto \langle p, \dot{1}_{\mathbb{Q}}\rangle$.

Then $i : \mathbb{P} \rightarrow \mathbb{P} * \dot{\mathbb{Q}}$ is a complete embedding, i.e.
(1) $\forall p, p' \in \mathbb{P}$ $(p' \leq p \rightarrow i(p') \leq i(p))$
(2) $\forall p, p' \in \mathbb{P}$ $(p' \perp p \rightarrow i(p') \perp i(p))$
(3) $\forall q \in \mathbb{P} * \dot{\mathbb{Q}}$ $\exists p \in \mathbb{P}$ $\forall p' \in \mathbb{P}$ $(p' \leq p \rightarrow i(p')$ and q are compatible in $\mathbb{P} * \dot{\mathbb{Q}})$.

In (3), we call p a reduction of q to $\mathbb{P}$.

Assume conversely that $i : \mathbb{P} \rightarrow \mathbb{Q}$ is a complete embedding. Then there exists a $\mathbb{P}$-name $\dot{\mathbb{B}}$ such that

$$\mathbb{Q} \simeq \mathbb{P} * \dot{\mathbb{B}}.$$

A finite support iteration of length $\lambda \in Lim$ is a system $\langle \sigma_{\eta\nu} \mid \eta < \nu \leq \lambda\rangle$ of complete embeddings $\sigma_{\eta\nu} : \mathbb{P}_\eta \rightarrow \mathbb{P}_\nu$ between partial orders which is
(1) commutative, i.e. $\sigma_{\eta\gamma} = \sigma_{\nu\gamma} \circ \sigma_{\eta\nu}$ for all $\eta < \nu < \gamma \leq \lambda$
(2) continuous, i.e. $\mathbb{P}_\gamma = \bigcup\{\sigma_{\eta\gamma}[\mathbb{P}_\eta] \mid \eta < \gamma\}$ for all $\gamma \leq \lambda$ with $\gamma \in Lim$.

Suppose that $\langle \sigma_{\eta\nu} : \mathbb{P}_\eta \rightarrow \mathbb{P}_\nu \mid \eta < \nu \leq \lambda\rangle$ is a finite support iteration. Then there exist sequences $\langle \mathbb{Q}_\nu \mid \nu \leq \lambda\rangle$ and $\langle \dot{\mathbb{B}} \mid \nu < \lambda\rangle$ such that:
(1) $\mathbb{P}_\nu \simeq \mathbb{Q}_\nu$ for all $\nu \leq \lambda$
(2) $\mathbb{Q}_\nu$ is a forcing notion for all $\nu \leq \lambda$ and $\dot{\mathbb{B}}_\nu$ is a $\mathbb{Q}_\nu$-name such that $\mathbb{Q}_\nu \Vdash (\dot{\mathbb{B}}_\nu$ is a forcing).
(3) If $\nu < \lambda$, then

$$\mathbb{Q}_{\nu+1} = \{p : \nu + 1 \rightarrow \nu \mid p \restriction \mathbb{Q}_\nu \ \wedge \ \mathbb{Q}_\nu \Vdash p(\nu) \in \dot{\mathbb{B}}_\nu\}$$

and

$$p \leq_{\nu+1} q \text{ iff } p \restriction \nu \leq_\nu q \restriction \nu \text{ and } p \restriction \nu \Vdash p(\nu) \leq q(\nu).$$

(4) If $\nu \leq \lambda$ and $\nu \in Lim$, then

$$\mathbb{Q}_\nu = \{p : \nu \to V \mid \forall \gamma < \nu \; p \restriction \gamma \in \mathbb{Q}_\gamma$$

$$\text{and for all but finitely many } \gamma \; \mathbb{Q}_\gamma \Vdash p(\gamma) = \dot{1}_\gamma\}$$

and

$$p \leq_\nu q \text{ iff } p \restriction \gamma \leq_\gamma q \restriction \gamma \text{ for all } \gamma < \nu.$$

This explains why $\langle \sigma_{\eta\nu} : \mathbb{P}_\eta \to \mathbb{P}_\nu \mid \eta < \nu \leq \lambda \rangle$ is called a finite support iteration. For $p \in \mathbb{Q}_\lambda$, the finite set

$$supp(p) = \{\gamma < \lambda \mid \mathbb{P}_\gamma \nVdash p(\gamma) = \dot{1}_\gamma\}$$

is called the support of p.

Theorem 1.5

Assume that $\mathbb{P}$ is a partial order which satisfies the κ-cc for a regular $\kappa \geq \omega$. Assume moreover that $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a partial order such that $\mathbb{P} \Vdash (\dot{\mathbb{Q}}$ satisfies the $\check{\kappa}$-cc). Then $\mathbb{P} * \dot{\mathbb{Q}}$ satisfies the κ-cc.

Proof: See for example Kunen's book. $\square$

Theorem 1.6

Assume that $\langle \sigma_{\eta\nu} : \mathbb{P}_\eta \to \mathbb{P}_\nu \mid \eta < \nu \leq \lambda \rangle$ is a finte support iteration of length $\lambda \in Lim$ such that every $\mathbb{P}_\nu$ with $\nu < \lambda$ satisfies the κ-cc for a regular $\kappa \geq \omega$. Then $\mathbb{P}_\lambda$ also satifies the κ-cc.

Proof: Let $\langle \mathbb{Q}_\nu \mid \nu \leq \lambda \rangle$, $\langle \dot{B}_\nu \mid \nu < \lambda \rangle$ be like above. We prove that $\mathbb{Q}_\lambda$ satisfies κ-cc. Assume that $card(A) = \kappa$, $A \subseteq \mathbb{Q}_\lambda$. We may assume by the Δ-system lemma, that $\{supp(p) \mid p \in A\}$ forms a Δ-system with root Δ. Set $\gamma = max(\Delta)$. Since $\mathbb{P}_\gamma$ satisfies the κ-cc, there are $p \neq q$ in A such that $p \restriction \gamma$ and $q \restriction \gamma$ are compatible in $\mathbb{Q}_\gamma$. Hence p, q are compatible in $\mathbb{Q}_\lambda$. $\square$

As we will see, there is also a proof which avoids the use of $\langle \mathbb{Q}_\nu \mid \nu \leq \lambda \rangle$ and $\langle \dot{B}_\nu \mid \nu < \lambda \rangle$.

We will call such an iteration sometimes a "linear" or "one-dimensional" iteration.
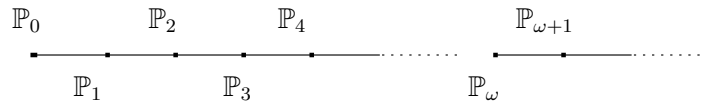
Higher-dimensional forcing generalizes this to higher dimensions:

$$\mathbb{P}_{\gamma\tau}$$
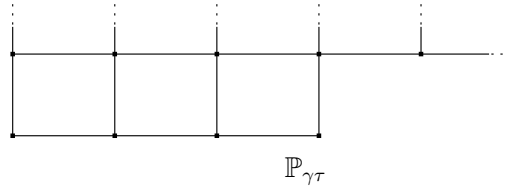
To do this we need an appropriate structure to replace the ordinal λ, i.e. a structure whose elements we can use to index the partial orders and along which we can define things by induction. The appropriate structures are simplified morasses. The "two-dimensional" morass is called gap-1 morass.

Historical remarks and references

The study of partition relations was started by B. Dushnik and E. W. Miller, who tried to generalize Ramsey's theorem. P. Erdös and R. Rado began a systematic exploration [10, 9, 7, 8] of the arrow relation $\kappa \rightarrow (\lambda)^n_\mu$, which was invented by Rado. The proof of the Erdös-Rado theorem which we presented here is the proof which is given by T. Jech in his textbook [21]. We used it because it is purely combinatorial and avoids the use of elementary substructures. An up to date overview of the field of partition relations is given in the chapter [15] of A. Hajnal and J. Larson for the Handbook of Set Theory. The reference for the Baumgartner-Hajnal theorem is [2].

The idea of iterated forcing was developed by S. Tennenbaum and R. M. Solovay [37] and is nowadays an indispensable tool for every set theorist. A modern approach is given in Kunen's textbook [31]. Another approach to non linear forcing iterations was developed by M. Groszek and T. Jech [13].
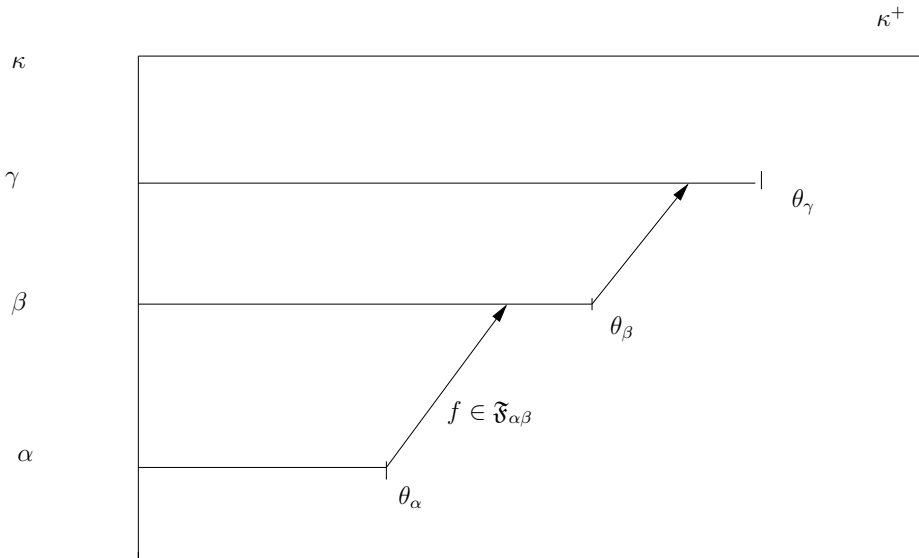
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Gap-1 morasses

A simplified $(\kappa, 1)$-morass is a structure $\mathfrak{M} = \langle\langle\theta_\alpha \mid \alpha \leq \kappa\rangle, \langle\mathfrak{F}_{\alpha\beta} \mid \alpha < \beta \leq \kappa\rangle\rangle$ satisfying the following conditions:

(P0) (a) $\theta_0 = 1$, $\theta_\kappa = \kappa^+$, $\forall \alpha < \kappa$ $0 < \theta_\alpha < \kappa$.

(b) $\mathfrak{F}_{\alpha\beta}$ is a set of order-preserving functions $f : \theta_\alpha \to \theta_\beta$.

(P1) $|\mathfrak{F}_{\alpha\beta}| < \kappa$ for all $\alpha < \beta < \kappa$.

(P2) If $\alpha < \beta < \gamma$, then $\mathfrak{F}_{\alpha\gamma} = \{f \circ g \mid f \in \mathfrak{F}_{\beta\gamma}, g \in \mathfrak{F}_{\alpha\beta}\}$.
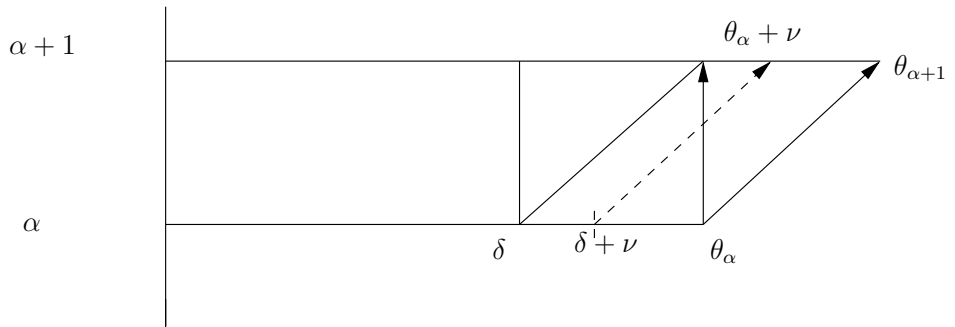

(P3) If $\alpha < \kappa$, then $\mathfrak{F}_{\alpha,\alpha+1} = \{id \restriction \theta_\alpha, f_\alpha\}$ where f_α is such that $f_\alpha \restriction \delta = id \restriction \delta$ and $f_\alpha(\delta) \geq \theta_\alpha$ for some $\delta < \theta_\alpha$.

(P4) If $\alpha \leq \kappa$ is a limit ordinal, $\beta_1, \beta_2 < \alpha$ and $f_1 \in \mathfrak{F}_{\beta_1\alpha}$, $f_2 \in \mathfrak{F}_{\beta_2\alpha}$, then
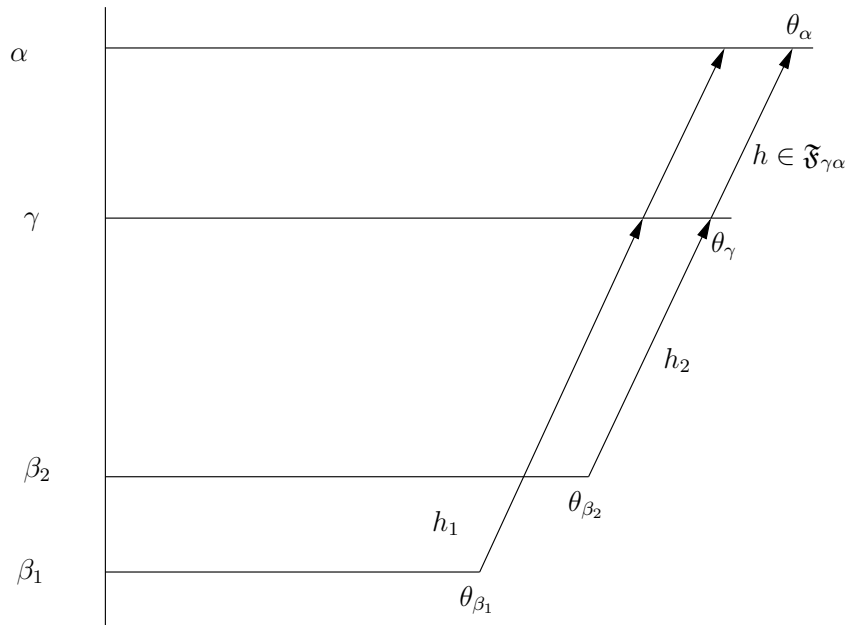
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there are a $\beta_1, \beta_2 < \gamma < \alpha$, $g \in \mathfrak{F}_{\gamma\alpha}$ and $h_1 \in \mathfrak{F}_{\beta_1\gamma}$, $h_2 \in \mathfrak{F}_{\beta_2\gamma}$ such that $f_1 = g \circ h_1$ and $f_2 = g \circ h_2$.

(P5) For all $\alpha > 0$, $\theta_\alpha = \bigcup\{f[\theta_\beta] \mid \beta < \alpha, f \in \mathfrak{F}_{\beta\alpha}\}$.

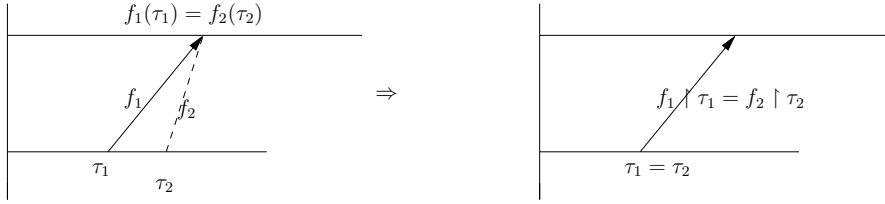By (P3) and (P5) our picture looks in the successor step as follows:

The picture for (P4) is the following:

Lemma 2.1

Let $\alpha < \beta \leq \kappa$, $\tau_1, \tau_2 < \theta_\alpha$, $f_1, f_2 \in \mathfrak{F}_{\alpha\beta}$ and $f_1(\tau_1) = f_2(\tau_2)$. Then $\tau_1 = \tau_2$ and $f_1 \restriction \tau_1 = f_2 \restriction \tau_2$.

Note, that this is clear for the case $\beta = \alpha + 1$.

Proof by induction over β. The base case of the induction is $\beta = \alpha + 1$. Then the claim follows immediately from (P3). So assume that $\beta = \gamma + 1$. Let, by (P2), $f_i = g_i \circ f_i'$ where $f_i' \in \mathfrak{F}_{\alpha\gamma}$, $g_i \in \mathfrak{F}_{\gamma\beta}$. Let $\tau_i' = f_i'(\tau_i)$. It follows like in the base case that $\tau_1' = \tau_2'$ and $f_1' \restriction \tau_1 = f_2' \restriction \tau_2$. So, by the induction hypothesis, $\tau_1 = \tau_2$ and $f_1' \restriction \tau_1 = f_2' \restriction \tau_2$. Hence $f_1 \restriction \tau_1 = f_2 \restriction \tau_2$.

Finally, let $\beta \in Lim$. Then there exists by (P4) $\alpha < \gamma < \beta$ and $g \in \mathfrak{F}_{\gamma\beta}$ such that $f_i = g \circ f_i'$, $f_i' \in \mathfrak{F}_{\alpha\gamma}$. So $f_1'(\tau_1) = f_2'(\tau_2)$. Hence $\tau_1 = \tau_2$ and $f_1' \restriction \tau_1 = f_2' \restriction \tau_2$ by the induction hypothesis. Therefore $f_1 \restriction \tau_1 = f_2 \restriction \tau_2$. $\square$
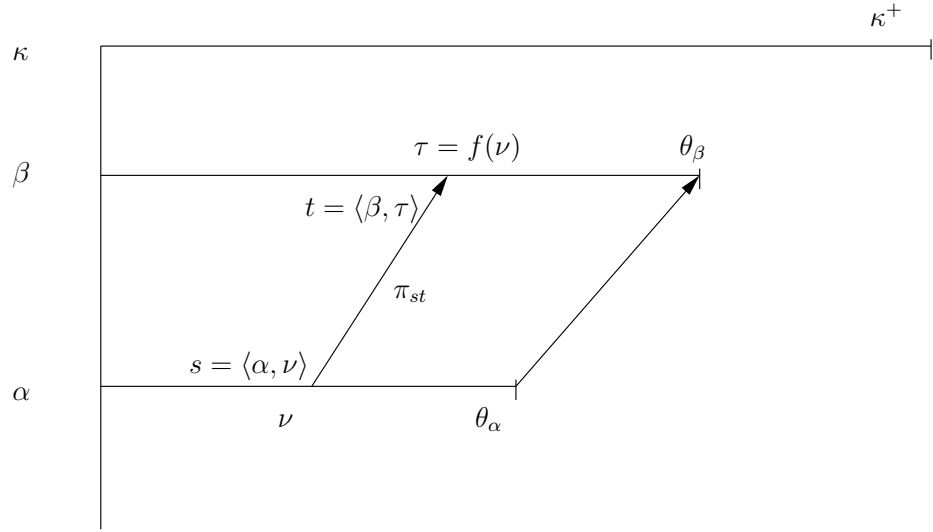
A simplified morass defines a tree $\langle T, \prec \rangle$:

Let $T = \{ \langle \alpha, \gamma \rangle \mid \alpha \leq \kappa, \gamma < \theta_\alpha \}$.

For $t = \langle \alpha, \nu \rangle \in T$ set $\alpha(t) = \alpha$ and $\nu(t) = \nu$.

Let $\langle \alpha, \nu \rangle \prec \langle \beta, \tau \rangle$ iff $\alpha < \beta$ and $f(\nu) = \tau$ for some $f \in \mathfrak{F}_{\alpha\beta}$.
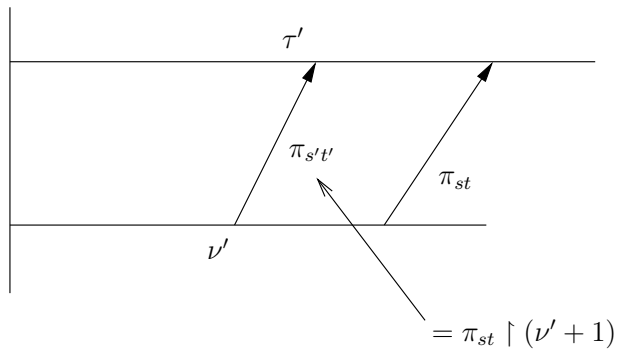
If $s := \langle \alpha, \nu \rangle \prec \langle \beta, \tau \rangle =: t$, $f \in \mathfrak{F}_{\alpha\beta}$ and $f(\nu) = \tau$, then $f \restriction (\nu(s) + 1)$ does not depend on f by lemma 3.1. So we may define $\pi_{st} := f \restriction (\nu(s) + 1)$.

Lemma 2.2
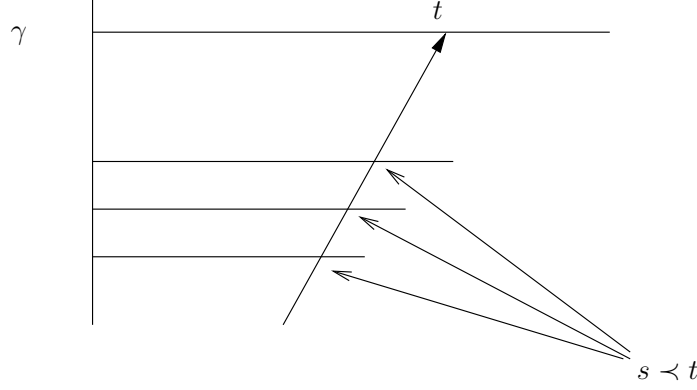
The following hold:

(a) $\prec$ is a tree, $ht_T(t) = \alpha(t)$.

(b) If $t_0 \prec t_1 \prec t_2$, then $\pi_{t_0 t_1} = \pi_{t_1 t_2} \circ \pi_{t_0 t_1}$.

(c) Let $s \prec t$ and $\pi = \pi_{st}$. If $\pi(\nu') = \tau'$, $s' = \langle \alpha(s), \nu' \rangle$ and $t' = \langle \alpha(t), \tau' \rangle$, then $s' \prec t'$ and $\pi_{s't'} = \pi \restriction (\nu' + 1)$.
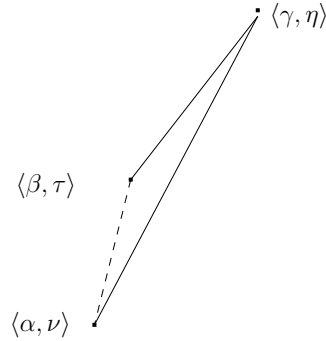
(d) Let $\gamma \leq \kappa$, $\gamma \in Lim$. Let $t \in T_\gamma$. Then $\nu(t) + 1 = \bigcup\{rng(\pi_{st}) \mid s \prec t\}$.

Proof: (a) First, we prove that $\prec$ is transitive. Let $\langle \alpha, \nu \rangle \prec \langle \beta, \tau \rangle$ be witnessed by $f \in \mathfrak{F}_{\alpha\beta}$ and $\langle \beta, \tau \rangle \prec \langle \gamma, \eta \rangle$ by $g \in \mathfrak{F}_{\alpha\beta}$. Set $h = g \circ f \in \mathfrak{F}_{\alpha\gamma}$ by (P2). Then $h(\nu) = \eta$. So $\langle \alpha, \nu \rangle \prec \langle \gamma, \eta \rangle$.

Now, let $\langle \alpha, \nu \rangle, \langle \beta, \tau \rangle \prec \langle \gamma, \eta \rangle$ and $\langle \alpha, \nu \rangle \neq \langle \beta, \tau \rangle$.

It follows from lemma 3.1 that $\alpha \neq \beta$. Let w.l.o.g. $\alpha < \beta$. Let $\langle \alpha, \nu \rangle \prec \langle \gamma, \eta \rangle$ be witnessed by $f \in \mathfrak{F}_{\alpha\gamma}$. By (P2) choose $g \in \mathfrak{F}_{\beta\gamma}$ and $h \in \mathfrak{F}_{\alpha\beta}$ such that $f = g \circ h$. Then $\langle \alpha, \nu \rangle \prec \langle \beta, h(\nu) \rangle \prec \langle \gamma, \eta \rangle$. However, $h(\nu) = \tau$ by lemma 3.1. Hence $\langle \alpha, \nu \rangle \prec \langle \beta, \tau \rangle$. This proves that $\prec$ is a tree.

Finally, by (P2), for all $t \in T$ there is $s \prec t$ such that $\alpha(s) = \beta$ if $\beta < \alpha(t)$. This shows the second claim.

(b) follows immediately from (a) and the definition.

(c) Let $s \prec t$ be witnessed by $f \in \mathfrak{F}_{\alpha\beta}$. Then $s' \prec t'$ is also witnessed by f and $\pi_{s't'} = \pi \restriction (\nu' + 1)$ holds by definition.

(d) It suffices to prove $\subseteq$. Let $\nu = \nu(t)$ and $\tau < \nu$. By (P5) choose $\alpha_1, \alpha_2 < \gamma$

and $f_i \in \mathfrak{F}_{\alpha_i\gamma}$ such that $\tau \in rng(f_1)$ and $\nu \in rng(f_2)$. By (P4) choose β such that $\alpha_1, \alpha_2 < \beta < \gamma$ and $f'_i \in \mathfrak{F}_{\alpha_i\beta}$, $g \in \mathfrak{F}_{\beta\gamma}$ where $f_i = g \circ f'_i$. Then $\tau, \nu \in rng(g)$. So let $g(\bar{\tau}) = \tau$ and $g(\bar{\nu}) = \nu$. Hence $\bar{\tau} < \bar{\nu}$, since g is order-preserving. Let $s = \langle \beta, \bar{\nu} \rangle$. Then $s \prec t$ and $\pi_{st}(\bar{\tau}) = \tau$. $\square$
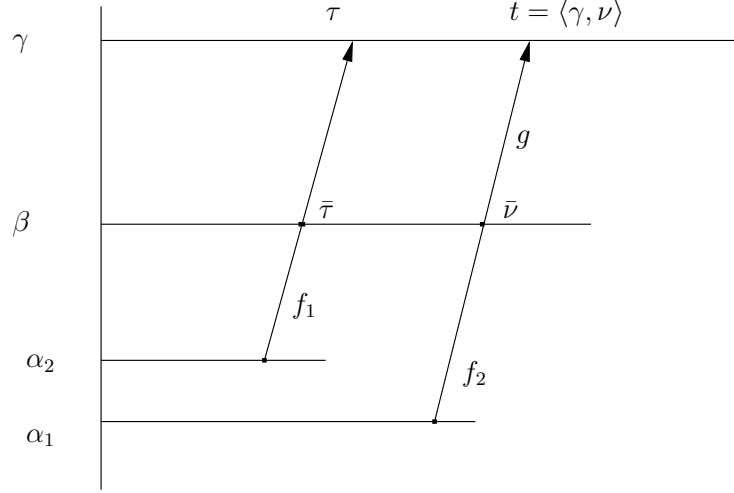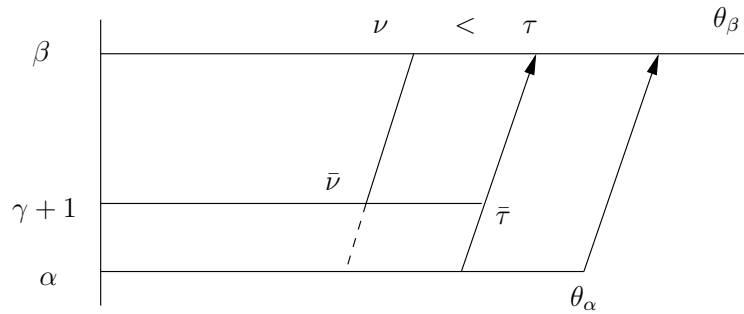

Lemma 2.3

Let $\alpha < \beta \leq \kappa$. Then $id \upharpoonright \theta_\alpha \in \mathfrak{F}_{\alpha\beta}$.

Proof by induction on β. The base case of the induction is $\beta = \alpha + 1$. Then the claim is part of (P3). So assume that $\beta = \gamma + 1$. By the induction hypothesis, $id \upharpoonright \theta_\alpha \in \mathfrak{F}_{\alpha\gamma}$. By (P3), $id \upharpoonright \theta_\gamma \in \mathfrak{F}_{\gamma\beta}$. Hence $id \upharpoonright \theta_\alpha = (id \upharpoonright \theta_\gamma) \circ (id \upharpoonright \theta_\alpha) \in \mathfrak{F}_{\alpha\beta}$ by (P2).

Finally, let $\beta \in Lim$. Assume towards a contradiction that $id \upharpoonright \theta_\alpha \notin \mathfrak{F}_{\alpha\beta}$. Let $f \in \mathfrak{F}_{\alpha\beta}$ be such that $sup(f[\theta_\alpha])$ is minimal. Since $f \neq id \upharpoonright \theta_\alpha$, there are $\nu < \tau \in \theta_\beta$ such that $\nu \notin rng(f)$ and $\tau \in rng(f)$.

Let $t = \langle \beta, \tau \rangle$. By lemma 3.2 (d), there is an $s \prec t$ such that $\nu \in rng(\pi_{st})$. Let $s = \langle \gamma + 1, \bar{\tau} \rangle$ be the minimal such s. Let $\nu = \pi_{st}(\bar{\nu})$. Furthermore, let $f = f_3 \circ f_2 \circ f_1$ where $f_3 \in \mathfrak{F}_{\gamma+1,\beta}$, $f_2 \in \mathfrak{F}_{\gamma,\gamma+1}$ and $f_1 \in \mathfrak{F}_{\alpha\gamma}$. Then $\pi_{st} = f_3 \upharpoonright \bar{\tau} + 1$. Hence by the minimality of s, $f_2 \neq id \upharpoonright \theta_\gamma$, $\bar{\nu} < \theta_\gamma$ and $\bar{\tau} \geq \theta_\gamma$.
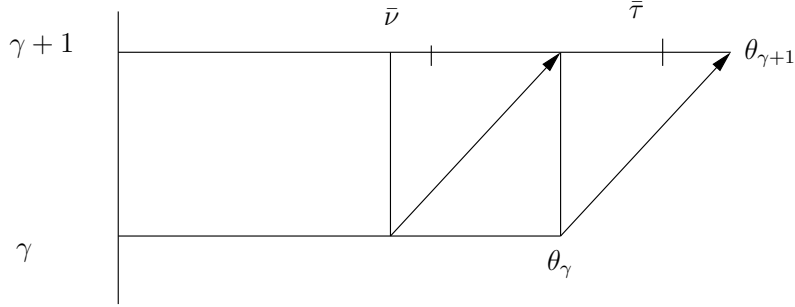
Define $g := f_3 \circ (id \upharpoonright \theta_\gamma) \circ f_1$. Then $g \in \mathfrak{F}_{\alpha\gamma}$ by (P2) and $rng(g) \subseteq f_3[\theta_\gamma] \subseteq f_3(\bar{\tau}) = \tau$. Hence $sup(f[\theta_\alpha])$ was not minimal. Contradiction! $\square$

Theorem 2.4

(a) If $V = L$, then there is a simplified $(\kappa, 1)$-morass for all regular $\kappa > \omega$.

(b) If κ is an uncountable regular cardinal such that κ^+ is not inaccessible in L, then there is a simplified $(\kappa, 1)$-morass.

(c) For every regular $\kappa > \omega$, there is a κ-complete (i.e. every decreasing sequence of length $< \kappa$ has a lower bound) forcing $\mathbb{P}$ satisfying κ^+-cc such that $\mathbb{P} \Vdash$ (there is a simplified $(\kappa, 1)$-morass).

Theorem 2.5

There exists a simplified $(\omega, 1)$-morass.

The consistency results of two-dimensional forcing can often be also obtained by forcing constructions which are based on ordinal walks as described in S. Todorcevic's book "Walks on ordinals and their characteristics. Such constructions assume $\square_{\omega_1}$ instead of the existence of a simplified $(\omega_1, 1)$-morass.

Theorem 2.6

Assume that there exists a simplified $(\omega_1, 1)$-morass. Then $\square_{\omega_1}$ holds, i.e. there exists a sequence $\langle C_\alpha \mid \alpha \in Lim \cap \omega_2 \rangle$ such that for all $\alpha \in Lim \cap \omega_2$:

(i) $C_\alpha \subseteq \alpha$ is club.

(ii) $\forall \beta \in Lim(C_\alpha) \; C_\beta = C_\alpha \cap \beta$.

(iii) If $cf(\alpha) < \omega_1$, then $card(C_\alpha) < \omega_1$.

Proof: Let $\mathfrak{M} = \langle \langle \theta_\alpha \mid \alpha \leq \omega_1 \rangle, \langle \mathfrak{F}_{\alpha\beta} \mid \alpha < \beta \leq \omega_1 \rangle \rangle$ be a simplified $(\omega_1, 1)$-morass. We construct a sequence $\langle C_{\omega\nu} \mid \nu \in \omega_2 \rangle$ by induction over the levels of $\mathfrak{M}$ which we enumerate by $\beta \leq \omega_1$ such that

(i) $C_{\omega\nu} \subseteq \omega\nu$ is unbounded in $\omega\nu$.

(ii) $\omega\beta \in acc(C_{\omega\nu}) := \{\gamma \mid \gamma = sup(C_{\omega\nu} \cap \gamma)\} \Rightarrow C_{\omega\beta} = C_{\omega\nu} \cap \omega\beta$

(iii) $cf(\omega\nu) < \omega_1 \Rightarrow card(C_{\omega\nu} < \omega_1$.

From $\langle C_{\omega\nu} \mid \nu \in \omega_2 \rangle$ we get a $\square_{\omega_1}$-sequence $\langle \tilde{C}_{\omega\nu} \mid \nu \in \omega_2 \rangle$ by setting $\tilde{C}_{\omega\nu} = C_{\omega\nu} \cup acc(C_{\omega\nu})$.
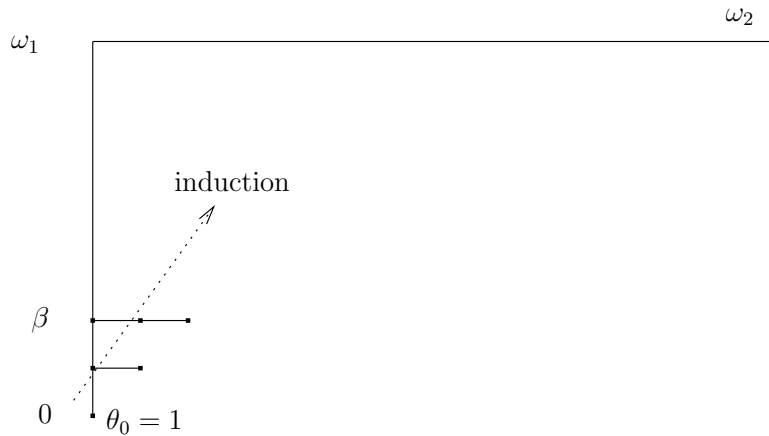
For $\pi_{st} : \nu(s) + 1 \rightarrow \nu(t) + 1$ define

$$\tilde{\pi}_{st} : \omega(\nu(s) + 1) \rightarrow \omega(\nu(t) + 1), \quad \omega\alpha + n \mapsto \omega\pi_{st}(\alpha) + n$$
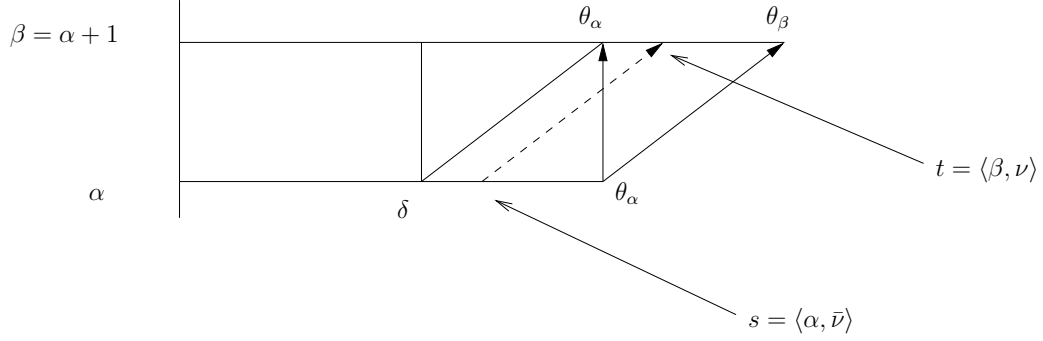
for all $n \in \omega$.

Base case: $\beta = 0$

Since by (P0) $\theta_0 = 1$, we only need to define C_0. Set $C_0 = \emptyset$.

Successor case: $\beta = \alpha + 1$

Hence $C_{\omega\nu}$ is already defined for all $\nu < \theta_\alpha$. For $\theta_\alpha < \nu < \theta_\beta$ set $t_{\langle\beta,\nu\rangle}$. Let $s \prec t$, $s \in T_\alpha$ and $s = \langle\alpha, \bar{\nu}\rangle$. Then we set $C_{\omega\nu} = \tilde{\pi}_{st}[C_{\omega\bar{\nu}}]$.

It remains to define $C_{\omega\nu}$ for $\nu = \theta_\alpha$. Let $f \in \mathfrak{F}_{\alpha\beta}$, $f \neq id$, $\delta = crit(f)$. In this case let $C \subseteq \omega\nu$ be any unbounded set with $C \subseteq [\omega\delta, \omega\theta_\alpha)$ and $otp(C) = \omega$. We set $C_{\omega\nu} = C_{\omega\delta} \cup C$.
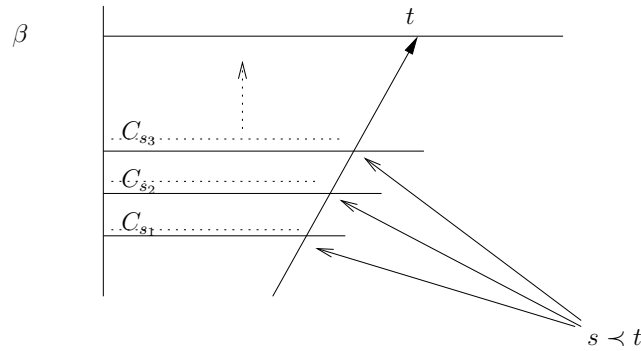
Limit case: $\beta \in Lim$

For $v < \theta_\beta$ set $t = \langle\beta, \nu\rangle$. Set

$$C_{\omega\nu} = \bigcup\{\tilde{\pi}_{st}[C_{\omega\nu(s)}] \mid s \prec t\}.$$

This is well-defined, because if $\nu < \theta_\alpha$ for some $\alpha < \beta$, then $\pi_{st} = id \upharpoonright \nu(s)+1$ for all $s \prec t$.

We have to prove that (i) - (iii) from above hold. We do this by induction over $\beta \leq \omega_1$. Moreover, we check

(iv): If $s \prec t$, then $C_{\omega\nu(t)}$ is an end-extension of $\tilde{\pi}_{st}[C_{\omega\nu(s)}]$, i.e. $\tilde{\pi}_{st}[C_{\omega\nu(s)}] = C_{\omega\nu(t)} \cap \gamma$ for some $\gamma \in On$.

Base case: $\beta = 0$
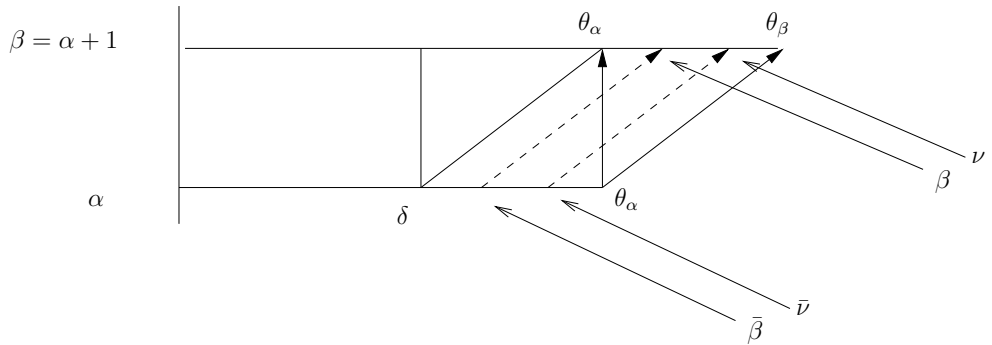
Trivial.

Sucessor case: $\beta = \alpha + 1$

Let $f \in \mathfrak{F}_{\alpha\beta}$, $f \neq id$, $\delta = crit(f)$.

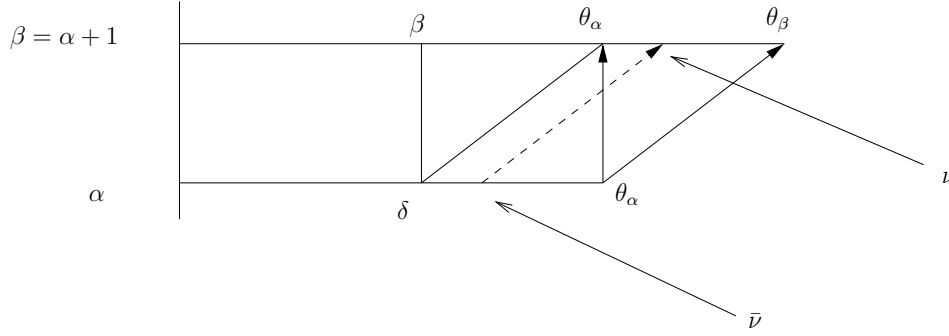We have to prove (i) - (iv) for $\theta_\alpha \leq \nu < \theta_{\alpha+1}$.

(i) If $\nu = \theta_\alpha$, then $C_{\omega\nu}$ is unbounded in $\omega\nu$ by definition. So let $\theta_\alpha < \nu < \theta_{\alpha+1}$. Let $f(\bar{\nu}) = \nu$. By the induction hypothesis $C_{\omega\bar{\nu}}$ is unbounded in $\omega\bar{\nu}$. Hence $C_{\omega\nu} = \tilde{f}[C_{\omega\bar{\nu}}]$ is unbounded in $\omega\nu$ by (P3) and (P5).

(ii) Assume $\nu = \theta_\alpha$ and $\omega\beta \in acc(C_{\omega\nu})$. Then either $\omega\beta = \omega\delta$ or $\omega\beta < \omega\delta$. If $\omega\beta = \omega\delta$, then by definition $C_{\omega\beta} = C_{\omega\nu} \cap \omega\beta$. So assume that $\omega\beta < \omega\delta$. Then $C_{\omega\nu} \cap \omega\beta = C_{\omega\delta} \cap \omega\beta$ and $\omega\beta \in acc(C_{\omega\delta})$. So by the induction hypothesis $C_{\omega\nu} \cap \omega\beta = C_{\omega\delta} \cap \omega\beta = C_{\omega\beta}$.

Now, let $\theta_\alpha < \nu < \theta_{\alpha+1}$ and $\omega\beta \in acc(C_{\omega\nu})$. Then by (P3) either $\omega\beta \in rng(\tilde{f})$ or $\omega\beta = \omega\delta$. Let $f(\bar{\nu}) = \nu$. Then $C_{\omega\nu} = \tilde{f}[C_{\omega\bar{\nu}}]$. If $\omega\beta \in rng(\tilde{f})$, let $f(\bar{\beta}) = \beta$. By the induction hypothesis $C_{\omega\bar{\nu}} \cap \omega\bar{\beta} = C_{\omega\bar{\beta}}$. Hence $C_{\omega\beta} = \tilde{f}[C_{\omega\bar{\beta}}] = \tilde{f}[C_{\omega\bar{\nu}} \cap \omega\bar{\beta}] = \tilde{f}[C_{\omega\bar{\nu}}] \cap \omega\beta = C_{\omega\nu} \cap \omega\beta$ where the first equality holds because $\omega\beta \neq \omega\theta_\alpha$.

Assume $\omega\beta = \omega\delta$. Let $f(\bar{\nu}) = \nu$.

Then by definition $C_{\omega\nu} = \tilde{f}[C_{\omega\bar{\nu}}]$. Hence by (P3) $C_{\omega\bar{\nu}} \cap \omega\beta = C_{\omega\nu} \cap \omega\beta$ and $\omega\beta \in acc(C_{\omega\bar{\nu}})$. By the induction hypothesis $C_{\omega\bar{\nu}} \cap \omega\beta = C_{\omega\beta}$. Hence $C_{\omega\nu} \cap \omega\beta = C_{\omega\bar{\nu}} \cap \omega\beta = C_{\omega\beta}$.

(iii) $\beta < \omega_1$. Hence (iii) is trivial by (P0).

(iv) Clear by the definition.

Limit case: $\beta \in Lim$

Let $\nu < \theta_\beta$, $t = \langle \beta, \nu \rangle$. We have to prove (i) - (iv) for

$$C_{\omega\nu} = \{\tilde{\pi}_{st}[C_{\omega\nu(s)}] \mid s \prec t\}.$$

For $s \prec t$ set

$$C_s = \tilde{\pi}[C_{\omega\nu(s)}].$$

If $s \prec s' \prec t$, then by (iv) of the induction hypothesis $C_{\omega\nu(s')}$ is an end-extension of $\tilde{\pi}_{ss'}[C_{\omega\nu(s)}]$. So $C_{s'} = \tilde{\pi}_{s't}[C_{\omega\nu(s')}]$ is an end-extension of $C_s = \tilde{\pi}_{s't}[\tilde{\pi}_{ss'}[C_{\omega\nu(s)}]]$. Hence $C_{\omega\nu} = \bigcup\{C_s \mid s \prec t\}$ is an end-extension of $C_{s'} = \tilde{\pi}_{s't}[C_{\omega\nu(s')}]$. This proves (iv).
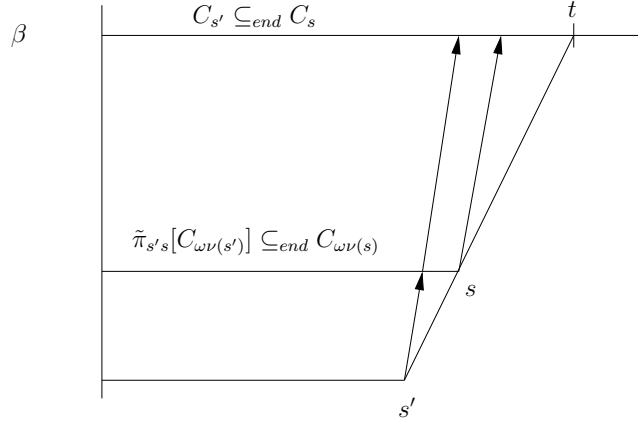
(i) Let $\eta < \omega\nu$. We have to show that there exists $\eta < \gamma \in C_{\omega\nu}$. Since $\nu = \bigcup\{\pi_{st}[\nu(s)] \mid s \prec t\}$, there exists $s \prec t$ such that $\eta \in rng(\tilde{\pi}_{st})$. By the induction hypothesis $C_{\omega\nu(s)}$ is unbounded in $\omega\nu(s)$.
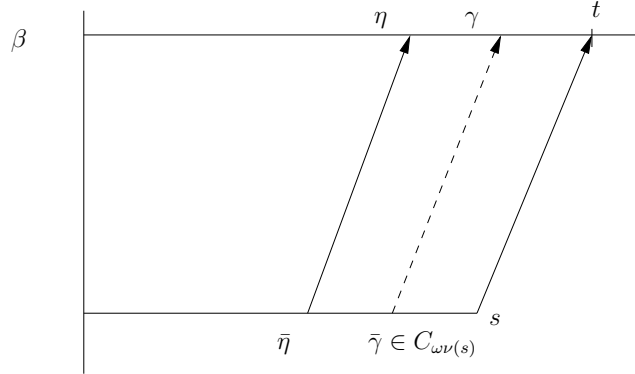
Let $\tilde{\pi}_{st}(\bar{\eta}) = \eta$. Choose $\bar{\gamma} \in C_{\omega\nu(s)}$ such that $\bar{\eta} < \bar{\gamma}$. Set $\gamma = \tilde{\pi}_{st}(\bar{\gamma})$. Then $\eta < \gamma \in \tilde{\pi}_{st}[C_{\omega\nu(s)}] \subseteq C_{\omega\nu}$.

(ii) Let $\omega\gamma \in acc(C_{\omega\nu})$. Since $\nu = \bigcup\{\pi_{st}[\nu(s)] \mid s \prec t\}$, we can pick $s \prec t$ such that $\omega\gamma \in rng(\tilde{\pi}_{st})$. By (iv), $\omega\gamma \in acc(\tilde{\pi}_{st}[C_{\omega\nu(s)}])$. Let $\pi_{st}(\bar{\gamma}) = \gamma$. Then $\omega\bar{\gamma} \in acc(C_{\omega\nu(s)})$. So $C_{\omega\bar{\gamma}} = C_{\omega\nu(s)} \cap \omega\bar{\gamma}$ by the induction hypothesis. Let $t' = \langle\beta, \gamma\rangle$ and $s' = \langle\alpha(s), \bar{\gamma}\rangle$. Then $\pi_{s't'} = \pi_{st} \upharpoonright \nu(s) + 1$. Hence $C_{\omega\gamma} = \tilde{\pi}_{s't'}[C_{\omega\bar{\gamma}}] = \tilde{\pi}_{st}[C_{\omega\bar{\gamma}}] = \tilde{\pi}_{st}[C_{\omega\nu(s)} \cap \omega\bar{\gamma}] = \tilde{\pi}[C_{\omega\nu(s)}] \cap \omega\gamma = C_{\omega\nu} \cap \omega\gamma$ where the first and last equality hold by (iv).

Finally, we prove (iii). Suppose $cf(\nu) < \omega_1$. If $\nu < \omega_1$, then (iii) is clear. So suppose that $\omega_1 \leq \nu$. Then $C_{\omega\nu}$ is defined in step $\beta = \omega_1$ in the recursion. Hence $t = \langle \omega_1, \nu \rangle$. By a previous lemma, $\omega\nu = \bigcup \{ \tilde{\pi}_{st}[\omega\nu(s)] \mid s \prec t \}$. Since $cf(\nu) 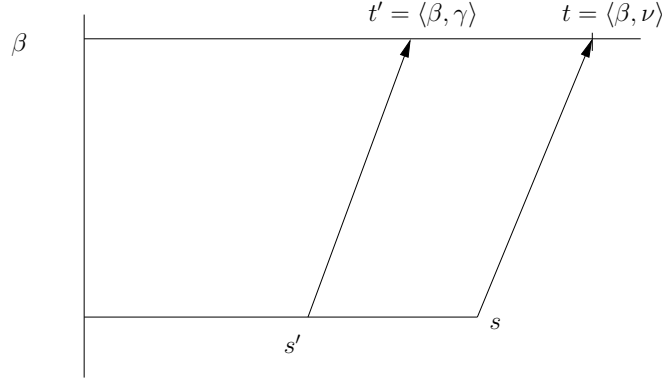< \omega_1$, there is $s \prec t$ such that $\tilde{\pi}_{st}[C_{\omega\nu(s)}]$ and therefore $|C_{\omega\nu}| < \omega_1$. $\square$

Remark 2.7

If $\kappa > \omega_1$, we only get the following **weak** $\square_\kappa$:

There is a set of limit ordinals $S \subseteq \kappa^+$, $\{ \alpha \in \kappa^+ \mid cf(\alpha) = \kappa \} \subseteq S$, and a sequence $\langle C_\alpha \mid \alpha \in S \rangle$ such that for all $\alpha \in S$:

(i) C_α is club in α

(ii) $\beta \in Lim(C_\alpha) \Rightarrow C_\beta = C_\alpha \cap \beta$ and $\beta \in S$

(iii) $cf(\alpha) < \kappa \Rightarrow |C_\alpha| < \kappa$.

Historical remarks and references

Morasses were introduced by R. Jensen in the early 1970's to solve the cardinal transfer problem of model theory in L (see e.g. Devlin [5]). For the proof of the gap-2 transfer theorem a gap-1 morass is used. For higher-gap transfer theorems Jensen has developed so-called higher-gap morasses [24]. In his Ph.D. thesis, the author generalized these to gaps of arbitrary size [19, 18, 16]. The theory of morasses is very far developed and very well examined. In particular it is known how to construct morasses in L [5, 12, 19, 16] and how to force them [38, 39].

Simplified morasses were introduced by D. Velleman [44]. He also proved that the existence of a classical $(\kappa, 1)$-morass is equivalent to the existence of a simplified $(\kappa, 1)$-morass for all regular $\kappa > \omega$ [44]. Along simplified morasses, morass constructions can be carried out very easily compared to

classical morasses. A good example for a construction along a simplified gap-1 morass is the proof of theorem 2.6. This result was proved by D. Velleman in [48]. He, however, does not directly use the morass, but he uses a forcing axiom which is equivalent to the existence of a simplified morass. An example for a direct construction along a simplified morass can be found in Devlin's book [5].

The definition of a simplified morass and lemmas 2.1 and 2.2 are from Velleman [44]. A direct proof of theorem 2.4 (a) was given by D. Donder [6]. A rather indirect way to prove theorem 2.4 (a) and (c) is taken by D. Velleman in [44]. Theorem 2.4 (b) was observed by Velleman in [45]. Theorem 2.5 was shown by Velleman [47]. Remark 2.7 was observed by L. Stanley [39].

3

Two-dimensional forcing which preserves GCH

Let $\mathfrak{M}$ be a simplified $(\kappa, 1)$-morass. We want to define a generalization of a FS iteration which is not indexed along an ordinal but along $\mathfrak{M}$. One way of doing this is the following definition:

We say that $\langle \langle \mathbb{P}_\eta \mid \eta \leq \kappa^+ \rangle, \langle \sigma_{st} \mid s \prec t \rangle, \langle e_\alpha \mid \alpha < \kappa \rangle \rangle$ is a FS system along $\mathfrak{M}$ if the following conditions hold:

(FS1) $\langle \mathbb{P}_\eta \mid \eta \leq \kappa^+ \rangle$ is a sequence of partial orders such that $\mathbb{P}_\eta \subseteq_\perp \mathbb{P}_\nu$ if $\eta \leq \nu$ and $\mathbb{P}_\lambda = \bigcup \{ \mathbb{P}_\eta \mid \eta < \lambda \}$ for $\lambda \in Lim$.

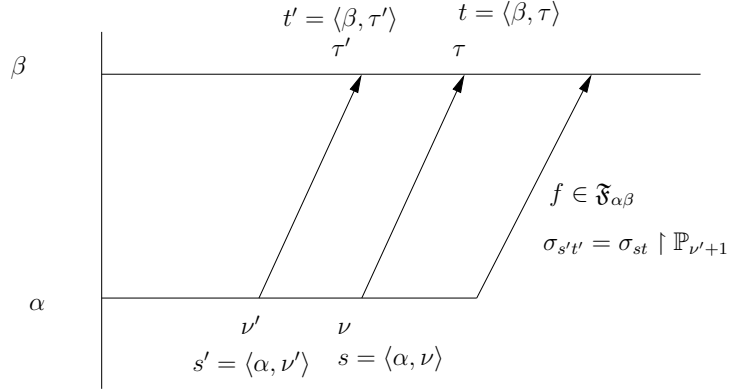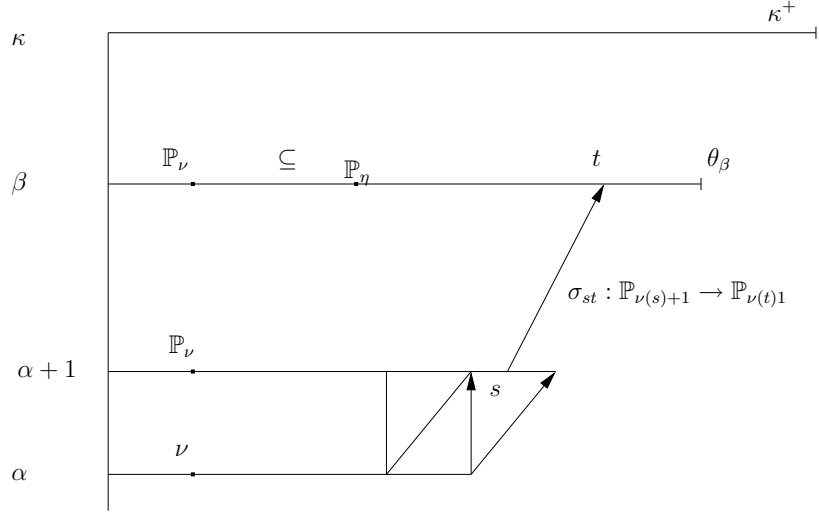(FS2) $\langle \sigma_{st} \mid s \prec t \rangle$ is a commutative system of injective embeddings $\sigma_{st} : \mathbb{P}_{\nu(s)+1} \to \mathbb{P}_{\nu(t)+1}$ such that if t is a limit point in $\prec$, then

$$\mathbb{P}_{\nu(t)+1} = \bigcup \{ \sigma_{st}[\mathbb{P}_{\nu(s)+1}] \mid s \prec t \}.$$

(FS3) $e_\alpha : \mathbb{P}_{\theta_{\alpha+1}} \to \mathbb{P}_{\theta_\alpha}$.

(FS4) Let $s \prec t$ and $\pi = \pi_{st}$. If $\pi(\nu') = \tau'$, $s' = \langle \alpha(s), \nu' \rangle$ and $t' = \langle \alpha(t), \tau' \rangle$, then $\sigma_{st} : \mathbb{P}_{\nu(s)+1} \to \mathbb{P}_{\nu(t)+1}$ extends $\sigma_{s't'} : \mathbb{P}_{\nu'+1} \to \mathbb{P}_{\tau'+1}$.

Hence for $f \in \mathfrak{F}_{\alpha\beta}$, we may define $\sigma_f = \bigcup \{ \sigma_{st} \mid s = \langle \alpha, \nu \rangle, t = \langle \beta, f(\nu) \rangle \}$.

κ κ^+

β $\mathbb{P}_\nu$ $\subseteq$ $\mathbb{P}_\eta$ t θ_β

$\sigma_{st} : \mathbb{P}_{\nu(s)+1} \to \mathbb{P}_{\nu(t)1}$

$\alpha + 1$ $\mathbb{P}_\nu$ s

α ν

$t' = \langle \beta, \tau' \rangle$ $t = \langle \beta, \tau \rangle$

β τ' τ

$f \in \mathfrak{F}_{\alpha\beta}$

$\sigma_{s't'} = \sigma_{st} \restriction \mathbb{P}_{\nu'+1}$

α ν' ν

$s' = \langle \alpha, \nu' \rangle$ $s = \langle \alpha, \nu \rangle$

(FS5) If $\pi_{st} = id \restriction \nu(s) + 1$, then $\sigma_{st} = id \restriction \mathbb{P}_{\nu(s)+1}$.

(FS6)(a) If $\alpha < \kappa$, then $\mathbb{P}_{\theta_\alpha}$ is completely contained in $\mathbb{P}_{\theta_{\alpha+1}}$ in such a way that $e_\alpha(p)$ is a reduction of $p \in \mathbb{P}_{\theta_{\alpha+1}}$.

(b) If $\alpha < \kappa$, then $\sigma_\alpha := \sigma_{f_\alpha} : \mathbb{P}_{\theta_\alpha} \to \mathbb{P}_{\theta_{\alpha+1}}$ is a complete embedding such that $e_\alpha(p)$ is a reduction of $p \in \mathbb{P}_{\theta_{\alpha+1}}$.

(FS7)(a) If $\alpha < \kappa$ and $p \in \mathbb{P}_{\theta_\alpha}$, then $e_\alpha(p) = p$.

(b) If $\alpha < \kappa$ and $p \in rng(\sigma_\alpha)$, then $e_\alpha(p) = \sigma_\alpha^{-1}(p)$.

To simplify notation, set $\mathbb{P} := \mathbb{P}_{\kappa^+}$.

Recall: $\sigma : \mathbb{P} \to \mathbb{Q}$ is called complete embedding, if

(1) $\forall p, p' \in \mathbb{P}\ (p' \leq p \to \sigma(p') \leq \sigma(p))$

(2) $\forall p, p' \in \mathbb{P}\ (p' \perp p \to \sigma(p') \perp \sigma(p))$

(3) $\forall q \in \mathbb{Q}\ \exists p \in \mathbb{P}\ \forall p' \in \mathbb{P}\ (p' \leq p \to \sigma(p') \parallel q)$.

q is called reduction.

If only (1) and (2) hold, we say that σ is an embedding. If $id \upharpoonright \mathbb{P} : \mathbb{P} \to \mathbb{Q}$ is an embedding, we also write $\mathbb{P} \subseteq_\perp \mathbb{Q}$.

Unlike in the case of FS iterations, it is unclear how a generic extension with respect to $\mathbb{P}_{\kappa^+}$ can be viewed as being obtained by successive extensions. This would justify to call a FS system along $\mathfrak{M}$ a FS *iteration* along $\mathfrak{M}$.

We want to prove that $\mathbb{P}$ satisfies ccc if all $\mathbb{P}_\nu$ satisfy ccc for $\nu < \kappa$. To do this, we want to apply an argument like in the introduction. For this we need something like the support of a condition $p \in \mathbb{P}$, i.e. we need to represent $p \in \mathbb{P}$ as a function $p^* : \kappa \to V$ such that $p^*(\alpha) \in \mathbb{P}_{\theta_\alpha}$ for all $\alpha < \kappa$.

To define such a function p^* from $p \in \mathbb{P}$ set recursively

$p_0 = p$

$\nu_n(p) = min\{\eta \mid p_n \in \mathbb{P}_{\eta+1}\}$

$t_n(p) = \langle \kappa, \nu_n(p) \rangle$

$p^{(n)}(\alpha) = \sigma_{st}^{-1}(p_n)$ if $s \in T_\alpha$, $s \prec t_n(p)$ and $p_n \in rng(\sigma_{st})$.

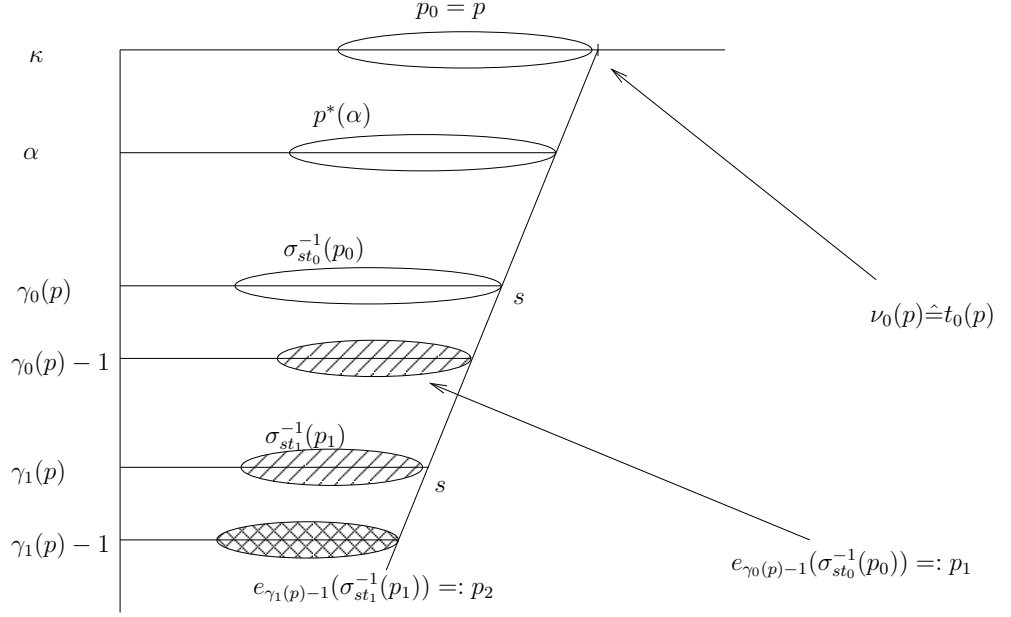Note that, since $\prec$ is atree, s is uniquely determined by α and $t_n(p)$. Hence we really define a function. Set

$\gamma_n(p) = min(dom(p^{(n)}))$.

By (FS2), $\gamma_n(p)$ is a successor ordinal or 0. Hence, if $\gamma_n(p) \neq 0$, we may define
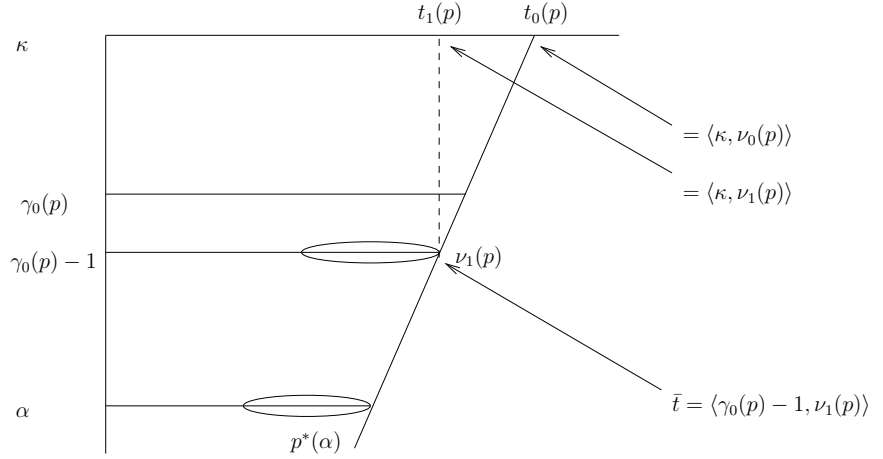
$p_{n+1} = e_{\gamma_n(p)-1}(p^{(n)}(\gamma_n(p)))$.

If $\gamma_n(p) = 0$, we let p_{n+1} be undefined.

Finally, set $p^* = \bigcup\{p^{(n)} \upharpoonright [\gamma_n(p), \gamma_{n-1}(p)[\mid n \in \omega\}$ where $\gamma_{-1}(p) = \kappa$.

Note: If $n > 0$ and $\alpha \in [\gamma_n(p), \gamma_{n-1}(p)[$, then $p^*(\alpha) = \sigma_{s\bar{t}}^{-1}(p_n)$ where $\bar{t} = \langle \gamma_n(p) - 1, \nu_n(p) \rangle$,

because $p^*(\alpha) = p^{(n)}(\alpha) = \sigma_{st}^{-1}(p_n) = (\sigma_{\bar{t}t} \circ \sigma_{s\bar{t}})^{-1}(p_n) = \sigma_{s\bar{t}}(p_n)$ with $s \prec t$ and $s \in T_\alpha$ where the first two equalities are just the definitions of p^* and $p^{(n)}$. For the third equality note that $\bar{t} \prec t$ since $id \upharpoonright \theta_\alpha \in \mathfrak{F}_{\alpha\beta}$ for all $\alpha < \beta \leq \kappa$. So the equality follows from the commutativity of $\langle \sigma_{st} \mid s \prec t \rangle$. The last equality holds by (FS5).

It follows from the previous observation that $\langle \gamma_n(p) \mid n \in \omega \rangle$ is decreasing.

So the recursive definition above breaks down at some point, i.e. $\gamma_n(p) = 0$ for some $n \in \omega$.

For $p \in \mathbb{P}$ define its support by

$$supp(p) = \{\gamma_n(p) \mid n \in \omega\}.$$

Hence $supp(p)$ is finite.

Lemma 3.1

If $p^*(\alpha)$ and $q^*(\alpha)$ are compatible for $\alpha = max(supp(p) \cap supp(q))$, then p and q are compatible.

Proof: Suppose that p and q are incompatible. Without loss of generality let $\nu := min\{\eta \mid p \in \mathbb{P}_{\eta+1}\} \leq min\{\eta \mid q \in \mathbb{P}_{\eta+1}\} =: \tau$. Set $s = \langle \kappa, \nu \rangle$ and $t = \langle \kappa, \tau \rangle$. Let $t' \prec t$ be minimal such that $\nu \in rng(\pi_{t't})$ and $p, q \in rng(\sigma_{t't})$. By (FS2), $t' \in T_{\alpha_0+1}$ for some $\alpha < \kappa$. Let $\pi_{t't}(\nu') = \nu$ and $s' = \langle \alpha+1, \nu' \rangle$. Let $\bar{s}, \bar{t}$ be the direct predecessors of s' and t' in $\prec$. Set $p' = \sigma_{s's}^{-1}(p)$, $q' = \sigma_{t't}^{-1}(q)$.
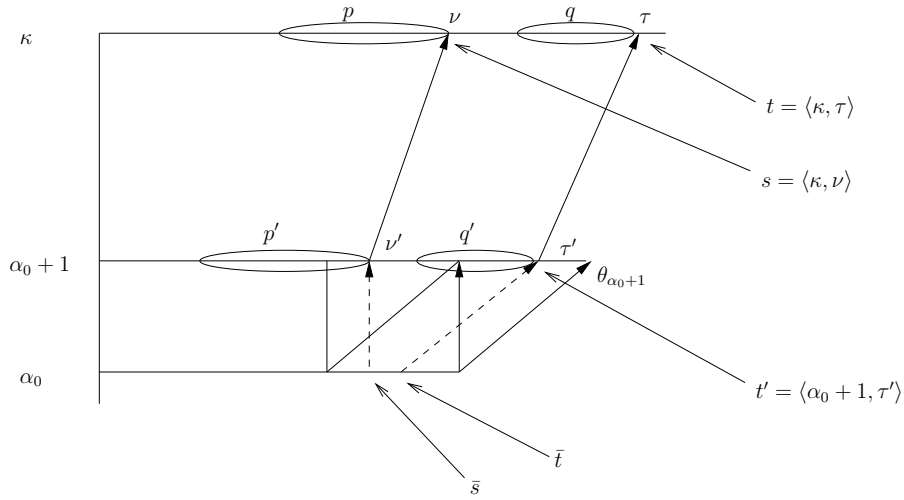

Then $p' = p^*(\alpha_0 + 1)$, $q' = q^*(\alpha_0 + 1)$ by the definition of p^*. Moreover, p' and q' are not compatible, because if $r \leq p', q'$, then $\sigma_{t't}(r) \leq p, q$ by (FS2). Now, we consider several cases.

Case 1: $\nu' \notin rng(\pi_{\bar{t}t'})$
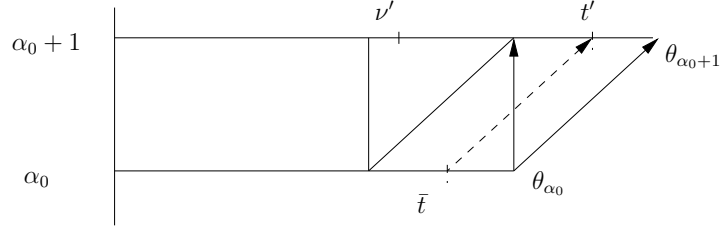
Then $\pi_{\bar{s}s'} = id \upharpoonright \nu(\bar{s}) + 1$ and $\sigma_{\bar{s}s'} = id \upharpoonright \mathbb{P}_{\nu(\bar{s})+1}$ by the minimality of α_0. Moreover, $\bar{p} := p'$ and $\bar{q} := e_\alpha(q')$ are not compatible, because if $r \leq p', e_\alpha(q')$, then there is $u \leq r, q', p'$ by (FS6)(a). There is no difference between compatibility in $\mathbb{P}_{\theta_{\alpha+1}}$ and in $\mathbb{P}_{\nu(t')+1}$ by (FS1). Finally, note that $\bar{p} = p^*(\alpha_0)$ and $\bar{q} = q^*(\alpha_0)$ by the definition of p^* and (FS7).

Case 2: $\nu' \in rng(\pi_{\bar{t}t'})$ and $\pi_{\bar{s}s'} = id \upharpoonright \nu(\bar{s}) + 1$
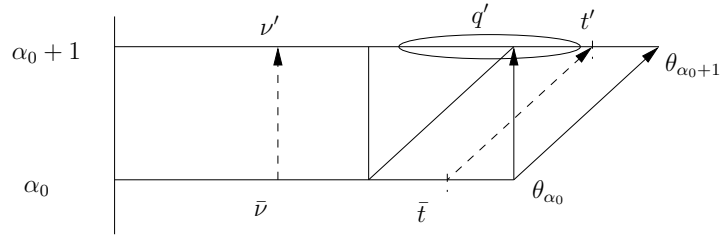
Then $\pi_{\bar{t}t'} \neq id \upharpoonright \nu(\bar{t}) + 1$ by the minimality of α_0 and $\bar{p} := p'$ and $\bar{q} := e_\alpha(q')$ are not compatible (like in case 1). However, $\bar{p} = p^*(\alpha_0)$ and $\bar{q} = q^*(\alpha_0)$ by the definition of p^* and (FS7).

Case 3: $\nu' \in rng(\pi_{\bar{t}t'})$, $\pi_{\bar{s}s'} \neq id \upharpoonright \nu(\bar{s}) + 1$ and $\alpha_0 + 1 \notin supp(p)$
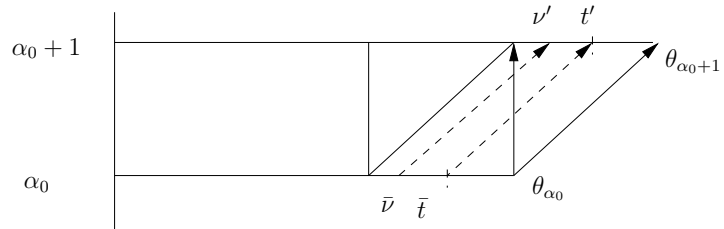
Then $\pi_{\bar{t}t'} \neq id \upharpoonright \nu(\bar{t}) + 1$ by the minimality of α_0. Set $\bar{p} := \sigma_{\bar{s}s'}^{-1}(p')$ and $\bar{q} = e_\alpha(q')$. Then $\bar{p}$ and $\bar{q}$ are not compatible, because if $r \leq \bar{p}, \bar{q}$, then there is $u \leq \sigma_\alpha(r), q', p'$ by (FS6)(b). However, $\bar{p} = p^*(\alpha_0)$ and $\bar{q} = q^*(\alpha_0)$ by the definition of p^* and (FS7).

Case 4: $\nu' \in rng(\pi_{\bar{t}t'})$, $\pi_{\bar{s}s'} \neq id \upharpoonright \nu(\bar{s}) + 1$ and $\alpha_0 + 1 \notin supp(q)$

Then $\pi_{\bar{t}t'} \neq id \upharpoonright \nu(\bar{t}) + 1$. Set $\bar{q} := \sigma_{\bar{s}s'}^{-1}(q')$ and $\bar{p} = e_\alpha(p')$. Then $\bar{q}$ and $\bar{p}$ are not compatible, because if $r \leq \bar{p}, \bar{q}$, then there is $u \leq \sigma_\alpha(r), p', q'$ by (FS6)(b).

Case 5: $\alpha_0 + 1 \in supp(p) \cap supp(q)$

Then $\alpha_0 + 1 = max(supp(p) \cap supp(q))$, since $\alpha_0 + 1 \geq max(supp(q))$ because by definition $q \in rng(\sigma_{rt})$ where $r \prec t$ and $r \in T_{max(supp(q))}$. However, $p' = p^*(\alpha_0 + 1)$, $q' = q^*(\alpha_0 + 1)$ are not compatible. Contradiction.

So in case 5 we are finished. If we are in cases 1 - 4, we define recursively α_{n+1} from $p^*(\alpha_n)$ and $q^*(\alpha_n)$ in the same way as we defined α_0 from p and q. Like in the previous proof that $\langle \gamma_n(p) \mid n \in \omega \rangle$ is decreasing, we see that $\langle \alpha_n \mid n \in \omega \rangle$ is decreasing. Hence the recursion breaks off, we end up in case 5 and get the desired contradiction. $\square$

Theorem 3.2

Let $\mu, \kappa > \omega$ be cardinals, κ regular. Let $\langle \langle \mathbb{P}_\eta \mid \eta \leq \kappa^+ \rangle, \langle \sigma_{st} \mid s \prec t \rangle, \langle e_\alpha \mid \alpha < \kappa \rangle \rangle$ be a FS system along a $(\kappa, 1)$-morass $\mathfrak{M}$. Assume that all $\mathbb{P}_\eta$ with $\eta < \kappa$ satisfy the μ-cc. Then $\mathbb{P}_{\kappa^+}$ also does.

Proof: Let $A \subseteq \mathbb{P}_{\kappa^+}$ be a set of size μ. Assume by the Δ-system lemma that $\{supp(p) \mid p \in A\}$ forms a Δ-system with root Δ. Set $\alpha = max(\Delta)$. Then $\mathbb{P}_{\theta_\alpha}$ satisfies the μ-cc by the hypothesis of the lemma. So there are $p \neq q \in A$ such that $p^*(\alpha)$ and $q^*(\alpha)$ are compatible. Hence p and q are compatible by the previous lemma. $\square$
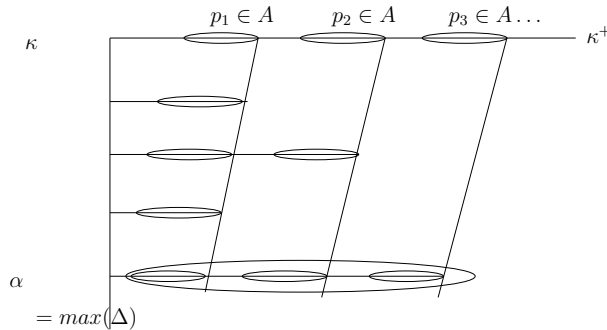
As an application, we will construct along an $(\omega_1, 1)$-morass a ccc forcing $\mathbb{P}$ of size ω_1 which adds an ω_2-Suslin tree. An ω_2-Suslin tree is a tree of size (or equivalently height) ω_2 which has neither a chain nor an antichain of size ω_2.

The natural forcing to do this with finite conditions is Tennenbaum's forcing (see S. Tennenbaum:"Souslin's problem", PNAS 59 (1968), 60 - 63).

Let $P(\theta)$ be the set of all finite trees $p = \langle x_p, <_p \rangle$, $x_p \subseteq \theta$, such that $\alpha < \beta$ if $\alpha <_p \beta$.

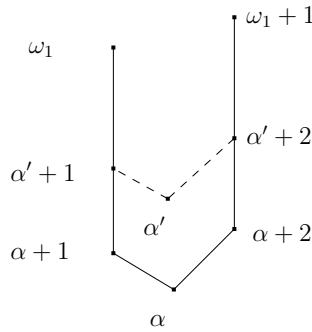Set $p \leq q$ iff $x_p \supset x_q$ and $<_q = <_p \cap x_q^2$.
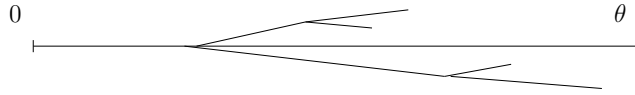
For $\theta = \omega_1$, $P(\theta)$ is Tennenbaum's forcing to add an ω_1-Suslin tree which satisfies ccc.

However, if $\theta > \omega_1 + 1$, then

$A = \{p \in P(\theta) \mid x_p = \{\alpha, \alpha + 1, \alpha + 2, \omega_1, \omega_1 + 1\}, \alpha < \omega_1,$

$\qquad \alpha <_p \alpha + 1 <_p \omega_1, \alpha <_p \alpha + 2 <_p \omega_1 + 1, \alpha + 1 \not<_p \alpha + 2\}$

is an antichain of size ω_1.

So $P(\theta)$ does not satisfy the ccc and in order to thin it out so that it obtains ccc, we have to restrict the possible values of the infima in our trees.

Let $\pi : \bar{\theta} \to \theta$ be a order-preserving map. Then $\pi : \bar{\theta} \to \theta$ induces maps $\pi : \bar{\theta}^2 \to \theta^2$ and $\pi : P(\bar{\theta}) \to P(\theta)$ in the obvious way:

$$\pi : \bar{\theta}^2 \to \theta^2, \quad \langle \alpha, \beta \rangle \mapsto \langle \pi(\alpha), \pi(\beta) \rangle$$

$$\pi : P(\bar\theta) \to P(\theta), \quad \langle x_p, <_p \rangle \mapsto \langle \pi[x_p], \pi[<_p] \rangle.$$

If $p \in P(\theta)$, then set

$$\pi^{-1}[p] := \langle \pi^{-1}[x_p \cap rng(\pi)], \pi^{-1}[<_p \cap rng(\pi)] \rangle.$$

It is easily seen that then $\pi^{-1}[p] \in P(\bar\theta)$.

We define our FS system by induction over $\beta \leq \omega_1$.

Base Case: $\beta = 0$

Then we need to define only $\mathbb{P}_1$. Set $\mathbb{P}_1 := P(1)$.
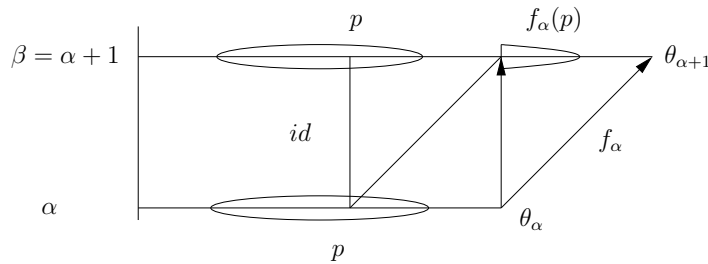
Successor Case: $\beta = \alpha + 1$

We first define $\mathbb{P}_{\theta_\beta}$. To do so, let

$$\mathbb{P}'_{\theta_\beta} := \{ \langle x_p \cup x_{f_\alpha(p)}, <_p \cup <_{f_\alpha(p)} \rangle \mid p \in \mathbb{P}_{\theta_\alpha} \}$$

$$\cup \{ \langle x_p \cup x_{f_\alpha(p)}, tc(<_p \cup <_{f_\alpha(p)} \cup \{ \langle \eta, min\{\gamma \in [\theta_\alpha, \theta_{\alpha+1}[\mid \gamma \leq_{f_\alpha(p)} f_\alpha(\eta)\} \rangle \}) \rangle$$

$$\mid p \in \mathbb{P}_{\theta_\alpha}, \eta \in x_p, \eta < f_\alpha(\eta) \}.$$

In this definition, $tc(x)$ denotes the transitive closure of the binary relation x.

Well, what does this definition say? Obviously there are two types of conditions in $\mathbb{P}'_{\theta_\beta}$.

Type 1: Let $p \in \mathbb{P}_{\varphi_{\theta_\beta}}$. Then a condition of type 1 is just the union of the two possible copies of p to the next level.

It is easy to see that $p \cup f_\alpha(p)$ is a tree again.

Type 2: Let $p \in \mathbb{P}_{\varphi_{\theta_\beta}}$. Then a condition of type 1 is just the union of the two possible copies of p to the next level plus one additional edge which connects one vertex η with the minimal element in $f_\alpha(p)$ below $f_\alpha(\eta)$ which was moved by f_α.
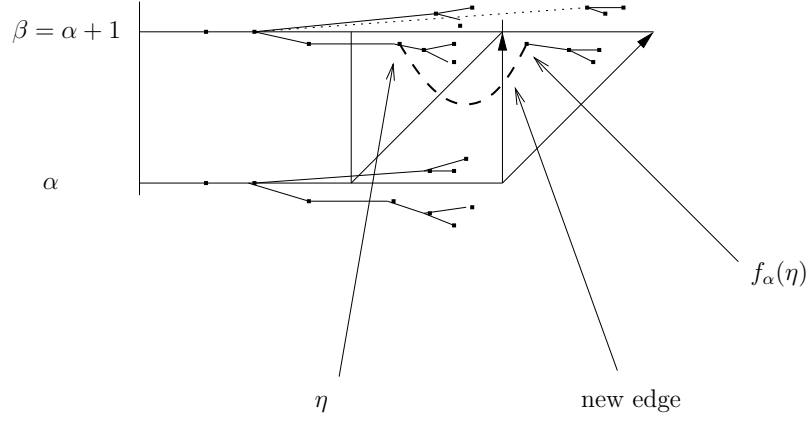
This complicated definition is necessary to get a tree again!

Now, define

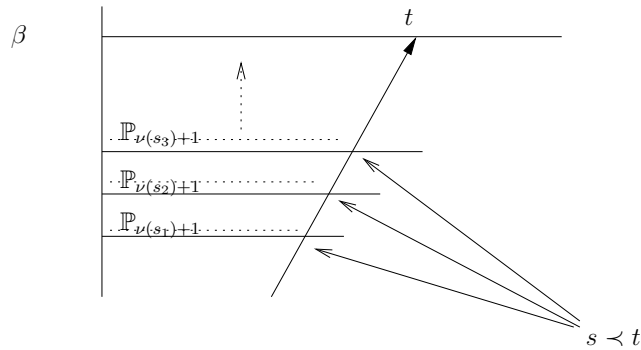$$\mathbb{P}_{\theta_\beta} := \{p \in P(\theta_\beta) \mid r \leq p \text{ for some } r \in \mathbb{P}'_{\theta_\beta}\}.$$

For $t \in T_\beta$ set $\mathbb{P}_{\nu(t)+1} = P(\nu(t) + 1) \cap \mathbb{P}_{\theta_\beta}$ and $\mathbb{P}_\lambda = \bigcup\{\mathbb{P}_\eta \mid \eta < \lambda\}$ for $\lambda \in Lim$. Let $\sigma_{st} : \mathbb{P}_{\nu(s)+1} \to \mathbb{P}_{\nu(t)+1}, p \mapsto \pi_{st}(p)$.

We still need to define e_α. If $p \in rng(\sigma_\alpha)$, then set $e_\alpha(p) = \sigma_\alpha^{-1}(p)$. If $p \in \mathbb{P}_{\theta_\alpha}$, then set $e_\alpha(p) = p$. Finally, if $p \notin rng(\sigma_\alpha) \cup \mathbb{P}_{\theta_\alpha}$, then pick an $r \in \mathbb{P}'_{\theta_\beta}$ such that $r \leq p$ and set $e_\alpha(p) = f_\alpha^{-1}[r]$.

Limit Case: $\beta \in Lim$

Then everything is already uniquely determined by (FS1) and (FS2). That is, for $t \in T_\beta$ set $\mathbb{P}_{\nu(t)+1} = \bigcup\{\sigma_{st}[\mathbb{P}_{\nu(s)+1}] \mid s \prec t\}$ and $\mathbb{P}_\lambda = \bigcup\{\mathbb{P}_\eta \mid \eta < \lambda\}$ for $\lambda \in Lim$. Let $\sigma_{st} : \mathbb{P}_{\nu(s)+1} \to \mathbb{P}_{\nu(t)+1}, p \mapsto \pi_{st}(p)$.

Lemma 3.3

$\mathbb{P}$ satisfies the ccc.

Proof: Since all $P(\theta)$ for $\theta < \omega_1$ have size $\leq \omega$, it suffices by theorem 4.2 to show that $\langle \langle \mathbb{P}_\eta \mid \eta \leq \kappa^+ \rangle, \langle \sigma_{st} \mid s \prec t \rangle, \langle e_\alpha \mid \alpha < \kappa \rangle \rangle$ is a FS system along the morass.

Most conditions of the definition of a FS system are clear. We only prove (FS6). Let $p \in \mathbb{P}_{\theta_\beta}$ and $\beta = \alpha + 1$. We may assume that $p \in \mathbb{P}'_{\theta_\beta}$, because by definition $\mathbb{P}_{\theta_\beta}$ is dense in $\mathbb{P}'_{\theta_\beta}$. We have to prove that $\sigma_\alpha^{-1}[p]$ is a reduction of p with respect to σ_α and $id \upharpoonright \mathbb{P}_{\theta_\alpha}$. To do so for σ_α, let $q \leq \sigma_\alpha^{-1}[p] =: s$. We have to find an $r \leq p, \sigma_\alpha(q)$ such that $r \in \mathbb{P}_{\theta_\beta}$. We consider two cases. If p is of the form $\langle x_s \cup x_{f_\alpha(s)}, <_s \cup <_{f_\alpha(s)} \rangle$, then define $r := \langle x_q \cup x_{f_\alpha(q)}, <_p \cup <_{f_\alpha(q)} \rangle$. It is easily seen that this is an extension of p and $\sigma_\alpha(q)$.
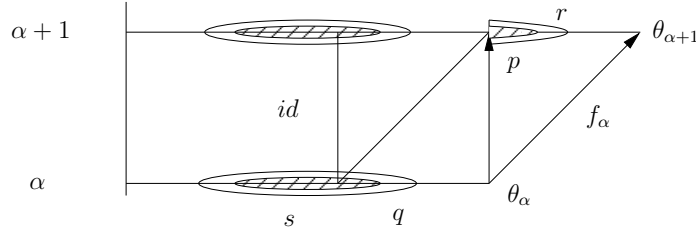
If p is of the form

$$\langle x_s \cup x_{f_\alpha(s)}, tc(<_s \cup <_{f_\alpha(s)} \cup \{\langle \eta, min\{\gamma \in [\theta_\alpha, \theta_{\alpha+1}[| \; \gamma \leq_{f_\alpha(s)} f_\alpha(\eta)\}\rangle\}) \rangle$$
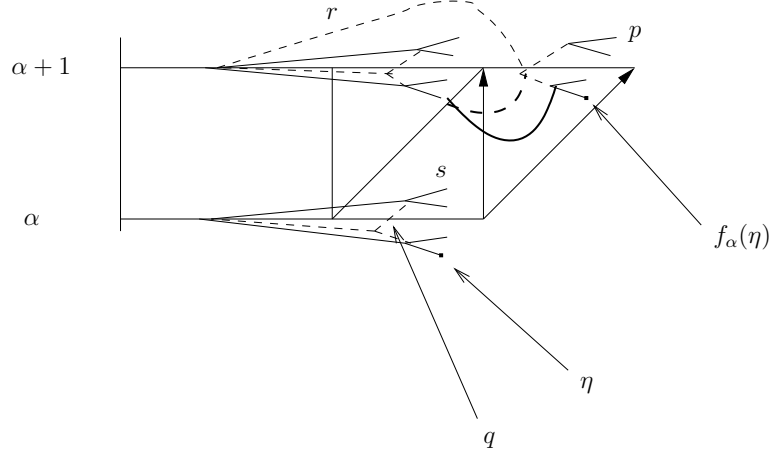
for some $\eta \in x_s$, then define r as

$$\langle x_q \cup x_{f_\alpha(q)}, tc(<_q \cup <_{f_\alpha(q)} \cup \{\langle \eta, min\{\gamma \in [\theta_\alpha, \theta_{\alpha+1}[| \; \gamma \leq_{f_\alpha(q)} f_\alpha(\eta)\}\rangle\}) \rangle.$$

It is not difficult to see that r is an extension of p and $\sigma_\alpha(q)$.

Let us sketch the proof of $r \leq p$. Assume $\alpha, \beta \in x_p$. Then we must show that $\alpha <_r \beta \Leftrightarrow \alpha <_p \beta$.

Case 1: $\alpha, \beta \in x_s$

Then $\alpha <_p \beta \Leftrightarrow \alpha <_s \beta \Leftrightarrow \alpha <_q \beta \Leftrightarrow \alpha <_r \beta$ where the middle eqivalence holds because $q \leq s$. For the first and last eqivalence note that $<_p \upharpoonright \theta_\alpha = <_s$ and $<_r \upharpoonright \theta_\alpha = <_q$.

Case 2: $\alpha, \beta \in x_{f_\alpha(s)}$

Then $\alpha <_p \beta \Leftrightarrow \alpha <_s \beta \Leftrightarrow \alpha <_q \beta \Leftrightarrow \alpha <_r \beta$ where the middle equivalence holds because $q \leq s$. For the first and last equivalence note that by the definition of the additional edge in r (or p respectively) α and β are connected if and only if they were before adding the edge.

Case 3: $\alpha \in x_s - x_{f_\alpha(s)}, \ \beta \in x_{f_\alpha(s)} - x_s$

Assume first that $\alpha <_p \beta$. Then $\alpha <_s \eta$ and $\gamma_0 <_{f_\alpha(s)} \beta$ where

$$\gamma_0 = min\{\gamma \in [\theta_\alpha, \theta_{\alpha+1}[|\ \gamma \leq_{f_\alpha(s)} f_\alpha(\eta)\}.$$

Hence $\alpha <_q \eta$ and $\gamma_1 <_{f_\alpha(q)} \beta$ where

$$\gamma_1 = min\{\eta \in [\theta_\alpha, \theta_{\alpha+1}[|\ \gamma \leq_{f_\alpha(q)} f_\alpha(\eta)\}.$$

However, $\eta <_r \gamma_1$ by definition. So $\alpha <_r \beta$.

Now, assume that $\alpha \not<_p \beta$. Then $\alpha \not<_s \eta$ or $\gamma_0 \not<_{f_\alpha(s)} \beta$. Hence $\alpha \not<_q \eta$ or $\gamma_1 \not<_{f_\alpha(q)} \beta$. So $\alpha \not<_r \beta$ by definition.

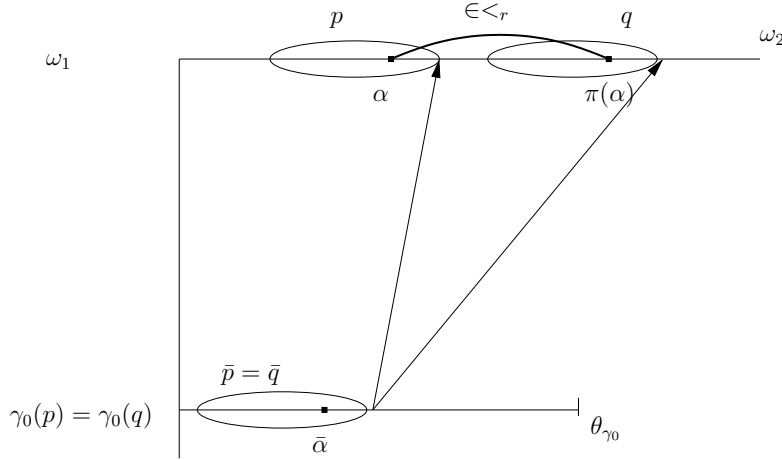*Case 4:* $\beta \in x_{f_\alpha(s)} - x_s, \ \alpha \in x_s - x_{f_\alpha(s)}$

Then both $\alpha <_r \beta$ and $\alpha <_p \beta$ are false, because the tree orders are supposed to be compatible with the order of the ordinals.

This proves that $\sigma_\alpha^{-1}[p]$ is a reduction of p with respect to σ_α. The proof that $\sigma_\alpha^{-1}[p]$ is a reduction of p with respect to $id \upharpoonright \mathbb{P}_{\theta_\alpha}$ is completely analogous. $\square$

Lemma 3.4

If $\gamma_0(p) = \gamma_0(q)$, $p^*(\gamma_0(p)) = q^*(\gamma_0(q))$, $\pi : p \cong q$ and $\alpha \leq \pi(\alpha)$, then there exists an $r \leq p, q$ such that $\langle \alpha, \pi(\alpha) \rangle \in \leq_r$.

Proof: Let p and q be as in the hypothesis of the lemma. We prove by induction over $\eta \in [\gamma_0(p), \omega_1]$ that if $\pi : p^*(\eta) \cong q^*(\eta)$ (where $p^*(\omega_1) := p$) and $\alpha \leq \pi(\alpha)$, then there exists an $r \leq p^*(\eta), q^*(\eta)$ such that $\langle \alpha, \pi(\alpha) \rangle \in \leq_r$.

Base Case: $\eta = \gamma_0(p) = \gamma_0(q)$

In this case the claim is trivial because $p^*(\eta) = q^*(\eta)$.
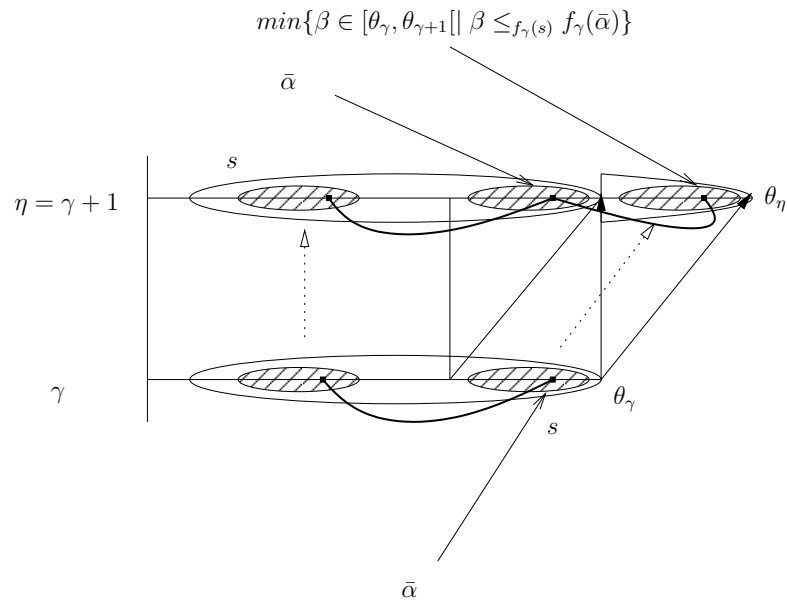
Successor Case: $\eta = \gamma + 1$

Let $\pi : p^*(\eta) \cong q^*(\eta)$ and $\alpha \leq \pi(\alpha)$. Let $\sigma_p : p^*(\gamma) \cong p^*(\eta)$, $\sigma_q : q^*(\gamma) \cong q^*(\eta)$ and $\sigma_p(\bar\alpha_p) = \alpha$, $\sigma_q(\bar\alpha_q) = \pi(\alpha)$.

By the induction hypothesis, there is an $s \leq p^*(\gamma), q^*(\gamma)$ such that $\langle \bar\alpha_q, \bar\alpha_p \rangle \in \leq_s$ or $\langle \bar\alpha_p, \bar\alpha_q \rangle \in \leq_s$. Let $\bar\alpha := max\{\bar\alpha_p, \bar\alpha_q\}$. Now, we consider two cases. If $\bar\alpha < f_\gamma(\bar\alpha)$, we define r as

$$\langle x_s \cup x_{f_\alpha(s)}, tc(<_s \cup <_{f_\alpha(s)} \cup\{\langle \bar\alpha, min\{\beta \in [\theta_\gamma, \theta_{\gamma+1}[|\ \beta \leq_{f_\alpha(s)} f_\gamma(\bar\alpha)\}\rangle\})\rangle.$$

If $\bar\alpha = f_\gamma(\bar\alpha)$, then we define

$$r := \langle x_s \cup x_{f_\alpha(s)}, <_s \cup <_{f_\alpha(s)} \rangle.$$
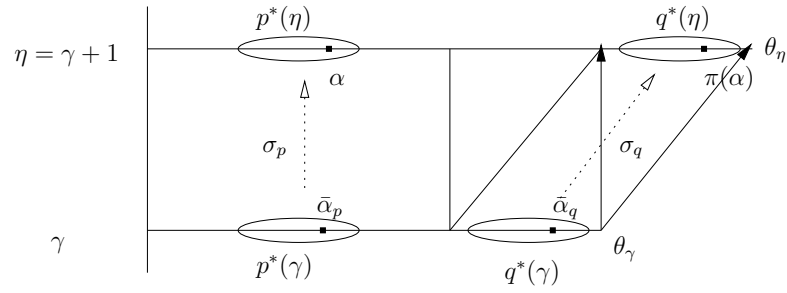
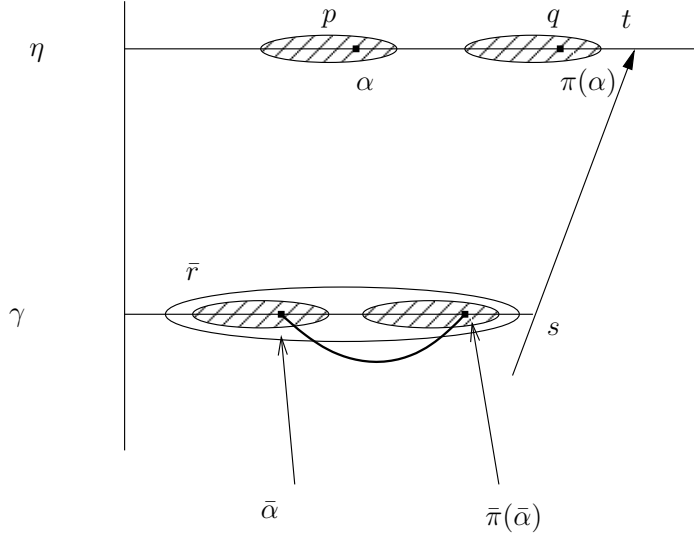In both cases, it is easily seen that $r \leq p^*(\eta), q^*(\eta)$ and $\langle \alpha, \pi(\alpha) \rangle \in \leq_r$.

Limit Case: $\eta \in Lim$

By (FS1) and (FS2), there are a $t \in T_\eta$ and an $s \prec t$ such that $p^*(\eta), q^*(\eta) \in rng(\sigma_{st})$.

Let $s \in T_\gamma$, $\sigma_{st}(\bar{\alpha}) = \alpha$ and $\sigma_{st} \circ \bar{\pi} = \pi \circ \sigma_{st}$. Then $\sigma_{st}(p^*(\eta)) = p^*(\gamma)$ and $\sigma_{st}(q^*(\eta)) = q^*(\gamma)$. Moreover, by the induction hypothesis, there is a $\bar{r} \le p^*(\gamma), q^*(\gamma)$ such that $\langle \bar{\alpha}, \bar{\pi}(\bar{\alpha}) \rangle \in \le_{\bar{r}}$. Set $r := \sigma_{st}(\bar{r})$. Then r is as desired. $\square$

Lemma 3.5

If $\gamma_0(p) = \gamma_0(q)$, $p^*(\gamma_0(p)) = q^*(\gamma_0(q))$, $\pi : p \cong q$ and $\alpha < \pi(\alpha)$, then there exists an $r \le p, q$ such that $\langle \alpha, \pi(\alpha) \rangle \notin <_r$.

Proof: Basically the proof proceeds like the previous proof. However, in the successor case, we always use common extensions of the form $\langle x_p \cup x_{f_\gamma(p)}, <_p \cup <_{f_\gamma(p)} \rangle$. $\square$

Theorem 3.6

If there is a simplified $(\omega_1, 1)$-morass, then there is a ccc forcing that adds an ω_2-Suslin tree.

Proof: Let G be a $\mathbb{P}$-generic filter. Let $T = \bigcup \{p \mid p \in G\}$. Well-known arguments show that T is a tree. We prove that T has neither an antichain nor a chain of size ω_2. First, assume towards a contradiction that there is an antichain of size ω_2. Then there is a $p \in \mathbb{P}$ and names $\dot{f}, \dot{A}$ such that

$$p \Vdash (\dot{A} \text{ is an antichain and } \dot{f} : \check{\omega}_2 \to \dot{A} \text{ is a bijection}).$$

Since $\mathbb{P}$ satisfies ccc, it preserves cardinals. Hence $p \Vdash (\dot{f} : \omega_2 \to \dot{A}$ is a bijection). Let $\langle \alpha_i \mid i \in \omega_2 \rangle$ and $\langle p_i \mid i \in \omega_2 \rangle$ be such that $p_i \le p$ for all $i \in \omega_2$ and $p_i \Vdash (\dot{x}_i = \check{\alpha}_i \wedge \dot{x}_i \in \check{x}_{p_i})$. Since $card(\mathbb{P}_{\omega_1}) = \omega_1$, there is $q \in \mathbb{P}_{\omega_1}$,

$\eta \in \omega_1$ and a subset $X \subseteq \omega_2$ of size ω_2 such that $\gamma_0(p_i) = \eta$ and $p_i^*(\gamma_0(p_i)) = q$ for all $i \in X$. Hence all p_i with $i \in X$ are isomorphic. Since x_q is finite, there are $i \neq j \in X$ such that $\pi(\alpha_i) = \alpha_j$ and $\alpha_i \leq \alpha_j$ where $\pi : p_i \cong p_j$. By a previous lemma, there exists an $r \leq p_i, p_j$ such that $\langle \alpha_i, \alpha_j \rangle \in \leq_r$. Hence $r \Vdash (\alpha_i$ and α_j are comparable). That contradicts the definition of p. The proof that there is no chain of size ω_2 works the same using the respective lemma. $\square$

This proves that there can be a ccc forcing which adds an ω_2-Suslin tree. However, we claimed that there can be such a forcing of size ω_1.

To this end, we define a forcing $\mathbb{Q}$ and an embedding $i : \mathbb{P} \to \mathbb{Q}$ such that $i[P] = \mathbb{Q}$. In particular, $i[\mathbb{P}]$ is dense in $\mathbb{Q}$. It is well-known (see e.g. Kunen's book), that in this case $\mathbb{P} \simeq \mathbb{Q}$, i.e. they generate the same generoc extensions. Hence also $\mathbb{Q}$ adds an ω_2-Suslin tree. Moreover, it also satisfies ccc, because if $A \subseteq \mathbb{Q}$ was an antichain of size ω_1, also $i^{-1}[A] \subseteq \mathbb{P}$ was one.

Set

$$\mathbb{Q} = \{p^* \restriction supp(p) \mid p \in \mathbb{P}\}.$$

For $p, q \in \mathbb{Q}$ set $p \leq q$ iff $dom(q) \subseteq dom(p)$ and $p(\eta) \leq q(\eta)$ for all $\eta \in dom(q)$.

Define

$$i : \mathbb{P} \to \mathbb{Q}, \ p \mapsto p^* \restriction supp(p).$$

Theorem 3.7

Assume that
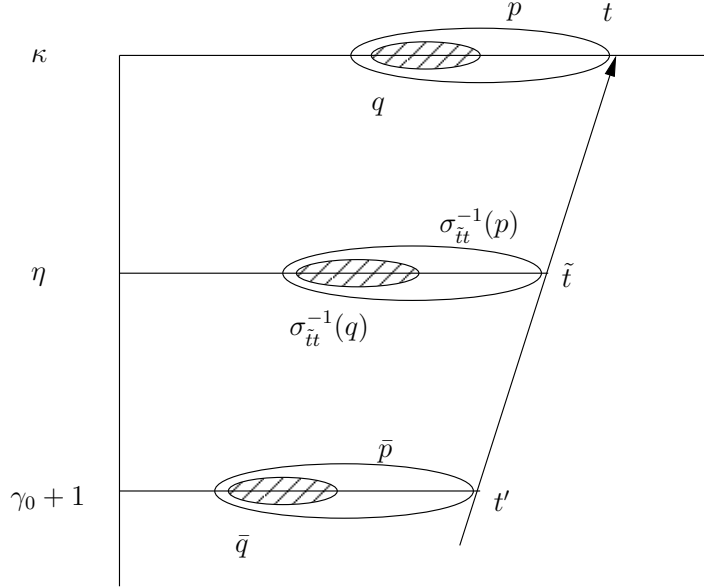
(i) $\forall p, q \in \mathbb{P} \ \forall \alpha \in \kappa : p \leq p \in \mathbb{P}_{\theta_{\alpha+1}} \to e_\alpha(p) \leq e_\alpha(q)$

(ii) $\forall p \leq q \in \mathbb{P} \ \forall s \prec t : p \in rng(\sigma_{st}) \to q \in rng(\sigma_{st})$.

Then $i : \mathbb{P} \to \mathbb{Q}$ is an embedding.

Proof: We must prove

(1) $\forall p, q \in \mathbb{P} \ (p \leq q \to i(p) \leq i(q))$

(2) $\forall p, q \in \mathbb{P} \ (p \perp q \leftrightarrow i(p) \perp i(q))$.

To show (1), let $p \leq q \in \mathbb{P}$. Let $\tau = min\{\eta \mid p \in \mathbb{P}_{\eta+1}\}$ and $t = \langle \kappa, \tau \rangle$. Let $t' \prec t$ be minimal such that $p \in rng(\sigma_{t't})$. By (FS2), $t' \in T_{\gamma_0+1}$ for some $\gamma_0 \in \kappa$. By definition of $supp(p)$, $\gamma_0 + 1 \in supp(p)$. By (FS4) and (ii), $p^*(\eta) = \sigma_{\tilde{t}t}^{-1}(p)$ and $q^*(\eta) = \sigma_{\tilde{t}t}^{-1}(q)$ where $\tilde{t} \in T_\eta$ for all $\gamma_0 < \eta < \kappa$. Hence $\eta \notin supp(p) \cup supp(q)$ for all $\gamma_0 + 1 < \eta < \kappa$.

Moreover, by (FS2), $\bar{p} := p^*(\gamma_0 + 1) \le q^*(\gamma_0 + 1) =: \bar{q}$. By (FS7), $p^*(\gamma_0) = e_\alpha(\bar{p})$ and $q^*(\gamma_0) = e_\alpha(\bar{q})$. Hence by (i), $p^*(\gamma_0) \le q^*(\gamma_0)$. Now, we repeat this argument finitely many times and get that $supp(q) \subseteq supp(p)$ and $p^*(\eta) \le q^*(\eta)$ for all $\eta \in supp(q)$.

It remains to prove (2). However, if $p \parallel q$, then there is an $r \le p, q$. Hence $i(r) \le i(p), i(q)$ by (1), i.e. $i(p) \parallel i(q)$. So assume that $p \perp q$. Let $\Delta = supp(p) \cap supp(q)$. Then by a previous lemma, $p^*(\alpha) \perp q^*(\alpha)$ for $\alpha = max(\Delta)$. Hence $i(p) \perp i(q)$. $\square$

Theorem 3.8

If there exists a simplified $(\omega_1, 1)$-morass, then there is a ccc forcing of size ω_1 which adds an ω_2-Suslin tree.

Proof: It is easily seen that the forcing $\mathbb{P}$ which we constructed previously satisfies (i) and (ii) of the previous theorem. Moreover, $\mathbb{Q}$ has size ω_1. Hence $\mathbb{Q}$ is as wanted. $\square$

Assume that $\langle \langle \mathbb{P}_\eta \mid \eta \le \omega_2 \rangle, \langle \sigma_{st} \mid s \prec t \rangle, \langle e_\alpha \mid \alpha < \omega_1 \rangle \rangle$ is a FS system along a simplified $(\omega_1, 1)$-morass, $\mathbb{P} := \mathbb{P}_{\omega_2}$ and all $\mathbb{P}_\eta$ are countable for $\eta < \omega_1$. Then we saw that under ery natural circumstances $i : \mathbb{P} \to \mathbb{Q}$ is a dense embedding and $\mathbb{Q}$ has size ω_1. Therefore, by the following lemma ,which is quoted from Kunen's textbook, our approach will usually produce forcings which preserve GCH.

Lemma

Assume that $\mathbb{P}$ satisfies ccc and $|\mathbb{P}| = \kappa \geq \omega$. Let $\lambda > \omega$ be a cardinal and $\theta = \kappa^\lambda$. Then $\mathbb{P} \Vdash 2^\lambda \leq \check{\theta}$.

This is sometimes useful. On the other hand, many statements in whose consistency we are interested imply $\neg GCH$. In the next section, we will discuss how we can change our approach to force such statements.

Historical remarks and references

It is a well known result by S. Shelah that adding a Cohen real also adds a Suslin tree. For a proof see for example theorem 28.12 in Jech's book [21]. A related result was proved by L. Stanley and S. Shelah [36]: If $2^\kappa = \kappa^+$ and there is a $(\kappa^+, 1)$-morass, then there exists a κ^{++}-Suslin tree. A similar result was proved by D. Velleman [45].

It is also possible to use S. Todorcevic's method of walks on ordinals [42] (theorem 7.5.1) to construct a ccc forcing which adds an ω_2-Suslin tree. This forcing, however, has size ω_2. On the other hand, it suffices to assume $\square_{\omega_1}$.
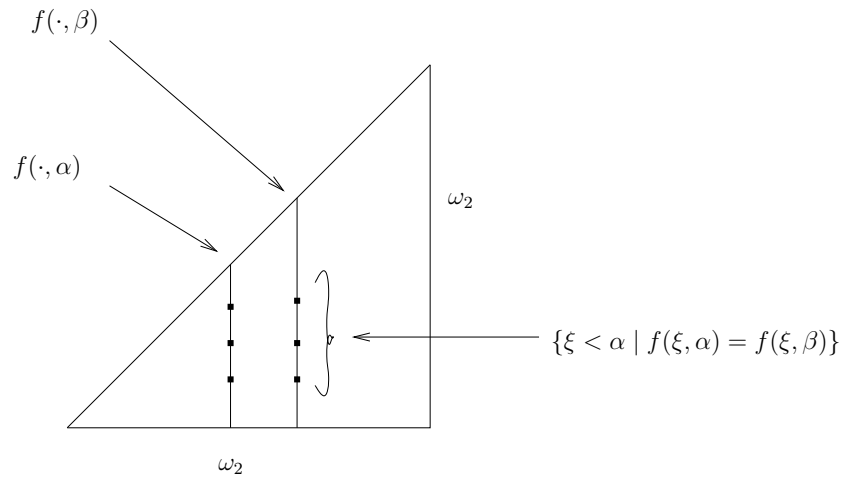
Tennenbaum's forcing was introduced in [40].

Most of the material in this section appeared first in the paper Irrgang [20].

4

Two-dimensional forcing which destroys GCH

We will now construct along a simplified gap-1 morass forcings which destroy GCH.

Our first example is a ccc forcing which adds a function $f : [\omega_2]^2 \to \omega$ such that $\{\xi < \alpha \mid f(\xi, \alpha) = f(\xi, \beta)\}$ is finite for all $\alpha < \beta < \omega_2$.

Such a forcing was constructed by S. Todorcevic using only the assumption that $\square_{\omega_1}$ holds. He uses ordinal walks and Δ-functions.

By the Erdös-Rado Theorem a function f like above implies $\neg GCH$. Because if CH holds, then $\omega_2 \to (\omega_1)^2_\omega$. Hence if $f : [\omega_2]^2 \to \omega$ is given, then it has a homogeneous set H of size ω_1. Therefore, if we take α to be the ω-th and β the $(\omega + 1)$-st one, $f(\xi, \alpha) = f(\xi, \beta)$ for all $\xi \in H \cap \alpha$.
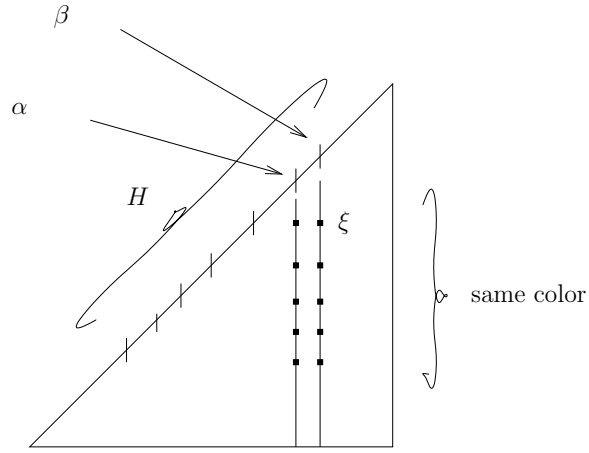
The existence of such a function is interesting for the partition calculus. If an $f.[\omega_2]^2 \to \omega$ exists such that $\{\xi < \alpha \mid f(\xi, \alpha) = f(\xi, \beta)\}$ is finite for all $\alpha < \beta < \omega_2$, then

$$\binom{\omega_2}{\omega_2} \not\to \binom{\omega}{\omega}_\omega .$$

To see this, define $g : \omega_2 \times \omega_2 \to \omega$ by

$$g(\alpha, \beta) = 2f(\alpha, \beta) \text{ if } \alpha < \beta$$

$$g(\alpha, \beta) = 2f(\alpha, \beta) + 1 \text{ if } \alpha > \beta$$

$$g(\alpha, \beta) = 0 \text{ if } \alpha = \beta.$$

Now, let $A, B \subseteq \omega_2$, $otp(A) = otp(B) = \omega$ Set $\gamma = sup(A)$, $\delta = sup(B)$. If $\gamma = \delta$, we can pick $\alpha_1, \alpha_2 \in A$ and $\beta \in B$ such that $\alpha_1 < \beta < \alpha_2$. Hence $g(\alpha_1, \beta) = 2f(\alpha_1, \beta) \neq 2f(\beta, \alpha_2) + 1 = g(\alpha_2, \beta)$, i.e. $A \times B$ is not homogeneous.
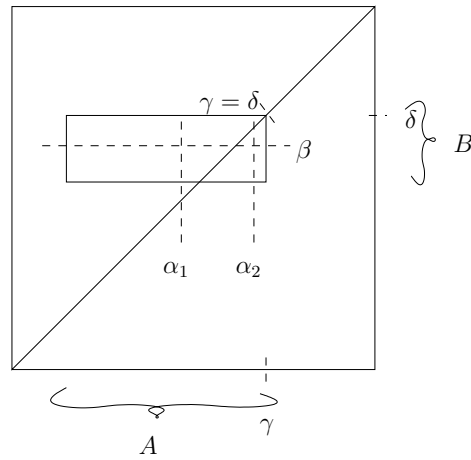
If $\gamma < \delta$, then we can find $\beta_1, \beta_2 \in B$ such that $\alpha < \beta_1, \beta_2$ for all $\alpha \in A$. However, $g(\alpha, \beta_1) = 2f(\alpha, \beta_1)$ and $g(\alpha, \beta_2) = 2f(\alpha, \beta_2)$. So $\{\alpha \in A \mid g(\alpha, \beta_1) = g(\alpha, \beta_2)\}$ is finite, i.e. $A \times B$ is not homogeneous.
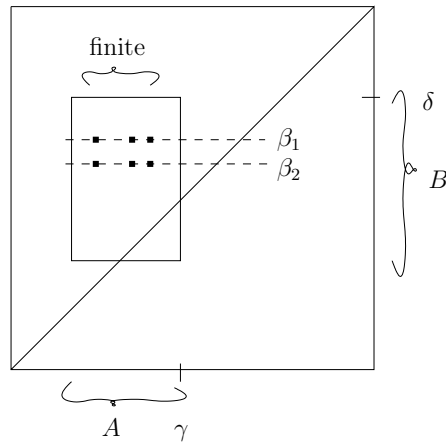
The case $\gamma > \delta$ is entirely symmetric to the case $\gamma < \delta$. It is open if

$$\binom{\omega_3}{\omega_3} \nrightarrow \binom{\omega}{\omega}_\omega$$

is consistent. Another related partition relation is the following. We write $\kappa \to (\sigma : \tau)_\gamma^2$ for: Every partition $f : [\kappa]^2 \to \gamma$ has a homogeneous set $[A; B] := \{\{\alpha, \beta\} \mid \alpha \in A, \beta \in B\}$ where $\alpha < \beta$ for all $\alpha \in A$, $\beta \in B$, $card(A) = \sigma$ and $card(B) = \tau$; i.e. f is constant on $[A; B]$.

As usual, we write $\kappa \nrightarrow (\sigma : \tau)_\gamma^2$ for the negation of this statement.
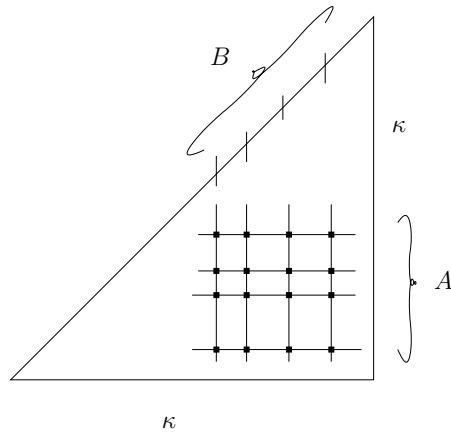
We used the consistency of $\omega_2 \to (\omega : 2)^2_\omega$ to prove that

$$\begin{pmatrix} \omega_2 \\ \omega_2 \end{pmatrix} \nrightarrow \begin{pmatrix} \omega \\ \omega \end{pmatrix}_\omega$$

is consistent. This will not work for the consistency of

$$\begin{pmatrix} \omega_3 \\ \omega_3 \end{pmatrix} \nrightarrow \begin{pmatrix} \omega \\ \omega \end{pmatrix}_\omega,$$

because $\omega_3 \nrightarrow (\omega : 2)^2_\omega$ is inconsistent, i.e. $\omega_3 \to (\omega : 2)^2_\omega$ is a theorem of ZFC:

Suppose that $f : [\omega_3]^2 \to \omega$ was a witness for $\omega_3 \nrightarrow (\omega : 2)^2_\omega$. Let $A \subseteq [\omega_2, \omega_3[$ be a set of size ω_1. For $\alpha \neq \beta \in A$ let $B(\alpha, \beta)$ be the set $\{\gamma \in \omega_2 \mid f(\alpha, \gamma) = f(\beta, \gamma)\}$.
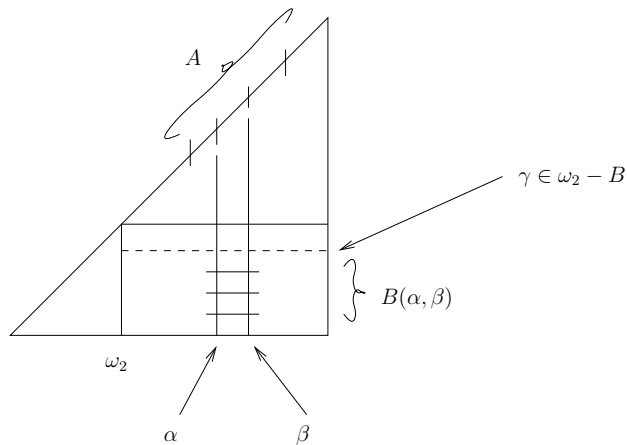
Since f witnesses $\omega_3 \nrightarrow (\omega : 2)^2_\omega$, every $B(\alpha, \beta)$ is countable. Hence $B = \bigcup\{B(\alpha, \beta) \mid \alpha \neq \beta \in A\}$ has size $\leq \omega_1$. Now, consider $\gamma \in \omega_2 - B$. Then by the definition of B, $f(\alpha, \gamma) \neq f(\beta, \gamma)$ for all $\alpha \neq \beta \in A$. However, this is impossible because $rng(s) \subseteq \omega$ while $|A| = \omega_1$.
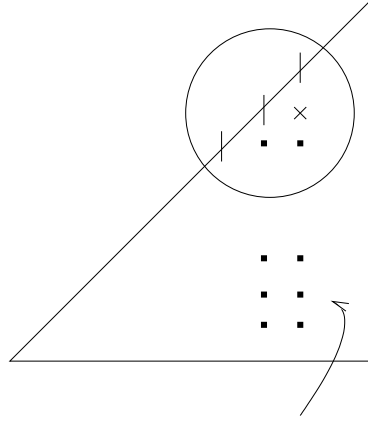
By the way,
$$\neg CH \quad \Rightarrow \quad \omega_2 \nrightarrow (3)^2_\omega.$$

In fact, $2^\kappa \nrightarrow (3)^2_\kappa$ for all κ. To see this, let $S = \{0, 1\}^\kappa$ and $F : [S] \to \kappa$ be defined by $F(\{f, g\}) = $ the least $\alpha < \kappa$ such that $f(\alpha) \neq g(\alpha)$. If f, g, h are distinct, it is impossible to have $F(\{f, g\}) = F(\{f, h\}) = F(\{g, h\})$.

However, same colors here

are allowed

Now, we are going to force an $f : [\omega_2]^2 \to \omega$ such that $\{\xi < \alpha \mid f(\xi, \alpha) = f(\xi, \beta)\}$ is finite for all $\alpha < \beta < \omega_2$.

The natural forcing to add such an f would be as follows: For $a, b \subseteq \omega_2$ let $[a, b] := \{\langle \alpha, \gamma \rangle \mid \alpha \in a, \beta \in b, \gamma < \alpha\}$. Set

$$P := \{\langle a_p, b_p, f_p \rangle \mid f_p : [a_p, b_p] \to \omega, \ a_p, b_p \subseteq \omega_2 \text{ finite}\}.$$

Note, that a_p, b_p are not determined by f_p. For example, if $a_p = \{\alpha\}$ and $b_p \subseteq]\alpha, \omega_2[$, then $[a_p, b_p] = \emptyset$ indepedently of what b_p exactly is. Nevertheless, we will abuse notation and just write $p : [a_p, b_p] \to \omega$ for the condition $\langle a_p, b_p, f_p \rangle$.

We set $p \leq q$ iff $a_q \subseteq a_p$, $b_q \subseteq b_p$ and $p(\alpha, \gamma) \neq p(\beta, \gamma)$ for all $\alpha < \beta \in a_q$ and all $\gamma \in b_p - b_q$ with $\gamma < \alpha$.
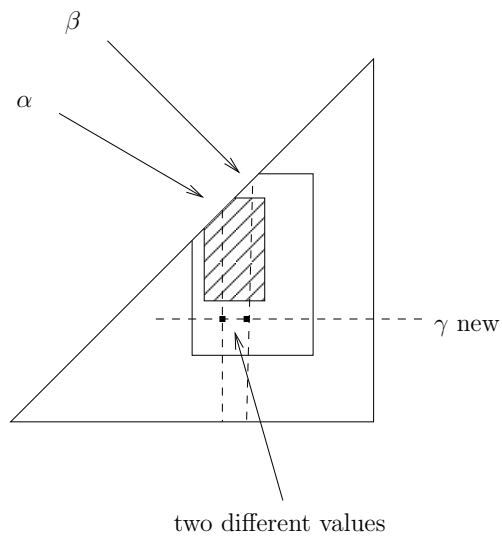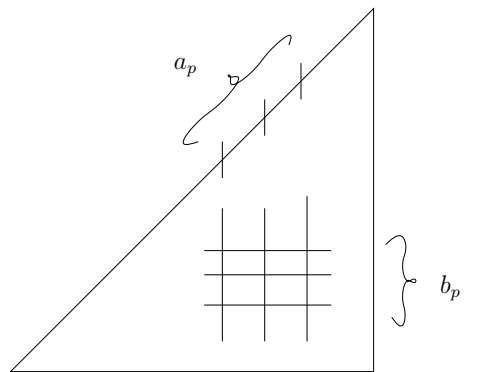
Like Tennenbaum's forcing on ω_2, P does not satisfy ccc: Let $\alpha \neq \beta \in [\omega_1, \omega_2[$. Then

$$A = \{p \in P \mid a_p = \{\alpha, \beta\}, b_p = \{\gamma\}, \gamma \in \omega_1, p(\alpha, \gamma) = p(\beta, \gamma)\}$$

is an antichain of size ω_1.
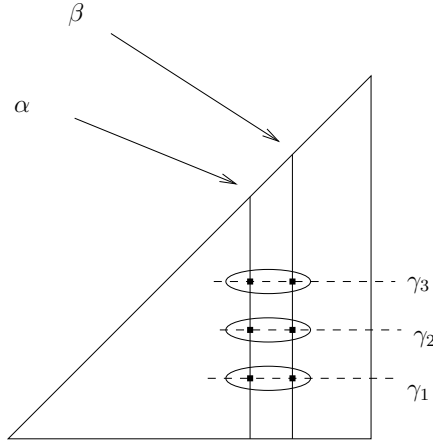
We want to thin out P to a forcing $\mathbb{P}$ which satisfies ccc. More precisely, we want to thin it out so that for every $\Delta \subseteq \omega_2$

$$\mathbb{P}_\Delta := \{p \in \mathbb{P} \mid a_p \subseteq \Delta\}$$

satisfies ccc. Moreover, we want that there remain enough conditions that a proof like the following still works: Let A be an uncountable set of conditions. Let w.l.o.g. $\{a_p \mid p \in A\}$ be a Δ-system with root Δ. Consider $\{p \restriction (\Delta \times \omega_2) \mid p \in A\}$. Then there are $p \neq q \in A$ such that $p \restriction (\Delta \times \omega_2)$ and $q \restriction (\Delta \times \omega_2)$ are compatible. Hence, p and q are compatible, too.

From now on, let $\mathfrak{M} = \langle \langle \theta_\alpha \mid \alpha \leq \omega_1 \rangle, \langle \mathfrak{F}_{\alpha\beta} \mid \alpha < \beta \leq \omega_1 \rangle \rangle$ be a simplified $(\omega_1, 1)$-morass.

In the recursive definition of $\mathbb{P}$, we use the morass tree $s \prec t$ and the mappings π_{st} to map conditions. Let more generally $\pi : \bar\theta \to \theta$ be any order-preserving map. Then $\pi : \bar\theta \to \theta$ induces maps $\pi : \bar\theta^2 \to \theta^2$ and $\pi : \bar\theta^2 \times \omega \to \theta^2 \times \omega$ in the obvious way:

$$\pi : \bar\theta^2 \to \theta^2, \quad \langle \gamma, \delta \rangle \mapsto \langle \pi(\gamma), \pi(\delta) \rangle$$

$$\pi : \bar\theta^2 \times \omega \to \theta^2 \times \omega, \quad \langle x, \epsilon \rangle \mapsto \langle \pi(x), \epsilon \rangle.$$

We define a system $\langle \langle \mathbb{P}_\eta \mid \eta \leq \omega_2 \rangle, \langle \sigma_{st} \mid s \prec t \rangle \rangle$ by induction on the levels of $\langle \langle \theta_\alpha \mid \alpha \leq \omega_1 \rangle, \langle \mathfrak{F}_{\alpha\beta} \mid \alpha < \beta \leq \omega_1 \rangle \rangle$ which we enumerate by $\beta \leq \omega_1$.

Base Case: $\beta = 0$

Then we need only to define $\mathbb{P}_1$.

Let $\mathbb{P}_1 := \{p \in P \mid a_p, b_p \subseteq 1\}$.

Successor Case: $\beta = \alpha + 1$

We first define $\mathbb{P}_{\theta_\beta}$. Let it be the set of all $p \in P$ such that:

(1) $a_p, b_p \subseteq \theta_\beta$

(2) $f_\alpha^{-1}[p], (id \upharpoonright \theta_\alpha)^{-1}[p] \in \mathbb{P}_{\theta_\alpha}$

(3) $p \upharpoonright ((\theta_\beta \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta))$ is injective

where f_α and δ are like in (P3) in the definition of a simplified gap-1 morass.
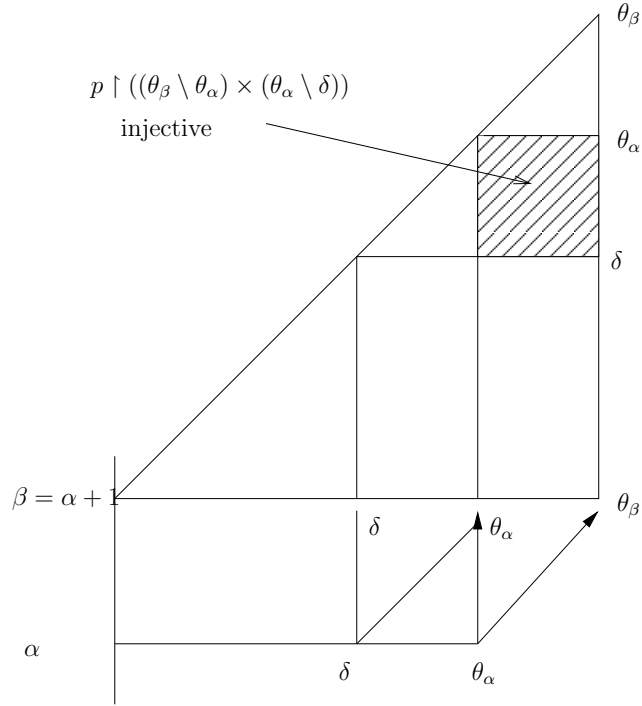
For $\nu \le \theta_\alpha$, $\mathbb{P}_\nu$ is already defined. For $\theta_\alpha < \nu \le \theta_\beta$ set $\mathbb{P}_\nu = \{p \in \mathbb{P}_{\theta_\beta} \mid a_p, b_p \subseteq \nu\}$.

Set

$$\sigma_{st} : \mathbb{P}_{\nu(s)+1} \to \mathbb{P}_{\nu(t)+1}, p \mapsto \pi_{st}[p].$$

Limit Case: $\beta \in Lim$

For $t \in T_\beta$ set $\mathbb{P}_{\nu(t)+1} = \bigcup \{\sigma_{st}[\mathbb{P}_{\nu(s)+1}] \mid s \prec t\}$ and $\mathbb{P}_\lambda = \bigcup \{\mathbb{P}_\eta \mid \eta < \lambda\}$ for $\lambda \in Lim$ where $\sigma_{st} : \mathbb{P}_{\nu(s)+1} \to \mathbb{P}_{\nu(t)+1}, p \mapsto \pi_{st}[p]$.

We set $\mathbb{P} := \mathbb{P}_{\omega_2}$.

Lemma 4.1

For $p \in P$, $p \in \mathbb{P}$ iff for all $\alpha < \omega_1$ and all $f \in \mathfrak{F}_{\alpha+1, \omega_1}$

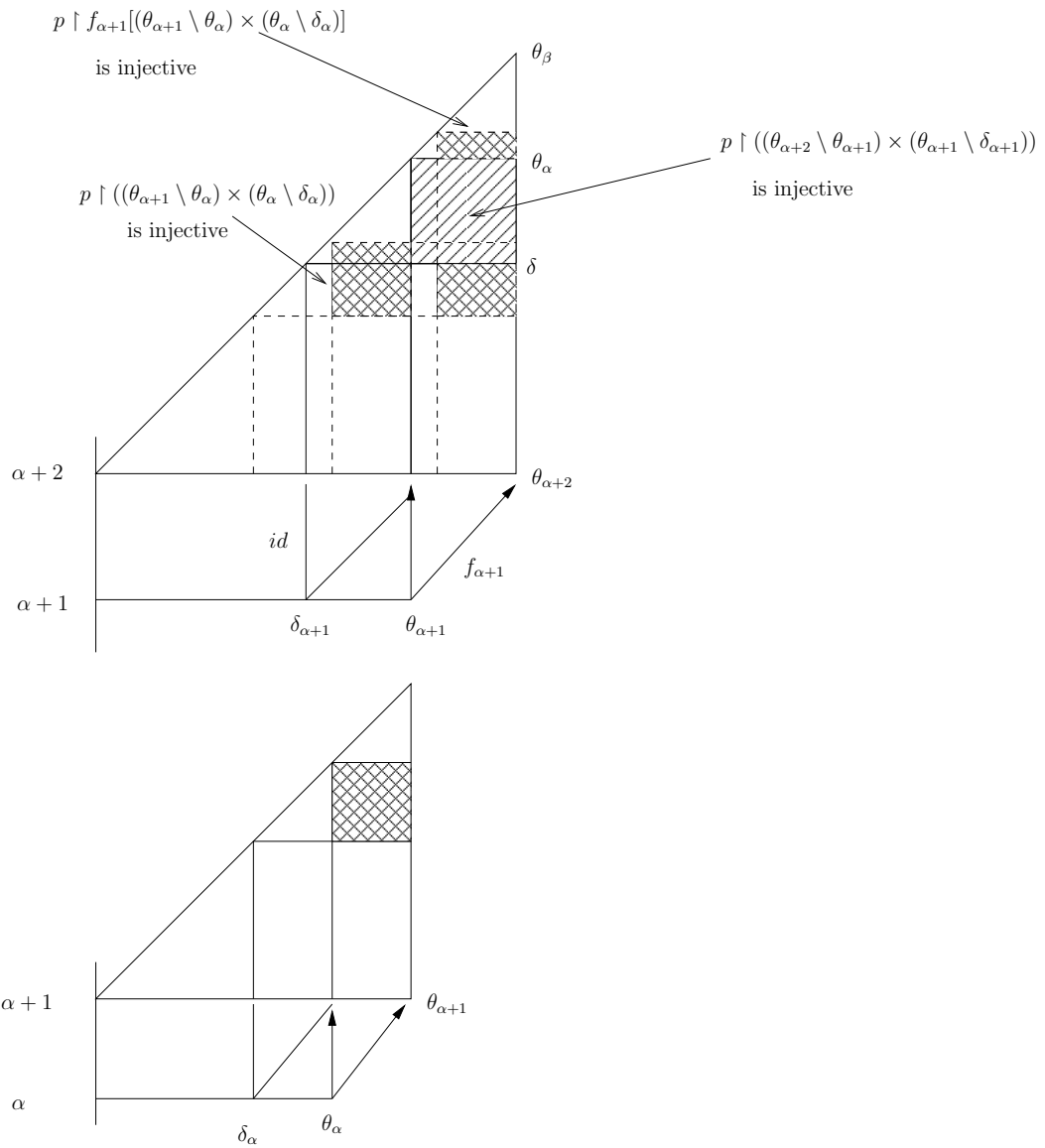$$f^{-1}[p] \upharpoonright ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \text{ is injective}$$

where δ_α is the critical point of f_α which is like in (P3) of the definition of a gap-1 morass.

Proof:

We prove by induction on $\gamma \leq \omega_1$ the following

Claim: $p \in \mathbb{P}_{\theta_\gamma}$ iff $p \in P$, $a_p \subseteq \theta_\gamma$, $b_p \subseteq \theta_\gamma$ and for all $\alpha < \gamma$ and all $f \in \mathfrak{F}_{\alpha+1,\gamma}$
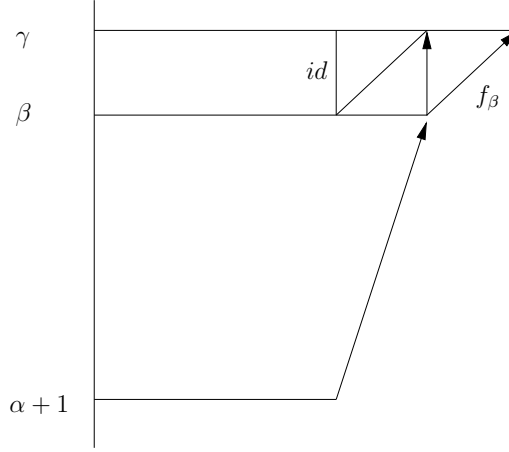
$$f^{-1}[p] \upharpoonright ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \text{ is injective.}$$

Base case: $\gamma = 0$

Then there is nothing to prove.

Successor case: $\gamma = \beta + 1$

Assume first that $p \in \mathbb{P}_{\theta_\gamma}$. Then, by (2) in the successor step of the definition of P_{ω_3}, $f^{-1}[p], (id \upharpoonright \theta_\beta)^{-1}[p] \in \mathbb{P}_{\theta_\beta}$. Now assume $f \in \mathfrak{F}_{\alpha+1,\gamma}$ and $\alpha < \beta$. Then $f = f_\beta \circ f'$ or $f = f'$ for some $f' \in \mathfrak{F}_{\alpha+1,\beta}$ by (P2) and (P3).

So by the induction hypothesis

$$f^{-1}[p] \upharpoonright ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \text{ is injective}$$

for all $f \in \mathfrak{F}_{\alpha+1,\gamma}$ and all $\alpha < \beta$. Moreover, if $\alpha = \beta$ then the identity is the only $f \in \mathfrak{F}_{\alpha+1,\gamma}$. In this case

$$f^{-1}[p] \upharpoonright ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \text{ is injective}$$

by (3) in the successor case of the definition of $\mathbb{P}$.

Now suppose that

$$f^{-1}[p] \upharpoonright ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \text{ is injective}$$

for all $\alpha < \gamma$ and all $f \in \mathfrak{F}_{\alpha+1,\gamma}$. We have to prove that (2) and (3) in the successor step of the definition of $\mathbb{P}$ hold. (3) obviously holds by the

assumption because the identity is the only function in $\mathfrak{F}_{\gamma\gamma} = \mathfrak{F}_{\beta+1,\gamma}$. For (2), it suffices by the induction hypothesis to show that

$$f^{-1}[f_\beta^{-1}[p]] \restriction ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \text{ is injective}$$

and

$$f^{-1}[(id \restriction \theta_\beta)^{-1}[p]] \restriction ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \text{ is injective}$$

for all $f \in \mathfrak{F}_{\alpha+1,\beta}$. This, however, holds by (P2) and the assumption.

Limit case: $\gamma \in Lim$

Assume first that $p \in \mathbb{P}_{\theta_\beta}$. Let $\alpha < \gamma$ and $f \in \mathfrak{F}_{\alpha+1,\gamma}$. We have to prove that

$$f^{-1}[p] \restriction ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \text{ is injective.}$$

By the limit step of the definition of $\mathbb{P}$, there are $\beta < \gamma$, $g \in \mathfrak{F}_{\beta\gamma}$ and $\bar{p} \in \mathbb{P}_{\theta_\beta}$ such that $p = g[\bar{p}]$. By (P4) there are $\alpha, \beta < \delta < \gamma$, $g' \in \mathfrak{F}_{\beta\delta}$, $f' \in \mathfrak{F}_{\alpha\delta}$ and $h \in \mathfrak{F}_{\delta\gamma}$ such that $g = h \circ g'$ and $f = h \circ f'$. Let $p' := g'[\bar{p}]$. Then, by the induction hypothesis
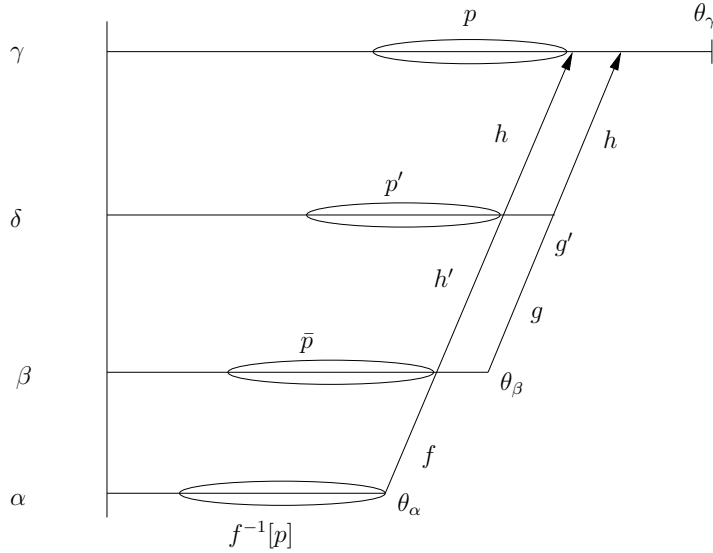
$$(f')^{-1}[p'] \restriction ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \text{ is injective.}$$

However, $(f')^{-1}[p'] = (f')^{-1}[h^{-1}[p]] = f^{-1}$ and we are done.

Now assume that

$$f^{-1}[p] \restriction ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \text{ is injective}$$

for all $\alpha < \gamma$ and all $f \in \mathfrak{F}_{\alpha+1,\gamma}$. We have to prove that $p \in \mathbb{P}_{\theta_\gamma}$, i.e. that there exists $t \in T_\gamma$ and $s \prec t$ such that $p = \pi_{st}[\bar{p}]$ for some $\bar{p} \in \mathbb{P}_{\nu(s)+1}$. To find such t, $s \prec t$ and $\bar{p}$, let $\nu < \theta_\gamma$ be such that $a_p, b_p \subseteq \nu$. Since $\nu = \{\pi_{st}[\nu(s)] \mid s \prec t\}$ and $p : [a_p, b_p] \to \omega$ is finite, there exist $s \prec t$ such that $a_p, b_p \subseteq rng(\pi_{st})$. Let $p = \pi_{st}[\bar{p}]$.

We need to prove that $\bar{p} \in \mathbb{P}_{\theta_\beta}$ where $\beta = \alpha(s)$. By the induction hypothesis it suffices to prove that

$$f^{-1}[\bar{p}] \upharpoonright ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \text{ is injective}$$
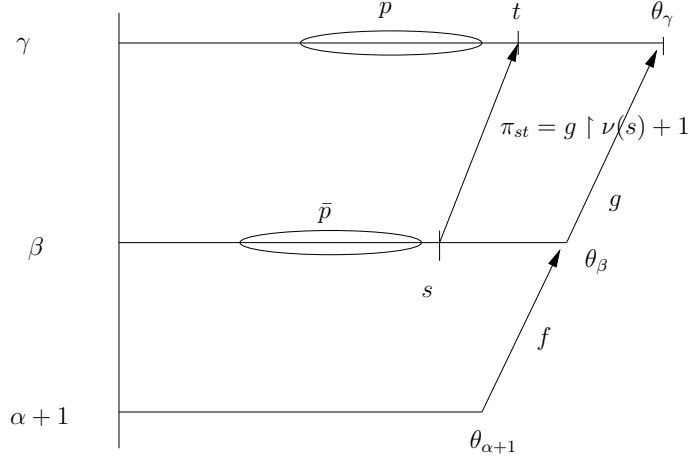
for all $\alpha < \beta$ and all $f \in \mathfrak{F}_{\alpha+1,\beta}$. So let $f \in \mathfrak{F}_{\alpha+1,\beta}$ and $g \in \mathfrak{F}_{\beta\gamma}$ such that $\pi_{st} = g \upharpoonright \nu(s) + 1$. Then

$$f^{-1}[\bar{p}] \upharpoonright ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) = f^{-1}[g^{-1}[p]] \upharpoonright ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) =$$

$$= (g \circ f)^{-1}[p] \upharpoonright ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha))$$

which is injective by our assumption. $\square$

For $p \in \mathbb{P}$ set

$$D_p = \{\alpha < \omega_1 \mid \exists f \in \mathfrak{F}_{\alpha+1,\omega_1} \ f^{-1}[p] \upharpoonright ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \neq \emptyset\}.$$
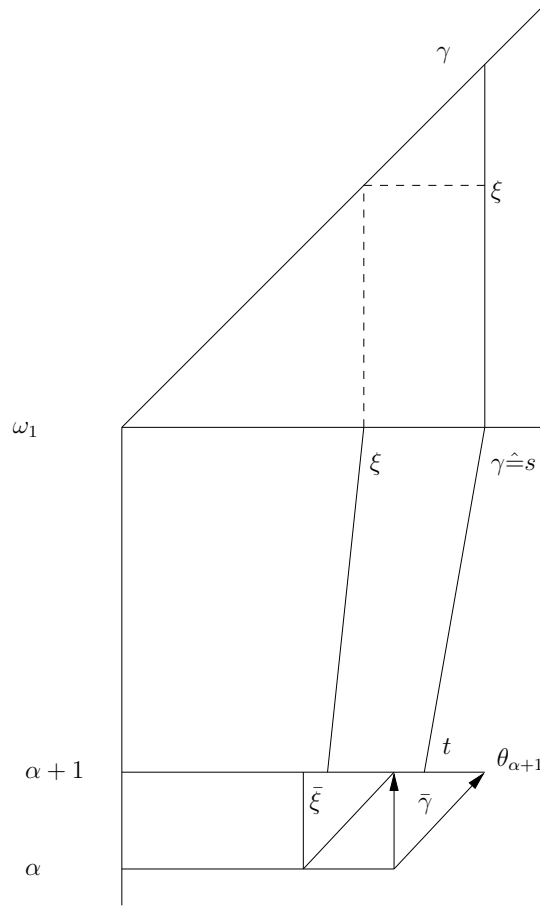
Lemma 4.2

D_p is finite for all $p \in \mathbb{P}$.

Proof: For every $\langle \gamma, \xi \rangle \in dom(p)$ set $s(\gamma, \xi) := \langle \omega_1, \gamma \rangle$ and let $t(\gamma, \xi)$ be the minimal $t \prec s(\gamma, \xi)$ such that $\xi \in rng(\pi_{t,s(\gamma,\xi)})$. Then

$$(*) \qquad D_p := \{\alpha \mid \exists \langle \gamma, \xi \rangle \in dom(p) \; t(\gamma, \xi) \in T_\alpha\}.$$

Hence D_p is finite because $dom(p)$ is finite.

Let us prove $(*)$.

Let $\langle \gamma, \xi \rangle \in dom(p)$, $s := s(\gamma, \xi)$, $t := t(\gamma, \xi)$ and $t \in T_{\alpha+1}$. Let $\pi_{st}(\bar{\gamma}) = \gamma$, $\pi_{st}(\bar{\xi}) = \xi$. Then $\bar{\gamma} \in \theta_{\alpha+1} \setminus \theta_\alpha$ and $\bar{\xi} \in \theta_\alpha \setminus \delta_\alpha$ by the minimality of $t \prec s$. Moreover, $\pi_{st} = f \restriction \nu(s) + 1$ for some $f \in \mathfrak{F}_{\alpha+1,\omega_1}$. Hence $\alpha \in D_p$.

Now, assume conversely that $\langle \gamma, \xi \rangle \in dom(p)$ and $f(\bar{\gamma}) = \gamma$, $f(\bar{\xi}) = \xi$ for some $f \in \mathfrak{F}_{\alpha+1,\omega_1}$ such that $\bar{\gamma} \in \theta_{\alpha+1} \setminus \theta_\alpha$, $\bar{\xi} \in \theta_\alpha \setminus \delta_\alpha$. Then $t \prec s$ and $\pi_{st}(\bar{\xi}) = \xi$ where $t := \langle \alpha + 1, \bar{\gamma} \rangle$, $s := \langle \omega_1, \gamma \rangle$. Moreover, there is no $t' \prec t$ such that $\bar{\xi} \in rng(\pi_{t't})$. Hence $t \prec s$ is minimal. $\square$

Let $\Delta \subseteq \omega_2$ be finite and $\mathbb{P}_\Delta = \{p \in \mathbb{P} \mid a_p \subseteq \Delta\}$. We want to represent every $p \in \mathbb{P}_\Delta$ as a function $p^* : [\alpha_0, \omega_1[\to \mathbb{P}$ such $p^*(\alpha) \in \mathbb{P}_{\theta_\alpha}$ for all $\alpha_0 \leq \alpha < \omega_1$:
Set

$\eta = max(\Delta)$

$t = \langle \omega_1, \eta \rangle$

$s_0 = min\{s \prec t \mid \Delta \subseteq rng(\pi_{st})\}$

$\alpha_0 = \alpha(s_0)$

$p^*(\alpha) = \pi_{st}^{-1}[p]$ for $\alpha_0 \leq \alpha < \omega_1$ where $s \in T_\alpha$, $s \prec t$

$supp(p) =$

$\{\alpha + 1 \mid \alpha_0 \leq \alpha < \omega_1, p^*(\alpha+1) \neq p^*(\alpha), p^*(\alpha+1) \neq f_\alpha[p^*(\alpha)]\} \cup \{\alpha_0\}$

where f_α is like in (P3) of the definition of a simplified gap-1 morass.

Note, that by $supp(p)$ is finite, since p is finite.
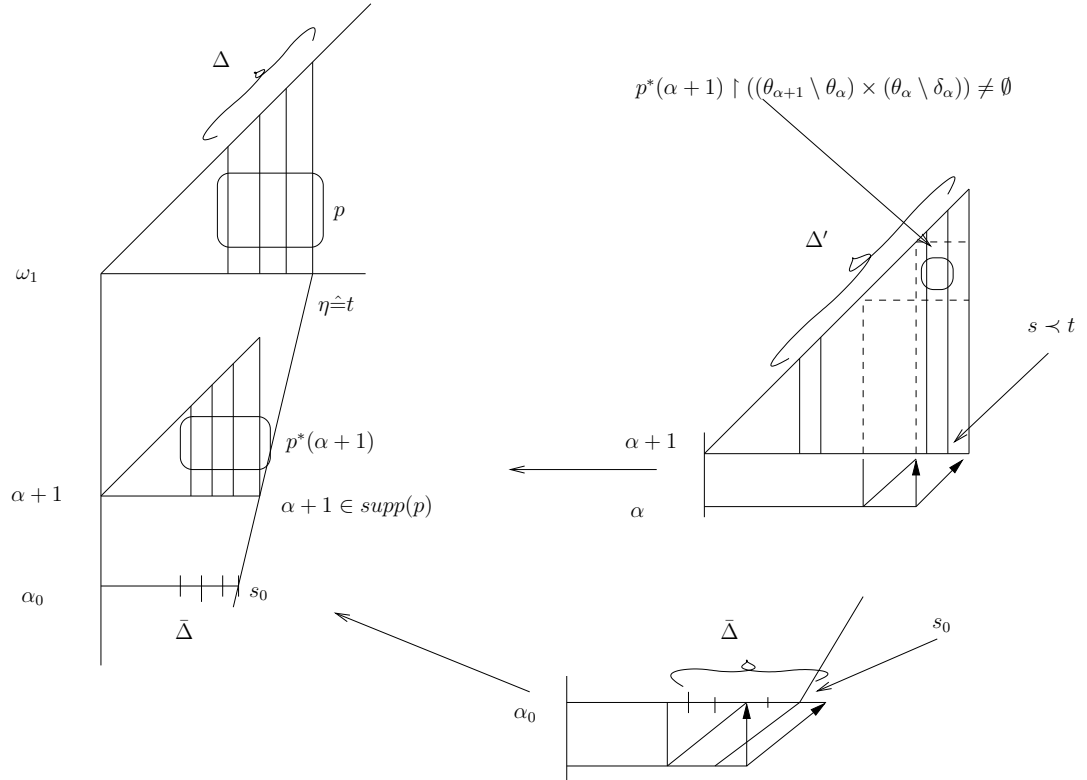
Lemma 4.3

If $p, q \in \mathbb{P}_\Delta$ and $p^*(\alpha), q^*(\alpha)$ are compatible in $\mathbb{P}_{\theta_\alpha}$ for $\alpha = max(supp(p) \cap supp(q))$, then p and q are compatible in $\mathbb{P}_\Delta$.

Proof: Suppose p and q are like in the lemma, but incompatible. Let $(supp(p) \cup supp(q)) - \alpha = \{\gamma_n < \ldots < \gamma_1\}$. We prove by induction on $1 \leq i \leq n$, that $p^*(\gamma_i)$ and $q^*(\gamma_i)$ are incompatible for all $1 \leq i \leq n$. Since $\gamma_n = \alpha$, this yields the desired contradiction.

Note first, that $p^*(\gamma_1)$ and $q^*(\gamma_1)$ are incompatible because otherwise $p = \pi_{st}[p^*(\gamma_1)]$ and $q = \pi_{st}[q^*(\gamma_1)]$ were incompatible (for $s \in T_{\gamma_1}$, $s \prec t$).
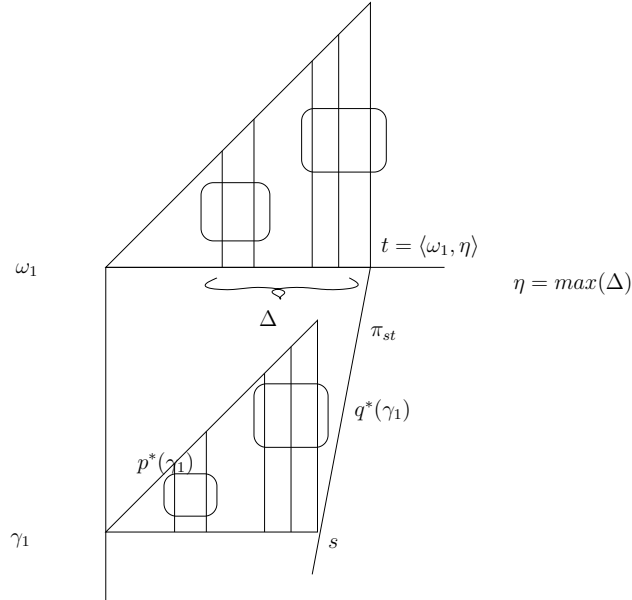
If $\gamma_1 = \alpha$, we are done. So assume that $\gamma_1 \neq \alpha$. Then either $p^*(\gamma_1) = \pi_{\bar{s}s}[p^*(\gamma_1 - 1)]$ or $q^*(\gamma_1) = \pi_{\bar{s}s}[q^*(\gamma_1 - 1)]$ where $\bar{s} \prec s \prec t$, $\bar{s} \in T_{\gamma_1 - 1}$ and $s \in T_{\gamma_1}$. We assume in the following that $p^*(\gamma_1) = \pi_{\bar{s}s}[p^*(\gamma_1 - 1)]$. Mutatis mutandis, the other case works the same.
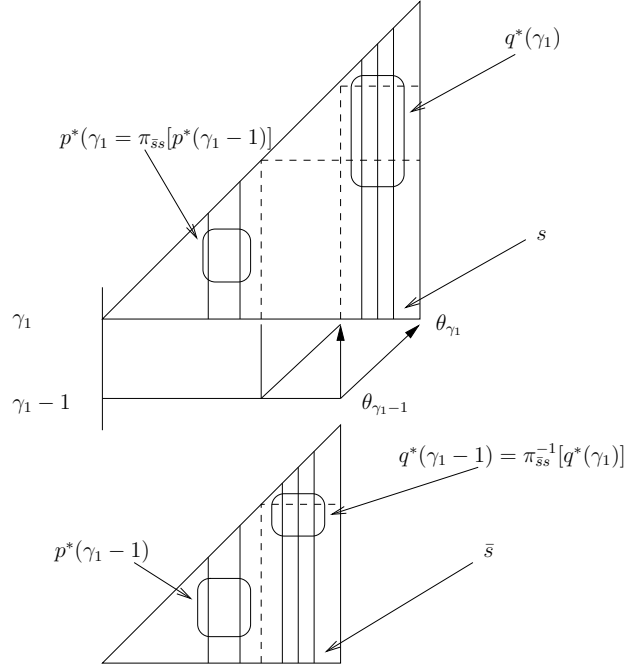
Claim: $p^*(\gamma_1 - 1)$ and $q^*(\gamma_1 - 1)$ are incompatible in $\mathbb{P}_{\theta_{\gamma_1 - 1}}$

Assume not. Then there is $\bar{r} \leq p^*(\gamma_1 - 1), q^*(\gamma_1 - 1)$ in $\mathbb{P}_{\theta_{\gamma_1 - 1}}$ such that $a_{\bar{r}} = a_{p^*(\gamma_1 - 1)} \cup a_{q^*(\gamma_1 - 1)}$. Let $r' := \pi_{\bar{s}s}[\bar{r}]$.
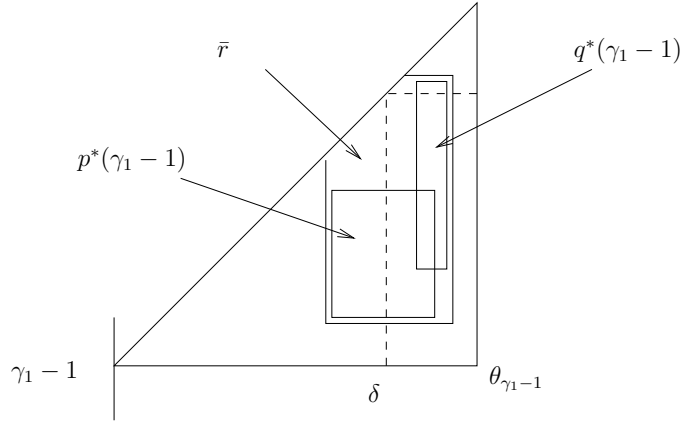
Then $r' \leq \pi_{\bar{s}s}[p^*(\gamma_1 - 1)] = p^*(\gamma_1)$ and $r' \leq \pi_{\bar{s}s}[q^*(\gamma_1 - 1)] = q^*(\gamma_1) \upharpoonright rng(\pi_{\bar{s}s})^2$. In the following we will construct an $r \leq p^*(\gamma_1), q^*(\gamma_1)$ which yields the contradiction we were looking for. Let $a_r := a_{q^*(\gamma_1)} \cup a_{p^*(\gamma_1)}$ and $b_r := b_{q^*(\gamma_1)} \cup b_{p^*(\gamma_1)}$. For $\langle \xi, \delta \rangle \in [a_{r'}, b_{r'}]$ set $r(\xi, \delta) := r'(\xi, \delta)$. For $\langle \xi, \delta \rangle \in [a_{q^*(\gamma_1)}, b_{q^*(\gamma_1)}]$ set $r(\xi, \delta) := q^*(\gamma_1)(\xi, \delta)$. Then $r(\xi, \delta)$ is defined for all $\langle \xi, \delta \rangle \in [a_r, b_r]$ except for those in $[a_{p^*(\gamma_1)} - a_{q^*(\gamma_1)}, b_{q^*(\gamma_1)} - rng(\pi_{\bar{s}s})]$.
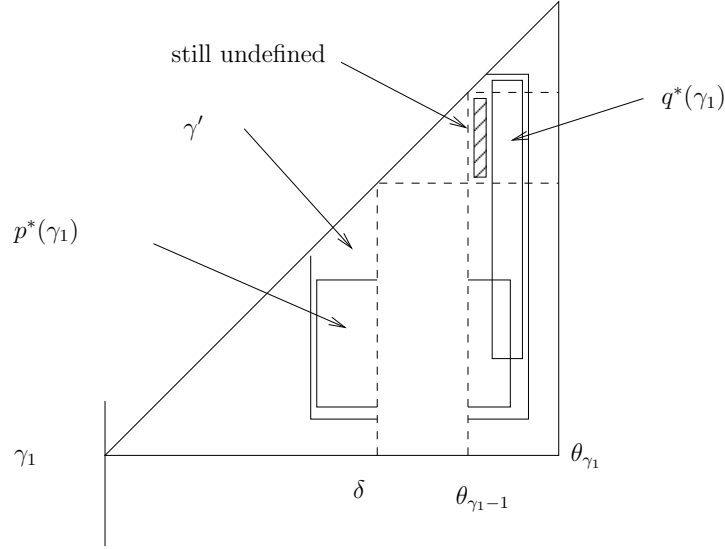
For those choose any values such that (3) in the successor step of the recursive definition of $\mathbb{P}$ holds. Then obviously $r \in \mathbb{P}_{\theta_{\gamma_1}}$. It remains to prove $r \leq p^*(\gamma_1), q^*(\gamma_1)$. That is, we must show that

(1) $r(\alpha, \xi) \neq r(\beta, \xi)$ for all $\alpha < \beta \in a_{p^*(\gamma_1)}$ and all $\xi \in b_r - b_{p^*(\gamma_1)}$ with $\xi < \alpha$

(2) $r(\alpha, \xi) \neq r(\beta, \xi)$ for all $\alpha < \beta \in a_{q^*(\gamma_1)}$ and all $\xi \in b_r - b_{q^*(\gamma_1)}$ with $\xi < \alpha$.

The first statement is clear if $\xi \in b_{r'}$ because $r' \leq p^*(\gamma_1)$. So assume $\xi \notin b_{r'}$. Then $\xi \notin rng(\pi_{\bar{s}s})$. Now, we use (P3) in the definition a simplified gap-1 morass. From (P3) and the fact that $\xi \notin rng(\pi_{\bar{s}s})$, $\alpha \in a_{p^*(\gamma_1)}$ and $\xi < \alpha$ it follows that $\pi_{\bar{s}s} \neq id \restriction \nu(\bar{s}) + 1$. Moreover, if δ is the critical point of f_{γ_1-1} like in (P3), then $\xi \in \theta_{\gamma_1-1} \setminus \delta$ and $\alpha, \beta \in \theta_{\gamma_1} \setminus \theta_{\gamma_1-1}$. Hence the first statement holds because of (3) in the successor step of the recursive definition of $\mathbb{P}$.

The proof of the second statement is mutatis mutandis the same. This proves the claim.

It follows from the claim, that $p^*(\gamma_2)$ and $q^*(\gamma_2)$ are incompatible. Hence we can prove the lemma by repeating this argument inductively finitely many times. $\square$

Lemma 4.4

$\mathbb{P}$ satisfies ccc.

Proof: Let $A \subseteq \mathbb{P}$ be a set of size ω_1. By the Δ-lemma, we may assume that $\{D_p \mid p \in A\}$ forms a Δ-system with root D. We may moreover assume that for all $\alpha \in D$, all $f \in \mathfrak{F}_{\alpha+1, \omega_1}$ and all $p, q \in A$

$$f^{-1}[p] \restriction ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \subseteq f^{-1}[q] \restriction ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha))$$

or

$$f^{-1}[p] \upharpoonright ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \supseteq f^{-1}[q] \upharpoonright ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)).$$

To see this assume that $X = \{a_p \mid p \in A\} \subseteq \omega_2$ forms a Δ-system with root D_1 and $Y = \{b_p \mid p \in A\} \subseteq \omega_2$ forms a Δ-system with root D_2. Fix $\alpha \in D$. By thinning out A, we can ensure that whenever $a \neq b \in X$, $\eta \in a - b$, $\nu \in b - a$, $\alpha < \beta$, $t = \langle \omega_1, \beta \rangle$, $s \prec t$, $s \in T_{\alpha+1}$, then $\eta \notin rng(\pi_{st})$. Moreover, we can guarantee that whenever $a \neq b \in Y$, $\eta \in a - b$, $\nu \in b - a$, $\alpha < \beta$, $t = \langle \omega_1, \beta \rangle$, $s \prec t$, $s \in T_{\alpha+1}$, then $\eta \notin rng(\pi_{st})$. This suffices.

By the Δ-system lemma, we may assume that $\{a_p \mid p \in A\} \subseteq \omega_2$ forms a Δ-system with root Δ_1. Consider $A' := \{p \upharpoonright (\Delta_1 \times \omega_2) \mid p \in A\}$. By the Δ-system lemma we may also assume that $\{supp(p) \mid p \in A'\} \subseteq \omega_1$ forms a Δ-system with root Δ_2. Let $\alpha = max(\Delta_2)$. Since $\mathbb{P}_{\theta_\alpha}$ is countable, there are $q_1 \neq q_2 \in A'$ such that $q_1^*(\alpha) = q_2^*(\alpha)$. Hence $q_1 \neq q_2 \in A'$ are compatible by a previous lemma. Assume that $q_1 = p_1^* \upharpoonright \Delta_2$ and $q_2 = p_2^* \upharpoonright \Delta_2$ with $p_1, p_2 \in A$.
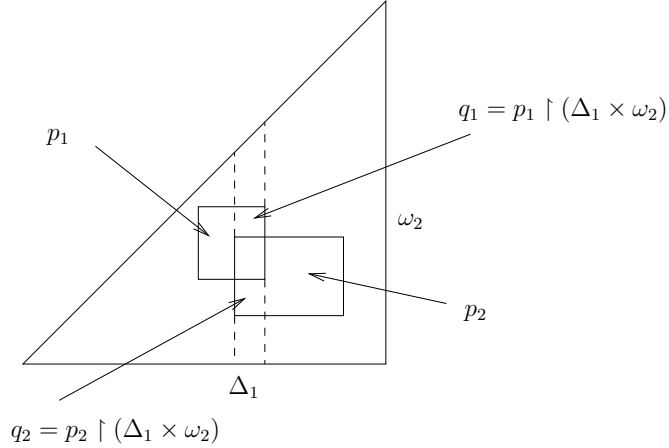
We can define $p \leq p_1, p_2$ as follows: $a_p = a_{p_1} \cup a_{p_2}$, $b_p = b_{p_1} \cup b_{p_2}$, $p \upharpoonright (a_{p_1} \times b_{p_1}) = p_1$, $p \upharpoonright (a_{p_2} \times b_{p_2}) = p_2$. We still need to define p on $[a_p, b_p] - ((a_{p_1} \times b_{p_1}) \cup (a_{p_2} \times b_{p_2}))$. We do this in such a way that the new values are different from the old ones and distinct among each other. Then $p \leq p_1, p_2$. We prove $p \leq p_1$. The other statement is showed similar. Let $\alpha < \beta \in a_p$ and $\xi \in b_p - b_{p_1}$. We have to show that $p(\alpha, \xi) \neq p(\beta, \xi)$. If $\alpha, \beta \in a_{q_1}$, then it holds because $q_1 = p \upharpoonright (\Delta_1 \times b_{q_1})$ and $q_2 = p \upharpoonright (\Delta_1 \times b_{q_2})$ are compatible. If either α or β is in a_{p_1}, while the other one is not, then $p(\alpha, \xi) \neq p(\beta, \xi)$ because the new values differ from the old ones. If $\alpha, \beta \in a_{p_1}$,

then $p(\alpha, \xi) \neq p(\beta, \xi)$ because the new values differ among each other.

It remains to prove that $p \in \mathbb{P}$. For this, we show that for all $\alpha < \omega_1$ and all $f \in \mathfrak{F}_{\alpha+1,\omega_1}$

$$f^{-1}[p] \restriction ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \text{ is injective,}$$

$$\text{i.e. } p \restriction f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)] \text{ is injective.}$$

Assume that $\alpha \in D$. Then by our second thinning-out

$$p_1 \restriction f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)] \subseteq p_2 \restriction f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)]$$

or

$$p_1 \restriction f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)] \supseteq p_2 \restriction f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)]$$

and hence

$$([a_p, b_p] - (a_{p_1} \times b_{p_1})) \cap f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)] =$$

$$= [a_p, b_p] - ((a_{p_1} \times b_{p_1}) \cup (a_{p_2} \times b_{p_2})) \cap f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)]$$

or

$$([a_p, b_p] - (a_{p_2} \times b_{p_2})) \cap f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)] =$$

$$= [a_p, b_p] - ((a_{p_1} \times b_{p_1}) \cup (a_{p_2} \times b_{p_2})) \cap f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)].$$

Assume w.l.o.g. the first. Let $\langle \alpha_1, \beta_1 \rangle \neq \langle \alpha_2, \beta_2 \rangle \in [a_p, b_p] \cap f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)]$. Then either $\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle \in (a_{p_1} \times b_{p_1})$ or at least one of both is not. In the first case $p(\alpha_1, \beta_1) \neq p(\alpha_2, \beta_2)$ because $p \leq p_1$ and $p_1 \in \mathbb{P}$. In the second case it is clear by our definition of p on $[a_p, b_p] - ((a_{p_1} \times b_{p_1}) \cup (a_{p_2} \times b_{p_2}))$.

Finally assume that $\alpha \notin D$. Then by our first thinning-out

$$p_1 \restriction f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)] = \emptyset$$

$$\text{or} \qquad p_2 \restriction f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)] = \emptyset.$$

Hence

$$([a_p, b_p] - (a_{p_1} \times b_{p_1})) \cap f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)] =$$

$$= [a_p, b_p] - ((a_{p_1} \times b_{p_1}) \cup (a_{p_2} \times b_{p_2})) \cap f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)]$$

or

$$([a_p, b_p] - (a_{p_2} \times b_{p_2})) \cap f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)] =$$

$$= [a_p, b_p] - ((a_{p_1} \times b_{p_1}) \cup (a_{p_2} \times b_{p_2})) \cap f[(\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)].$$

From this the injectivity follows like in the case $\alpha \in D$. $\square$

Lemma 4.5

Let $p \in \mathbb{P}$ and $\alpha, \beta \in \omega_2$. Then there exists $q \leq p$ such that $\alpha \in a_q$ and $\beta \in b_q$.

Proof: Let $a_q = a_p \cup \{\alpha\}$, $b_q = b_p \cup \{\beta\}$ and $q \upharpoonright (a_p \times b_p) = p$. We have to define $q(\alpha, \beta)$ on $[a_q, b_q] - ((a_{p_1} \times b_{p_1}) \cup (a_{p_2} \times b_{p_2}))$. We do this in such a way that the new values are different from the old ones and distinct among each other. Then obviously $q \leq p$ and

$$f^{-1}[p] \upharpoonright ((\theta_{\alpha+1} \setminus \theta_\alpha) \times (\theta_\alpha \setminus \delta_\alpha)) \text{ is injective}$$

for all $\alpha < \omega_1$ and all $f \in \mathfrak{F}_{\alpha+1,\omega_1}$. Hence also $q \in \mathbb{P}$. □

Theorem 4.6

Assume that there exists a simplified $(\omega_1, 1)$-morass. Then there is a ccc forcing which forces $\omega_2 \not\rightarrow (\omega : 2)_\omega^2$.

Proof: Of course, $\mathbb{P}$ is the forcing which we defined above. Let G be $\mathbb{P}$-generic and set $f = \bigcup\{p \mid p \in G\}$. Since $\mathbb{P}$ satisfies ccc, cardinals are preserved. By the previous lemma, f is defined on all of $[\omega_2]^2$. By the definition of $\leq$, f is as wanted. □

There are other applications of two-dimensional forcing which also require destroying GCH.

Let us commence with considering strongly almost disjoint subsets of ω_1. Assume that $\langle X_\alpha \mid \alpha < \omega_2 \rangle$ is a family of infinite subsets $X_\alpha \subseteq \omega_1$ such that $|X_\alpha \cap X_\beta| < \omega$ for all $\alpha \neq \beta \in \omega_2$. Then there exists a family $\langle X'_\alpha \mid \alpha < \omega_2 \rangle$ of infinite subsets $X'_\alpha \subseteq \omega_1$ such that $|X'_\alpha \cap X'_\beta| < \omega$ and $|X'_\alpha| = \omega$ for all $\alpha \neq \beta \in \omega_2$. To see this, let $X'_\alpha \subseteq X_\alpha$ be any subset of X_α with size ω.
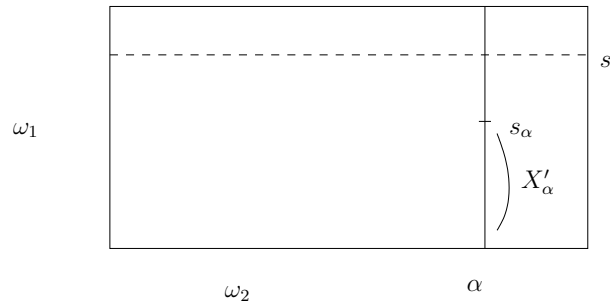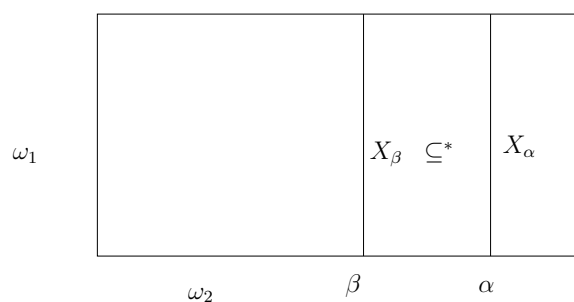

For every $\alpha \in \omega_2$ set $s_\alpha = sup(X'_\alpha)$. Since $s_\alpha \in \omega_1$ for all $\alpha \in \omega_2$, there exists $D \subseteq \omega_2$ with $|D| = \omega_2$ and $s \in \omega_1$ such that $s_\alpha = s$ for all $\alpha \in D$. Hence

$X_\alpha \neq X_\beta$ and $X_\alpha \subseteq s$ for all $\alpha \neq \beta \in D$. However, s is countable. Therefore, $2^\omega \geq \omega_2$.

In the following, we will use two-dimensional forcing to add a chain $\langle X_\alpha \mid \alpha \in \omega_2 \rangle$ such that $X_\alpha \subseteq \omega_1$, $X_\beta - X_\alpha$ is finite and $X_\alpha - X_\beta$ has size ω_1 for all $\beta < \alpha < \omega_2$. The consistency of such a chain was first proved by P. Koszmider using ordinal walks / Δ-functions.

Note, that if we set $Y_\alpha = X_{\alpha+1} - X_\alpha$ then $\langle Y_\alpha \mid \alpha < \omega_2 \rangle$ forms a family of uncountable subsets of ω_1 such that $|Y_\alpha \cap Y_\beta| < \omega$ for all $\alpha \neq \beta \in \omega_2$. Hence by the argument above, if such a family exists, CH cannot hold.

The natural forcing would be

$$ P := \{p : a_p \times b_p \to 2 \mid a_p \times b_p \subseteq \omega_2 \times \omega_1 \text{ finite } \} $$

where we set $p \leq q$ iff $p \subseteq p$ and

$$ \forall \alpha_1 < \alpha_2 \in a_p \ \forall \beta \in b_p - b_q \quad p(\alpha_1, \beta) \leq p(\alpha_2, \beta). $$
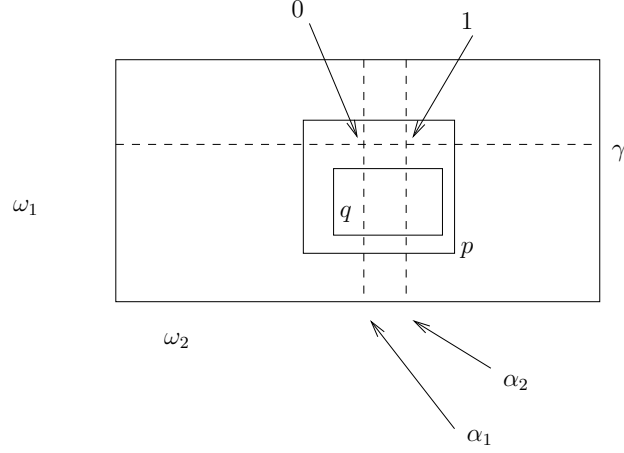
Obviously, we will set $X_\alpha = \{\beta \in \omega_1 \mid p(\alpha_1, \beta) = 1$ for some $p \in G\}$ where G is P-generic. However, like the forcing for $\omega_2 \not\to (\omega : 2)_\omega^2$, P does not satisfy ccc. If $\alpha < \beta \in \omega_2$, then the set

$$A = \{p \in P \mid a_p = \{\alpha, \beta\}, b_p = \{\gamma\}, \gamma \in \omega_1, p(\alpha, \gamma) = 1, p(\beta, \gamma) = 0\}$$

is an antichain of size ω_1.
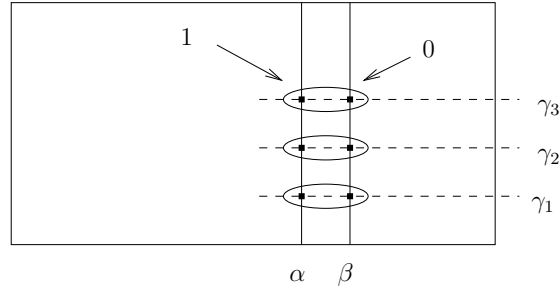
To thin P out to a forcing $\mathbb{P}$ we proceed like before: Let $\mathfrak{M} = \langle\langle \theta_\alpha \mid \alpha \leq \omega_1\rangle, \langle \mathfrak{F}_{\alpha\beta} \mid \alpha < \beta \leq \omega_1\rangle\rangle$ be a simplified $(\omega_1, 1)$-morass. We define $\langle\langle \mathbb{P}_\eta \mid \eta \leq \omega_2\rangle, \langle \sigma_{st} \mid s \prec t\rangle\rangle$ by induction on the levels of $\mathfrak{M}$ which we enumerate by $\beta \leq \omega_2$.

Base case: $\beta = 0$

Then we only need to define $\mathbb{P}_1$. Set $\mathbb{P}_1 := \{p \in P \mid a_p \times b_p \subseteq 1 \times 1\}$.

Successor case: $\beta = \alpha + 1$.

We first define $\mathbb{P}_{\theta_\beta}$. Let it be the set of all $p \in P$ such that

(1) $a_p \times b_p \subseteq \theta_\beta \times \beta$

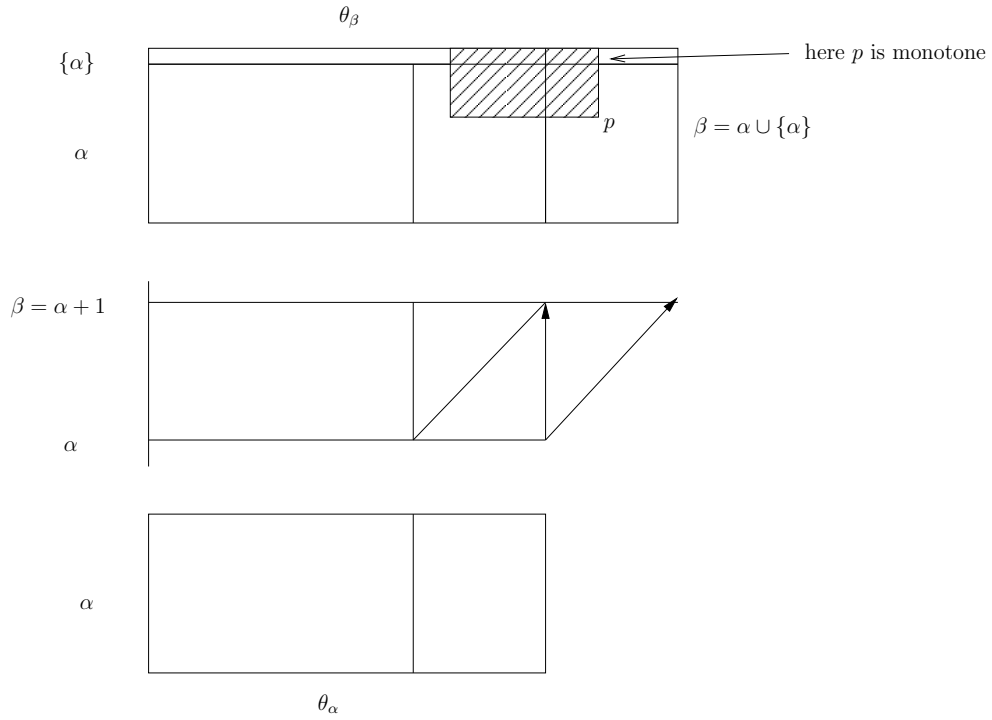(2) $f_\alpha^{-1}[p] \upharpoonright (\theta_\alpha \times \alpha) \in \mathbb{P}_{\theta_\alpha}$, $p \upharpoonright (\theta_\alpha \times \alpha) \in \mathbb{P}_{\theta_\alpha}$ where f_α is like in (P3) in the definition of a simplified gap-1 morass. For its extension to P see below.

(3) If $\alpha \in b_p$, then $p(\gamma, \alpha) \le p(\delta, \alpha)$ for all $\gamma < \delta \in a_p$, i.e.

$$p \upharpoonright (\theta_\beta \times \{\alpha\}) \text{ is monotone.}$$

For all $\nu \le \theta_\alpha$, $\mathbb{P}_\nu$ is already defined. For $\theta_\alpha < \nu \le \theta_\beta$ set

$$\mathbb{P}_\nu := \{p \in \mathbb{P}_{\theta_\beta} \mid a_p \times b_p \subseteq \nu \times \beta\}.$$

Set

$$\sigma_{st} : \mathbb{P}_{\nu(s)+1} \to \mathbb{P}_{\nu(t)+1}, \quad p \mapsto \pi_{st}[p].$$

For the definition of $\pi_{st}[p]$, note that any map $\pi : \bar{\theta} \to \theta$ induces maps $\pi : \bar{\theta} \times \omega_1 \to \theta \times \omega_1$ and $\pi : (\bar{\theta} \times \omega_1) \times 2 \to (\theta \times \omega_1) \times 2$ by:

$$\pi : \bar{\theta} \times \omega_1 \to \theta \times \omega_1, \quad \langle \gamma, \delta \rangle \mapsto \langle \pi(\gamma), \delta \rangle$$

$$\pi : (\bar{\theta} \times \omega_1) \times 2 \to (\theta \times \omega_1) \times 2, \quad \langle x, \epsilon \rangle \mapsto \langle \pi(x), \epsilon \rangle.$$

Limit case: $\beta \in Lim$

For $t \in T_\beta$ set $\mathbb{P}_{\nu(t)+1} = \bigcup \{\sigma_{st}[\mathbb{P}_{\nu(s)+1}] \mid s \prec t\}$ and $\mathbb{P}_\lambda = \bigcup \{\mathbb{P}_\eta \mid \eta < \lambda\}$ for $\lambda \in Lim$ where

$$\sigma_{st} : \mathbb{P}_{\nu(s)+1} \to \mathbb{P}_{\nu(t)+1}, \quad p \mapsto \pi_{st}[p].$$

Like before we can prove the following

Lemma 4.7

For $p \in P$, $p \in \mathbb{P}$ iff for all $\alpha < \omega_1$ and all $f \in \mathfrak{F}_{\alpha+1,\omega_1}$

$$f^{-1}[p] \restriction (\theta_{\alpha+1} \times \{\alpha\}) \quad \text{is monotone.}$$

If we follow the consistency proof for $\omega_2 \nrightarrow (\omega : 2)^2_\omega$, we finally get:

Theorem 4.8

Assume that there exists a simplified $(\omega_1, 1)$-morass. Then there exists a ccc forcing which adds a chain $\langle X_\alpha \mid \alpha < \omega_2 \rangle$ such that $X_\alpha \subseteq \omega_1$, $X_\beta - X_\alpha$ is finite and $X_\alpha - X_\beta$ is uncountable for all $\beta < \alpha < \omega_2$.

Closely related to $\omega_2 \nrightarrow (\omega : 2)^2_\omega$ are families of ω_2-many strongly almost disjoint functions $f_\alpha : \omega_1 \to \omega$. Of course, our proof of the consistency of $\omega_2 \nrightarrow (\omega : 2)^2_\omega$ also shows that there can be consistently ω_2-many strongly almost disjoint functions $f_\alpha : \omega_1 \to \omega$. To force it directly, consider

$$P =: \{p : a_p \times b_p \to \omega \mid a_p \times b_p \subseteq \omega_2 \times \omega_1 \text{ finite}\}$$

with $p \leq q$ iff $q \subseteq p$ and

$$\forall \alpha_1 < \alpha_2 \in a_p \ \forall \beta \in b_p - b_q \quad p(\alpha_1, \beta) \neq p(\alpha_2, \beta).$$

We replace (3) in the recursive definition of $\mathbb{P}$ in our last example by

(3)′ If $\alpha \in b_p$, then $p(\gamma, \alpha) \neq p(\delta, \alpha)$ for all $\gamma < \delta \in a_p$, i.e.

$$p \restriction (\theta_\beta \times \{\alpha\}) \quad \text{is injective.}$$
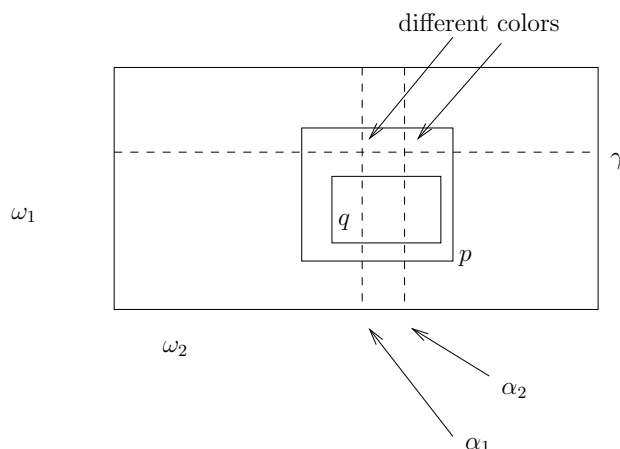
Then we get:

Theorem 4.9

Assume that there exists a simplified $(\omega_1, 1)$-morass. Then there is a ccc forcing which adds a family $\langle f_\alpha \mid \alpha < \omega_2 \rangle$ of functions $f_\alpha : \omega_1 \to \omega$ such that $\{\xi < \omega_1 \mid f_\alpha(\xi) = f_\beta(\xi)\}$ is finite for all $\alpha < \beta < \omega_2$.

It is known that there can be families $\{f_\alpha : \omega_1 \to \omega \mid \alpha \in \kappa\}$ of arbitrary pre-scribed size κ such that $\{\xi < \omega_1 \mid f_\alpha(\xi) = f_\beta(\xi)\}$ is finite for all $\alpha < \beta < \kappa$. This was proved by J. Zapletal using proper forcing and Todorcevic's method of models as side conditions.

Note, that such a family of functions is a family of ω_2-many uncountable sub-sets of $\omega_1 \times \omega$ such that the intersection of two distinct members is always finite. Hence by our above observations $2^\omega \geq \omega_2$.

Our final application of two-dimensional forcing is to add an (ω, ω_2)-superatomic Boolean algebra.

Let us recall some facts about Boolean algebras which can be found in the Handbook of Boolean Algebras:

$\mathbb{B} = \langle B, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra if the following axioms hold for all $x, y, z \in B$:

(B1) $x + (y + z) = (x + y) + z$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

(B2) $x + y = y + x$, $x \cdot y = y \cdot x$
(B3) $x + (x \cdot y) = x$, $x \cdot (x + y) = x$
(B4) $x \cdot (y + z) = x \cdot y + x \cdot z$, $x + (y \cdot z) = (x + y) \cdot (x + z)$
(B5) $x + (-x) = 1$, $x \cdot (-x) = 0$.

A homomorphism $f : \mathbb{B}_1 \to \mathbb{B}_2$ is a map such that for all $x, y \in \mathbb{B}_1$:
$f(0) = 0$, $f(1) = 1$
$f(x + y) = f(x) + f(y)$, $f(x \cdot y) = f(x) \cdot f(y)$
$f(-x) = -f(x)$.

Define a relation on $\mathbb{B}$ by setting

$$x \leq y \quad \text{iff} \quad x + y = y.$$

This is a partial order.

An $a \in \mathbb{B}$ is called atom if $0 < a$ but there is no $x \in A$ with $0 < x < a$.

$\mathbb{B}$ is called atomic if for each $0 < x$ there exists an atom $a < x$.

If $\mathbb{B}$ is atomic and $At(\mathbb{B})$ is its set of atoms, then

$$f(x) = \{a \in At(\mathbb{B}) \mid a \leq x\}$$

is an injective homomorphism $f : \mathbb{B} \to \mathfrak{P}(At(\mathbb{B}))$.

Hence every atomic Boolean algebra can be represented as an algebra of sets. This is possible in general.

A filter F on $\mathbb{B}$ is a subset $F \subseteq \mathbb{B}$ such that for all $u, v \in \mathbb{B}$:
(i) $0 \notin F$, $1 \in F$
(ii) $u \in F$, $v \in F \Rightarrow u \cdot v \in F$
(iii) $u, v \in \mathbb{B}$, $u \in F$, $u \leq v \Rightarrow v \in F$.

A filter F on $\mathbb{B}$ is an ultrafilter if for all $u \in \mathbb{B}$ either $u \in F$ or $-u \in F$.

Set $S(\mathbb{B}) := \{F \subseteq \mathbb{B} \mid F \text{ is an ultrafilter on } \mathbb{B}\}$.

Then the map

$$f : \mathbb{B} \to \mathfrak{P}(S(\mathbb{B})), \ x \mapsto \{F \in S(\mathbb{B}) \mid x \in F\}$$

is an injective homomorphism.

Note, that $f[\mathbb{B}]$ is a family of subsets of $S(\mathbb{B})$ which is closed under finite intersections. Hence it is the base of a unique topology. $S(\mathbb{B})$ equipped with this topology is called the Stone space of $\mathbb{B}$.

An ideal I on $\mathbb{B}$ is a subset $I \subseteq \mathbb{B}$ such that for all $u, v \in \mathbb{B}$:

(i) $0 \in I$, $1 \notin I$

(ii) $u \in I$, $v \in I \Rightarrow u + v \in I$

(iii) $u, v \in \mathbb{B}$, $u \in I$, $v \leq u \Rightarrow v \in I$.

Let I be an ideal on $\mathbb{B}$. Consider the equivalence relation

$$u \sim v \quad \text{iff} \quad u \Delta v := u \cdot (-v) + v \cdot (-u) \in I$$

on $\mathbb{B}$. Let $[u] = \{v \in \mathbb{B} \mid u \sim v\}$. Then $\mathbb{B}/I := \{[u] \mid u \in \mathbb{B}\}$ with
$0 = [0]$, $1 = [1]$
$[u] + [v] = [u + v]$, $[u] \cdot [v] = [u \cdot v]$
$-[u] = [-u]$
forms a Boolean algebra. $\mathbb{B}/I$ is called a quotient algebra.

A Boolean algebra is called superatomic if every quotient algebra is atomic.

Assume that $\mathbb{B}$ is superatomic. Then we can define recursively:
$I_0 = \{0\}$
$I_{\alpha+1} =$ the ideal generated by $I_\alpha \cup \{u \in \mathbb{B} \mid [u] \text{ is an atom in } \mathbb{B}/I_\alpha\}$
$I_\lambda = \bigcup\{I_\alpha \mid \alpha < \lambda\}$ for $\lambda \in Lim$.
Finally, $I_\alpha = \mathbb{B}$ for some α. Conversely, if $I_\alpha = \mathbb{B}$ for some α, then $\mathbb{B}$ is superatomic. The recursive definition of I_α corresponds to recursively taking Cantor-Bendixon derivatives in the Stone space.

The least α such that $At(\mathbb{B}/I_\alpha)$ is finite, or equivalently that $I_{\alpha+1} = \mathbb{B}$, is called the height $ht(\mathbb{B})$ of $\mathbb{B}$. For every $\alpha < ht(\mathbb{B})$ the width $wd_\alpha(\mathbb{B})$ is the cardinality of $At(\mathbb{B}/I_\alpha)$. We say that $\mathbb{B}$ is a (κ, α)-superatomic Boolean algebra (sBa), if $ht(\mathbb{B}) = \alpha$ and $wd_\beta(\mathbb{B}) \leq \kappa$ for all $\beta < \alpha$.

It was shown by Baumgartner and Shelah that there can be consistently an (ω, ω_2)-sBa $\mathbb{B}$. Note that in this case $2^\omega \geq \omega_2$ because by definition $\mathbb{B}$ has size ω_2 but only ω-many atoms.

We obtain an (ω, ω_2)-sBa by reversing the recursive definition of the I_α.

Assume that there is a partial order $<_B$ on ω_2 such that:

(a) $\forall \alpha, \beta \in \omega_2 \quad \alpha <_B \beta \rightarrow \alpha < \beta$

(b) If $\alpha, \beta \in [\omega\gamma, \omega\gamma + \omega[$, then α and β are incompatible.

(c) If α, β are compatible in $<_B$, then they have an infimum $i(\alpha, \beta)$.

(d) If $\alpha \in [\omega\gamma, \omega\gamma + \omega[$, then there exist for all $\delta < \gamma$ infinitely many $\beta <_B \alpha$ such that $\beta \in [\omega\delta, \omega\delta + \omega[$.
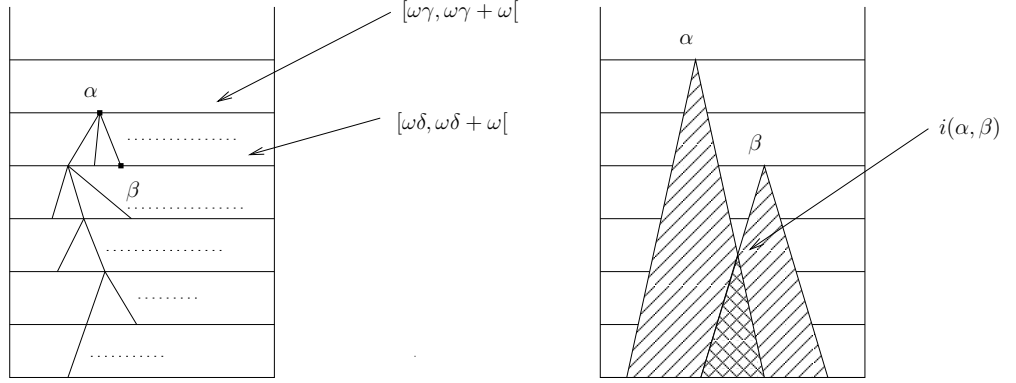
For $\gamma < \omega_2$ let $x_\gamma = \{n \in \omega \mid n \leq_B \gamma\}$.

Let $\mathbb{B}$ be the subalgebra of $\mathfrak{P}(\omega)$ which is generated by $\{x_\gamma \mid \gamma < \omega_2\}$.

Then

$$(*) \quad At(\mathbb{B}/I_\alpha) = \{[x_\gamma] \mid \omega\alpha \leq \gamma < \omega\alpha + \omega\} \quad \text{for all } \alpha < \omega_2$$

where $[x_\gamma] \neq [x_{\gamma'}]$ if $\gamma \neq \gamma'$.

Hence $|At(\mathbb{B}/I_\alpha)| = \omega$ for all $\alpha < \omega_2$ and $\mathbb{B}$ is an (ω, ω_2)-sBa.

We prove $(*)$ by induction over $\alpha < \omega_2$.

We have to show that

(i) $[x_\gamma] \neq 0$ for all $\omega\alpha \leq \gamma < \omega\alpha + \omega$

(ii) $[x_\gamma] \neq [x_{\gamma'}]$ for all $\gamma \neq \gamma' \in [\omega\alpha, \omega\alpha + \omega[$

(iii) $\forall 0 \neq u \in \mathbb{B}/I_\alpha \ \exists \omega\alpha \leq \gamma < \omega\alpha + \omega: [x_\gamma] \leq u$.

This suffices because $[x_\gamma] \cdot [x'_\gamma] = 0$ for all $\gamma \neq \gamma' \in [\omega\alpha, \omega\alpha + \omega[$ by (b).

Proof: (i) If $\alpha = 0$, then the claim is is obvious. So assume that $\alpha \neq 0$. Then by the induction hypothesis I_α is the ideal on $\mathbb{B}$ which is generated by $\{x_\gamma \mid \gamma < \omega\alpha\}$. Let $\gamma \in [\omega\alpha, \omega\alpha + \omega[$. Assume towards a contradiction that $[x_\gamma] = 0$, i.e. $x_\gamma \Delta \emptyset = (x_\gamma - \emptyset) + (\emptyset - x_\gamma) = x_\gamma \in I_\alpha$. So there are $\gamma_1, \ldots, \gamma_n \in \omega_\alpha$ such that $x_\gamma - (x_{\gamma_1} \cup \ldots \cup x_{\gamma_n}) = \emptyset$. Hence we are done, if we can find a $\delta \in x_\gamma$ such that $\delta \notin x_{\gamma_i}$ for all $1 \leq i \leq n$. We find such a δ by induction. Note first that $x_\gamma - (x_{\gamma_1} \cup \ldots \cup x_{\gamma_n}) = x_\gamma - ((x_\gamma \cap x_{\gamma_1}) \cup \ldots \cup (x_\gamma \cap x_{\gamma_n})) = x_\gamma - (x_{i(\gamma,\gamma_1)} \cup \ldots \cup x_{i(\gamma,\gamma_n)})$. Let

$$\{\eta_1 > \eta_2 > \ldots > \eta_k\} = \{\eta \mid \exists 1 \leq i \leq n \ i(\gamma, \gamma_n) \in [\omega\eta, \omega\eta + \omega[\}.$$

Then we can find by (d) a $\delta_1 \in [\omega\eta_1, \omega\eta_1 + \omega[$ such that $\delta_1 <_B \gamma$ and $\delta_1 \neq i(\gamma, \gamma_i)$ for all $1 \leq i \leq n$. In a second step, we can find by (d) a $\delta_2 \in [\omega\eta_2, \omega\eta_2 + \omega[$ such that $\delta_2 <_B \delta_1$ and $\delta_2 \neq i(\gamma, \gamma_i)$ for all $1 \leq i \leq n$.

By (b), we have $\delta_2 \not<_B i(\gamma, \gamma_i)$ for all $1 \leq i \leq n$. Continuing in this way we finally find a $\delta \in [0, \omega[$ such that $\delta <_B \gamma$ and $\delta \not<_B i(\gamma, \gamma_i)$ for all $1 \leq i \leq n$. Hence we are done.
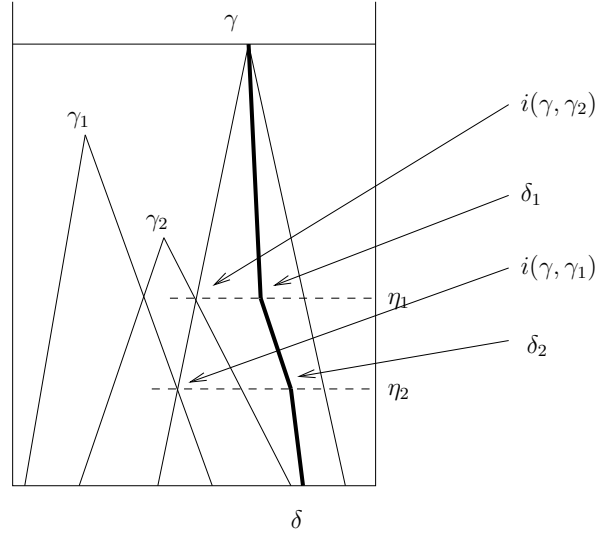
(ii) We want to show that $[x_\gamma] \neq [x_{\gamma'}]$ for $\gamma \neq \gamma'$, i.e. $(x_\gamma - x_{\gamma'}) \cup (x_{\gamma'} - x_\gamma) \notin I_\alpha$. By (b), $(x_\gamma - x_{\gamma'}) \cup (x_{\gamma'} - x_\gamma) = (x_\gamma \cup x_{\gamma'}) - (x_\gamma \cap x_{\gamma'}) = x_\gamma \cup x_{\gamma'}$. By (i), $0 \neq [x_\gamma] \leq [x_\gamma] + [x_{\gamma'}]$. So $[x_\gamma] + [x_{\gamma'}] = [x_\gamma + x_{\gamma'}] \neq 0$, i.e. $x_\gamma \cup x_{\gamma'} \notin I_\alpha$.

(iii) Every $0 \neq [u] \in \mathbb{B}/I_\alpha$ can be written in disjunctive normal form as

$$[u] = [x_{\gamma_1'}] \cdot [x_{\gamma_2'}] \cdot \ldots \cdot [x_{\gamma_n'}] \cdot \ldots (-[x_{\gamma_1}]) \cdot \ldots \cdot (-[x_{\gamma_n}]) + \ldots \quad .$$

Assume w.l.o.g. that $[u'] := [x_{\gamma_1'}] \cdot [x_{\gamma_2'}] \cdot \ldots \cdot [x_{\gamma_n'}] \cdot \ldots (-[x_{\gamma_1}]) \cdot \ldots \cdot (-[x_{\gamma_n}]) \neq 0$. Hence it suffices to find $[x_\delta] \leq [u']$ with $\omega\alpha \leq \delta < \omega\alpha + \omega$. By (c), $x_{\gamma_1'} \cap x_{\gamma_2'} = x_{i(\gamma_1', \gamma_2')}$. So there exists a γ such that $[x_{\gamma_1'}] \cdot \ldots \cdot [x_{\gamma_n'}] = [x_\gamma]$. So

$$[u'] = [x_\gamma] \cdot (-[x_{\gamma_1}]) \cdot \ldots \cdot (-[x_{\gamma_n}]) = [x_\gamma] - ([x_{\gamma_1}] + \ldots + [x_{\gamma_n}]) =$$

$$= [x_\gamma] - ([x_\gamma] \cdot [x_{\gamma_1}] + \ldots + [x_\gamma] \cdot [x_{\gamma_n}]) = [x_\gamma] - ([x_{i(\gamma, \gamma_1)}] + \ldots + [x_{i(\gamma, \gamma_n)}]).$$

By the induction hypothesis, I_α is the ideal on $\mathbb{B}$ which is generated by $\{x_\gamma \mid \gamma < \omega\alpha\}$. Hence $\omega\alpha \leq \gamma$ because $[u'] \neq 0$. We are done if we can find a δ such that $\delta \in [\omega\alpha, \omega\alpha + \omega[$, $[x_\delta] \leq [x_\gamma]$ and $[x_\delta] \cdot [x_{i(\gamma, \gamma_i)}] = 0$ for all $1 \leq i \leq n$. Hence by (b) it suffices to find a $\delta \leq_B \gamma$ such that $\delta \in [\omega\alpha, \omega\alpha + \omega[$ and $\delta \not<_B i(\gamma, \gamma_i)$ for all $1 \leq i \leq n$. We find such a δ very much like in (i). $\square$
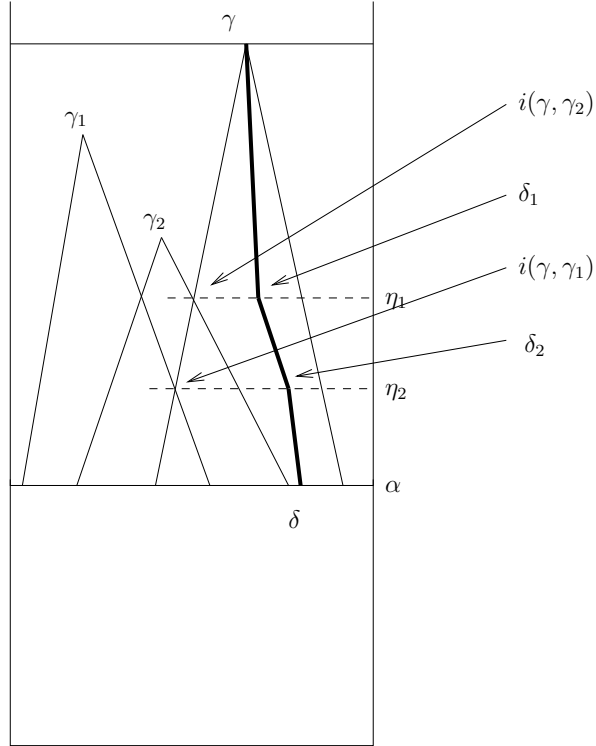
The natural forcing to add a partial ordering like above is the set P of all finite (strict) partial orders $p = \langle x_p. <_p \rangle$ such that

(a) $\forall \alpha, \beta \in \omega_2 \quad \alpha <_B \beta \rightarrow \alpha < \beta$

(b) If $\alpha, \beta \in [\omega\gamma, \omega\gamma + \omega[$, then α and β are incompatible.

(c) If α, β are compatible in $<_p$, then they have an infimum $i_p(\alpha, \beta)$.

For $p, q \in P$, we set $p \leq q$ iff

(i) $x_q \subseteq x_p$, $<_p \upharpoonright x_q = <_q$

(ii) If α, β are compatible in $<_p$, then they are compatible in $<_q$ and $i_p(\alpha, \beta) = i_q(\alpha, \beta)$.

This forcing is very similar to the ones we considered before. For $p \in P$ define $f_p : [a_p, b_p] \rightarrow 2$ by

$$a_p = \{\beta \mid \exists \alpha \ \alpha <_p \beta\} \qquad b_p = \{\alpha \mid \exists \beta \ \alpha <_p \beta\}$$

$$f_p(\alpha, \beta) = 1 \ \Leftrightarrow \ \alpha <_p \beta.$$

Then the forcing colors a triangle
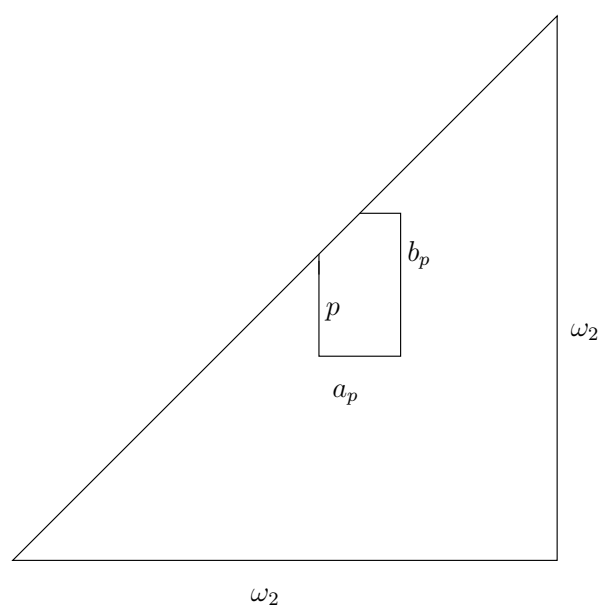
Like in the previous examples, it does not satisfy ccc. To thin it out so that it obtains ccc, we proceed exactly like in our consistency proof of $\omega_2 \nrightarrow (\omega : 2)^2_\omega$, except that we change (3) in the successor step of the definition of $\mathbb{P}$. We replace it by

(3): $\forall \gamma \in \theta_\alpha \setminus \delta \ card(\{\delta \in \theta_\beta \setminus \theta_\alpha \mid f_p(\gamma, \delta) = 1\}) \leq 1$.

There is at most one δ such that

$$f_p(\gamma, \delta) = 1$$
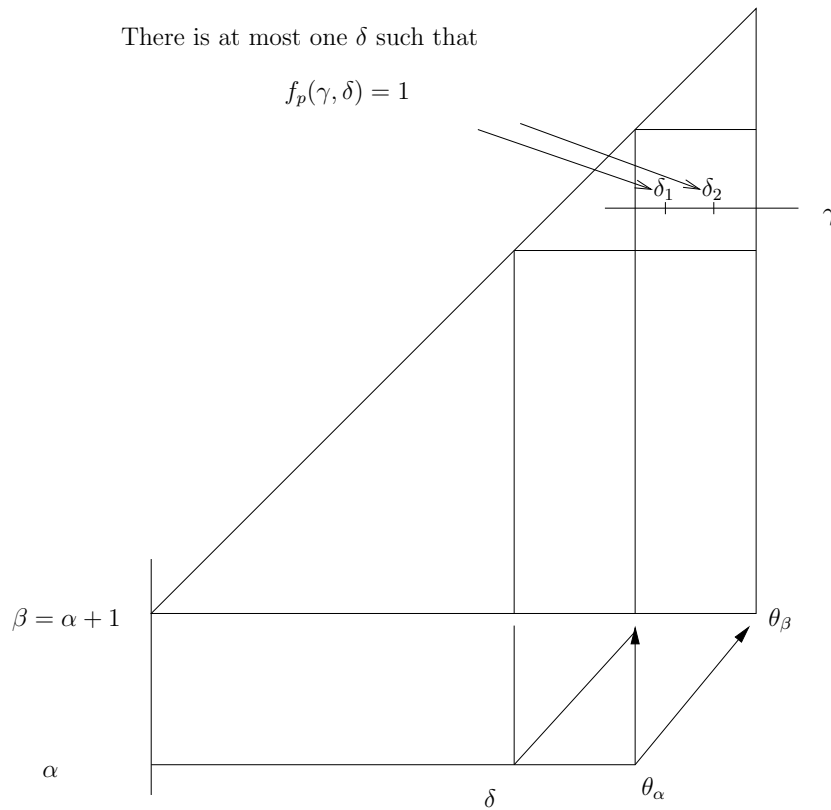
If we now proceed like in the consistency proof for $\omega_2 \nrightarrow (\omega : 2)^2_\omega$, we obtain:

Theorem 4.10

Assume that there is a simplified $(\omega_1, 1)$-morass. Then there exists a ccc forcing $\mathbb{P}$ which adds an (ω, ω_2)-sBa.

Historical remarks and references

P. Koszmider [30] proved that it is consistent that there exists a sequence $\langle X_\alpha \mid \alpha < \omega_2 \rangle$ of subsets $X_\alpha \subseteq \omega_1$ such that $X_\beta - X_\alpha$ is finite and $X_\alpha - X_\beta$ is uncountable for all $\beta < \alpha < \omega_2$. He uses S. Todorcevic's method of ordinal walks [42]. It is also known as the method of ρ-functions [29] and provides a powerful tool to construct ccc forcings in the presence of $\square_{\omega_1}$. Other applications are a ccc forcing that adds an ω_2-Suslin tree [42], a ccc forcing for $\omega_2 \nrightarrow (\omega : 2)^2_\omega$ [42], a ccc forcing to add a Kurepa tree [42, 43] and a ccc forcing to add a thin-very tall superatomic Boolean algebra [42]. The last

forcing was first found by Baumgartner and Shelah [3] independently from ρ-functions. That there can be a ccc forcing for $\omega_2 \not\rightarrow (\omega : 2)^2_\omega$ was first observed by Galvin [27]. That $\square_{\omega_1}$ implies the existence of a ccc forcing which adds a Kurepa tree was first proved by Jensen [23, 22]. Note, that we do not need a forcing at all in the context of morasses, because the morass tree $\langle T, \prec \rangle$ which was defined in section 2 is a Kurepa tree.

How ρ-fuctions can be constructed from morasses is explained by C. Morgan [32]. P. Koepke and J. C. Martinez [28] proved that the existence of a simplified $(\kappa, 1)$-morass implies the existence of a (κ, κ^+)-superatomic Boolean algebra. J. Rebholz [34] showed that $\omega_2 \not\rightarrow (\omega_1 : \omega)^2_2$ and $\omega_3 \not\rightarrow (\omega_2 : \omega_1)^2_2$ hold in L using morasses and diamond.

The fact that the existence of ω_2-many strongly almost disjoint subsets of ω_1 implies $2^\omega \geq \omega_2$ is from Baumgartner's paper [1]. The existence of an (ω, ω_1)-sBa follows from ZFC. This was proved by I. Juhasz and W. Weiss [26], and, independently, by M. Rajagopalan [33].

That there can be families $\{f_\alpha : \omega_1 \rightarrow \omega \mid \alpha \in \kappa\}$ of arbitrary prescribed size κ such that $\{\xi < \omega_1 \mid f_\alpha(\xi) = f_\beta(\xi)\}$ is finite for all $\alpha < \beta < \kappa$ was proved by J. Zapletal in [49].

The reference for the Handbook of Boolean Algebras is [35].

The question of whether

$$\begin{pmatrix} \omega_3 \\ \omega_3 \end{pmatrix} \not\rightarrow \begin{pmatrix} \omega \\ \omega \end{pmatrix}_\omega$$

is consistent was first asked by S. Todorcevic [41].

The idea to construct forcings like in this chapter was first presented in Irrgang [17].

5

Gap-2 morasses

In what follows, we will generalize our approach to three dimensions and therefore we will need a three-dimensional system along which we can index our forcings. An appropriate structure is a so-called gap-2 morass. The idea behind it is that we want to approximate a simplified $(\omega_2, 1)$-morass in the same way as we approximated the ordinal ω_2 before.
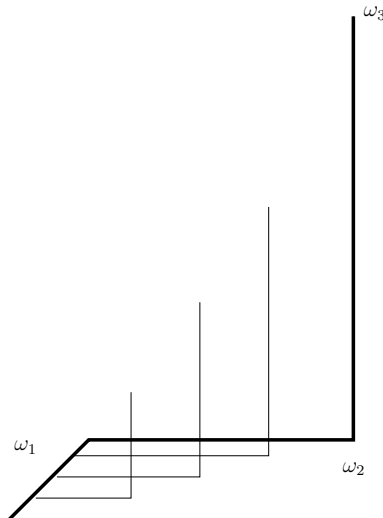
We imagine the situation to be as follows. First, we turn around the simplified $(\omega_2, 1)$-morass.
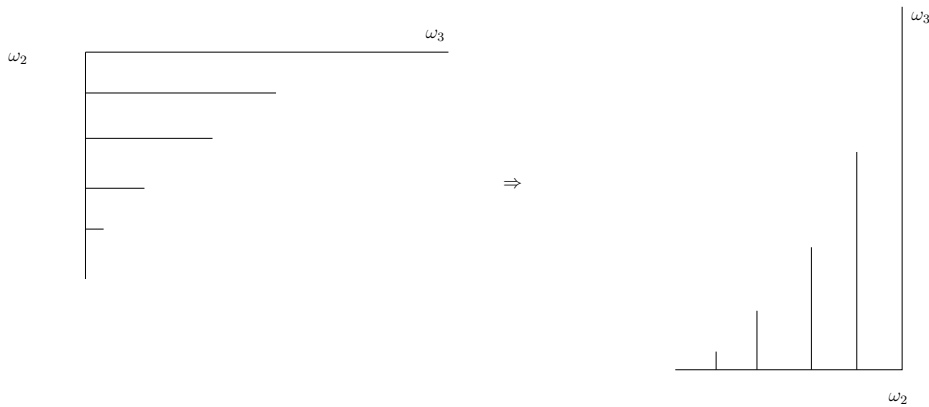
Then we think of it as the backside of a box which is ω_3 high, ω_2 wide and ω_1 deep.
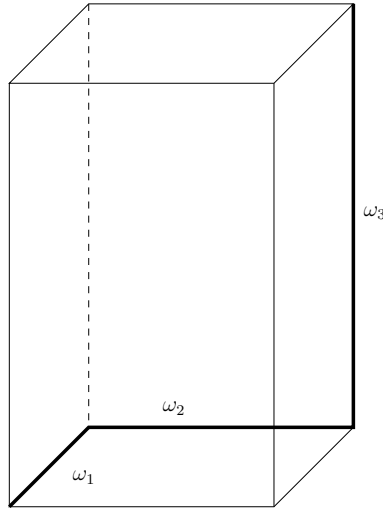
The ω_1-many slices of the box which are parallel to this backside are thought of as approximations to the gap-1 morass on the backside.

Of course, these approximations will not fill the whole box.

They look themseves like morasses and are called fake gap-1 morasses.

A fake gap-1 morass is a structure $\langle\langle\varphi_\zeta \mid \zeta \leq \theta\rangle, \langle\mathfrak{G}_{\zeta\xi} \mid \zeta < \xi \leq \theta\rangle\rangle$ which satisfies the definition of simplified gap-1 morass, except that θ need not be a cardinal and there is no restriction on the cardinalities of φ_ζ and $\mathfrak{G}_{\zeta\xi}$. Let $\mathfrak{G}_{\zeta,\zeta+1} = \{id, b\}$. Then the critical point of b is denoted by σ_ζ and called the split point of $\mathfrak{G}_{\zeta,\zeta+1} = \{id, b\}$.
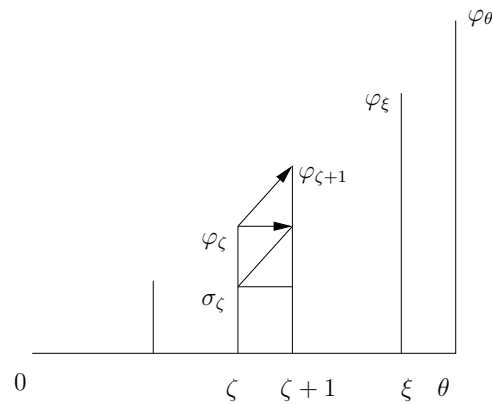
The fake morass $\langle\langle\varphi_\zeta \mid \zeta \leq \theta\rangle, \langle\mathfrak{G}_{\zeta\xi} \mid \zeta < \xi \leq \theta\rangle\rangle$ will replace in the gap-2 morass the ordinal levels θ_α of the gap-1 morass.
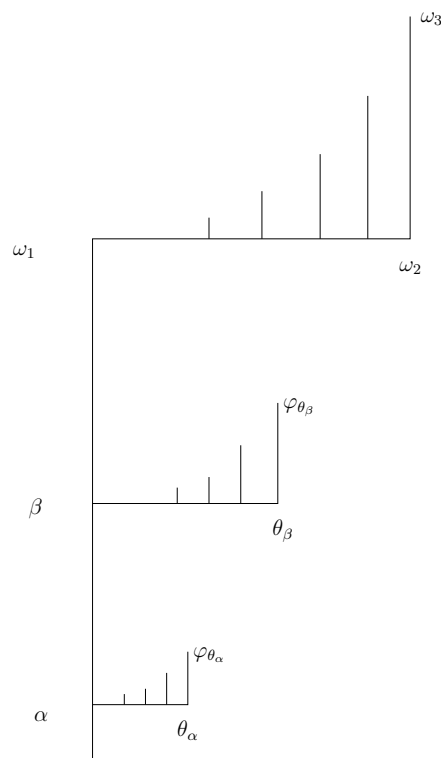
We need of course also replacements for the order-preserving maps $f : \theta_\alpha \to \theta_\beta$, i.e. maps which preserve the order of the fake morasses. The appropriate maps are called embeddings.

Suppose that $\langle\langle\varphi_\zeta \mid \zeta \leq \theta\rangle, \langle\mathfrak{G}_{\zeta\xi} \mid \zeta < \xi \leq \theta\rangle\rangle$ and $\langle\langle\varphi'_\zeta \mid \zeta \leq \theta'\rangle, \langle\mathfrak{G}'_{\zeta\xi} \mid \zeta <$

$\xi \leq \theta'\rangle\rangle$ are fake gap-1 morasses. An embedding from the first one to the second will be a function f with domain

$$(\theta + 1) \cup \{\langle \zeta, \tau \rangle \mid \zeta \leq \theta, \tau < \varphi_\zeta\} \cup \{\langle \zeta, \xi, b \rangle \mid \zeta < \xi \leq \theta, b \in \mathfrak{G}_{\zeta\xi}\}$$

satisfying certain requirements. We will write $f_\zeta(\tau)$ for $f(\langle \zeta, \tau \rangle)$ and $f_{\zeta\xi}(b)$ for $f(\langle \zeta, \xi, b \rangle)$.

The properties are the following ones:

(1) $f \upharpoonright (\theta + 1)$ is an order preserving function from $\theta + 1$ to $\theta' + 1$ such that $f(\theta) = \theta'$.
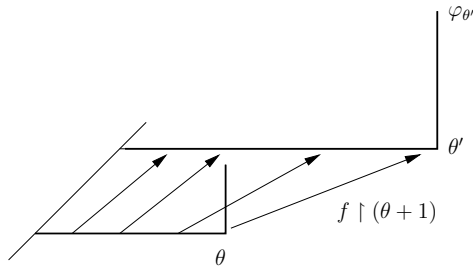
(2) For all $\zeta \leq \theta$, f_ζ is an order preserving function from φ_ζ to $\varphi'_{f(\zeta)}$.
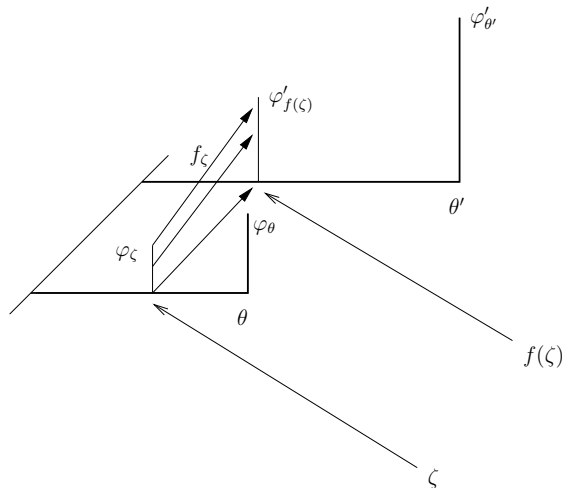
(3) For all $\zeta < \xi \leq \theta$, $f_{\zeta\xi}$ maps $\mathfrak{G}_{\zeta\xi}$ to $\mathfrak{G}'_{f(\zeta)f(\xi)}$.
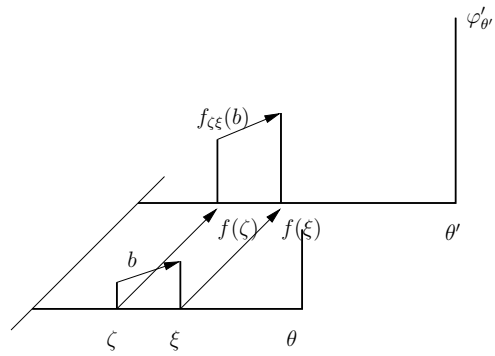
(4) If $\zeta < \theta$, then $f_\zeta(\sigma_\zeta) = \sigma'_{f(\zeta)}$.

(5) If $\zeta < \xi \leq \theta$, $b \in \mathfrak{G}_{\zeta\xi}$ and $c \in \mathfrak{G}_{\xi\eta}$, then $f_{\zeta\eta}(c \circ b) = f_{\xi\eta}(c) \circ f_{\zeta\xi}(b)$.
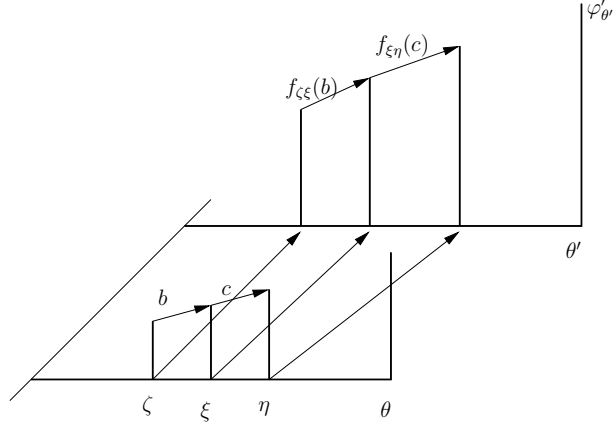
(6) If $\zeta < \xi \leq \theta$ and $b \in \mathfrak{G}_{\zeta\xi}$, then $f_\xi \circ b = f_{\zeta\xi}(b) \circ f_\zeta$.
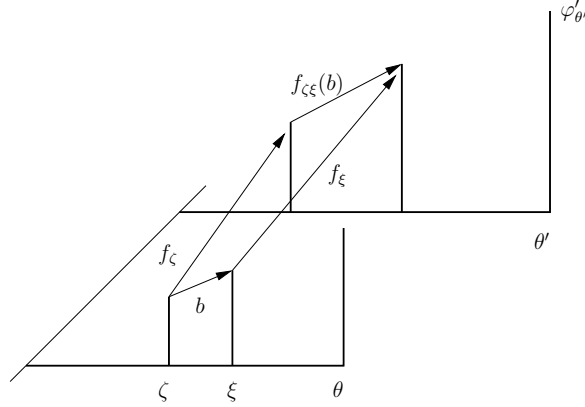
An embedding from one fake gap-1 morass to another preserves its whole structure. The notion of embedding will play in the definition of gap-2 morasses the role which the order-preserving maps $f \in \mathfrak{F}_{\alpha\beta}$ played in the gap-1 case. Note, however, one difference. The levels θ_α of the gap-1 morass did not necessarily have a greatest element, while all fake gap-1 morasses have a top-level, namely $\va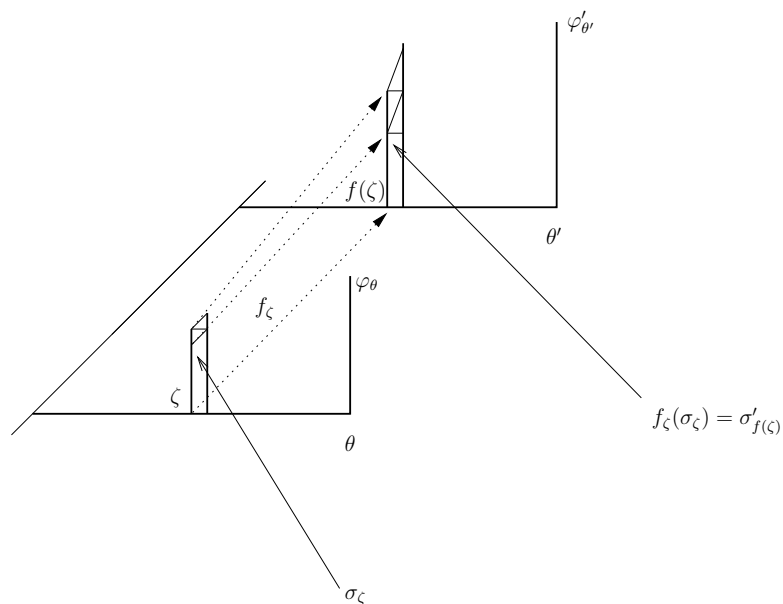rphi_\theta$. This correspondes to the fact that we approximate with a gap-1 morass the ordinal κ^+, while we approximate a simplified $(\kappa^+, 1)$-morass with a gap-2 morass. And κ^+ does not have a greatest element, while the simplified $(\kappa^+, 1)$-morass has a top-level. This results in the condition $f(\theta) = \theta'$ in (1) in the definition of embedding.

Before we can give the definition of a simplified gap-2 morass, we still need to replace the identity and the splitting map of (P3) in the definition of gap-1 morasses by appropriate notions.

Let like before $\langle\langle\varphi_\zeta \mid \zeta \leq \theta\rangle, \langle\mathfrak{G}_{\zeta\xi} \mid \zeta < \xi \leq \theta\rangle\rangle$ to $\langle\langle\varphi'_\zeta \mid \zeta \leq \theta'\rangle, \langle\mathfrak{G}'_{\zeta\xi} \mid \zeta < \xi \leq \theta'\rangle\rangle$ be fake gap-1 morasses. Assume moreover that $\langle\langle\varphi_\zeta \mid \zeta \leq \theta\rangle, \langle\mathfrak{G}_{\zeta\xi} \mid \zeta < \xi \leq \theta\rangle\rangle$ is an initial-segment of $\langle\langle\varphi'_\zeta \mid \zeta \leq \theta'\rangle, \langle\mathfrak{G}'_{\zeta\xi} \mid \zeta < \xi \leq \theta'\rangle\rangle$, i.e. $\theta < \theta'$, $\varphi'_\zeta = \varphi_\zeta$ for $\zeta \leq \theta$ and $\mathfrak{G}'_{\zeta\xi} = \mathfrak{G}_{\zeta\xi}$ for $\zeta < \xi \leq \theta$. We define an embedding, which is as close to the identity as we can get, as follows: Let $f \restriction \theta = id$, $f_\zeta = id$ for all $\zeta < \theta$ and $f_{\zeta\xi} = id$ for all $\zeta < \xi < \theta$. Then we can define an embedding by picking $f_\theta \in \mathfrak{G}'_{\theta\theta'}$ and setting $f_{\zeta\theta}(b) = f_\theta \circ b$ for all $\zeta < \theta$ and all $b \in \mathfrak{G}_{\zeta\theta}$.
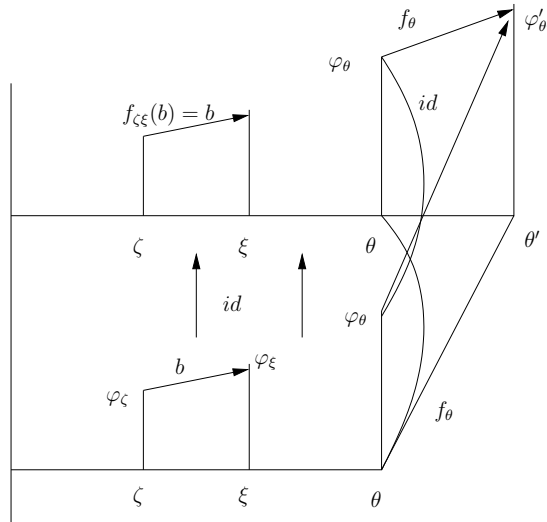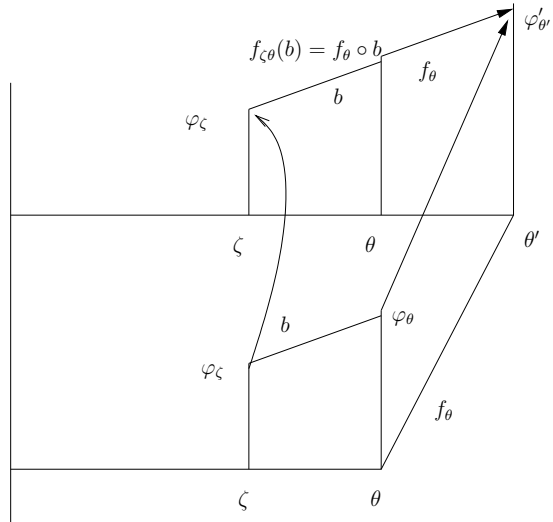
We call such an embedding f a left-branching embedding. There are many
left-branching embeddings, one for every choice of f_θ.

An embedding f is right-branching if for some $\eta < \theta$,
(1) $f \upharpoonright \eta = id$
(2) $f(\eta + \zeta) = \theta + \zeta$ if $\eta + \zeta \leq \theta$
(3) $f_\zeta = id$ for $\zeta < \eta$
(4) $f_{\zeta\xi} = id$ for $\zeta < \xi < \eta$
(5) $f_\eta \in \mathfrak{G}_{\eta\theta}$
(6) $f_{\zeta\xi}[\mathfrak{G}_{\zeta\xi}] = \mathfrak{G}'_{f(\zeta)f(\xi)}$ if $\eta \leq \zeta < \xi \leq \theta$.
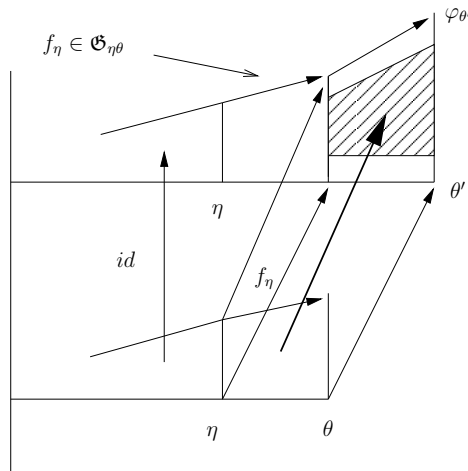
An amalgamation is a family of embeddings that contains all possible left-branching embeddings, exactly one right-branching embedding and nothing else.

Now, we are ready to define gap-2 morasses.

Let $\kappa \geq \omega$ be regular and $\langle\langle\varphi_\zeta \mid \zeta \leq \kappa^+\rangle, \langle\mathfrak{G}_{\zeta\xi} \mid \zeta < \xi \leq \kappa^+\rangle\rangle$ a simplified $(\kappa^+, 1)$-morass such that $\varphi_\zeta < \kappa$ for all $\zeta < \kappa$. Let $\langle\theta_\alpha \mid \alpha < \kappa\rangle$ be a sequence such that $0 < \theta_\alpha < \kappa$ and $\theta_\kappa = \kappa^+$. Let $\langle\mathfrak{F}_{\alpha\beta} \mid \alpha < \beta \leq \kappa\rangle$ be such that $\mathfrak{F}_{\alpha\beta}$ is a family of embeddings from $\langle\langle\varphi_\zeta \mid \zeta \leq \theta_\alpha\rangle, \langle\mathfrak{G}_{\zeta\xi} \mid \zeta < \xi \leq \theta_\alpha\rangle\rangle$ to $\langle\langle\varphi_\zeta \mid \zeta \leq \theta_\beta\rangle, \langle\mathfrak{G}_{\zeta\xi} \mid \zeta < \xi \leq \theta_\beta\rangle\rangle$.

This is a simplified $(\kappa, 2)$-morass if it has the following properties:

(1) $|\mathfrak{F}_{\alpha\beta}| < \kappa$ for all $\alpha < \beta < \kappa$.
(2) If $\alpha < \beta < \gamma$, then $\mathfrak{F}_{\alpha\gamma} = \{f \circ g \mid f \in \mathfrak{F}_{\beta\gamma}, g \in \mathfrak{F}_{\alpha\beta}\}$.
Here $f \circ g$ is the composition of the embeddings f and g, which are defined in the obvious way: $(f \circ g)_\zeta = f_{g(\zeta)} \circ g_\zeta$ for $\zeta \leq \theta_\alpha$ and $(f \circ g)_{\zeta\xi} = f_{g(\zeta)g(\xi)} \circ g_{\zeta\xi}$ for $\zeta < \xi \leq \theta_\alpha$.
(3) If $\alpha < \kappa$, then $\mathfrak{F}_{\alpha,\alpha+1}$ is an amalgamation.
(4) If $\alpha \leq \kappa$ is a limit ordinal, $\beta_1, \beta_2 < \alpha$ and $f_1 \in \mathfrak{F}_{\beta_1\alpha}$, $f_2 \in \mathfrak{F}_{\beta_2\alpha}$, then there are a $\beta_1, \beta_2 < \gamma < \alpha$, $g \in \mathfrak{F}_{\gamma\alpha}$ and $h_1 \in \mathfrak{F}_{\beta_1\gamma}$, $h_2 \in \mathfrak{F}_{\beta_2\gamma}$ such that $f_1 = g \circ h_1$ and $f_2 = g \circ h_2$.
(5) For all $\alpha \leq \kappa$, $\alpha \in Lim$:
(a) $\theta_\alpha = \bigcup\{f[\theta_\beta] \mid \beta < \alpha, f \in \mathfrak{F}_{\beta\alpha}\}$.
(b) For all $\zeta \leq \theta_\alpha$, $\varphi_\zeta = \bigcup\{f_{\bar\zeta}[\varphi_{\bar\zeta}] \mid \exists\beta < \alpha \ (f \in \mathfrak{F}_{\beta\alpha} \text{ and } f(\bar\zeta) = \zeta)\}$.
(c) For all $\zeta < \xi \leq \theta_\alpha$, $\mathfrak{G}_{\zeta\xi} = \bigcup\{f_{\bar\zeta\bar\xi}[\mathfrak{G}_{\bar\zeta\bar\xi}] \mid \exists\beta < \alpha \ (f \in \mathfrak{F}_{\beta\alpha}, f(\bar\zeta) = \zeta \text{ and } f(\bar\xi) = \xi)\}$.

The notion of simplified gap-2 morass was introduced by D. Velleman in his elegant paper "Simplified gap-2 morasses" (APAL 34, 171 - 208). He also proved almost all of the following results.

Theorem 5.1
(a) If $V = L$, then there is a simplified $(\kappa, 2)$-morass for all regular $\kappa > \omega$.
(b) If $\kappa > \omega$ is regular, then there is a forcing $\mathbb{P}$ which preserves cardinals and cofinalities such $\mathbb{P} \Vdash$ (there is a simplified $(\kappa, 2)$-morass).

Since $\langle\langle\varphi_\zeta \mid \zeta \leq \kappa^+\rangle, \langle\mathfrak{G}_{\zeta\xi} \mid \zeta < \xi \leq \kappa^+\rangle\rangle$ is a simplified $(\kappa^+, 1)$-morass, there is a tree $\langle T, \prec\rangle$ with levels T_η for $\eta \leq \kappa^+$. And there are maps π_{st} for $s \prec t$. Moreover, if we set $\mathfrak{F}'_{\alpha\beta} = \{f \upharpoonright \theta_\alpha \mid f \in \mathfrak{F}_{\alpha\beta}\}$, then $\langle\langle\theta_\alpha \mid \alpha \leq \kappa\rangle, \langle\mathfrak{F}'_{\alpha\beta} \mid \alpha < \beta \leq \kappa\rangle\rangle$ is a simplified $(\kappa, 1)$-morass. So there is also a tree $\langle T', \prec'\rangle$ with levels T'_η for $\eta \leq \kappa$.
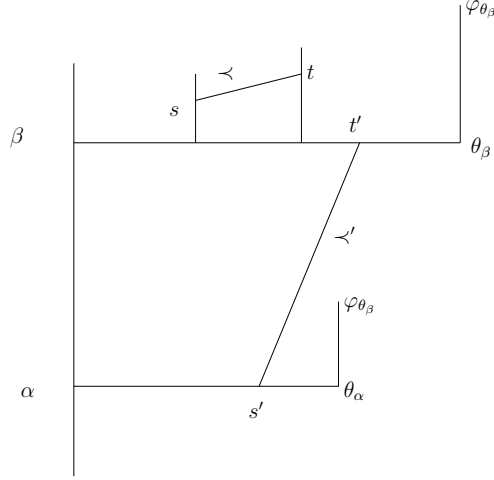
Lemma 5.2

Suppose $\alpha < \beta \le \kappa$, $f_1, f_2 \in \mathfrak{F}_{\alpha\beta}$, $\zeta_1, \zeta_2 < \theta_\alpha$ and $f_1(\zeta_1) = f_2(\zeta_2)$.
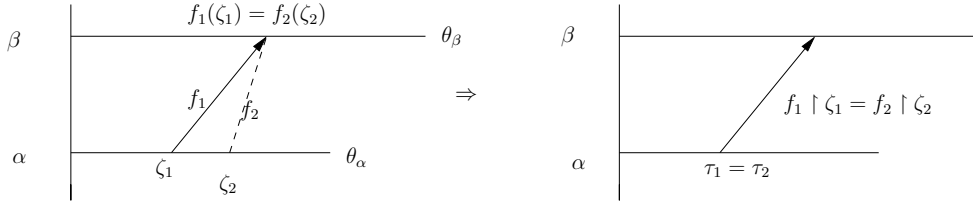
Then $\zeta_1 = \zeta_2$, $f_1 \upharpoonright \zeta_1 = f_2 \upharpoonright \zeta_1$, $(f_1)_\xi = (f_2)_\xi$ for all $\xi \le \zeta_1$, and $(f_1)_{\xi\eta} = (f_2)_{\xi\eta}$ for all $\xi < \eta \le \zeta_1$.

Now, let $s = \langle \alpha, \nu \rangle \in T'_\alpha$, $t = \langle \beta, \tau \rangle \in T'_\beta$ and $s \prec' t$. Then there is some $f \in \mathfrak{F}'_{\alpha\beta}$ such that $f(\nu) = \tau$. By the previous lemma

$$f \upharpoonright ((\nu + 1) \cup \{\langle \zeta, \tau \rangle \mid \zeta \le \nu, \tau < \varphi_\zeta\} \cup \{\langle \zeta, \xi, b \rangle \mid \zeta < \xi \le \nu, b \in \mathfrak{G}_{\zeta\xi}\})$$

does not depend on f. So we may call it π'_{st}.

Lemma 5.3

If $\alpha < \beta \le \kappa$, then there is an $f \in \mathfrak{F}_{\alpha\beta}$ such that $f \upharpoonright \theta_\alpha = id \upharpoonright \theta_\alpha$, $f_\eta = id \upharpoonright \varphi_\eta$ for all $\eta < \theta_\alpha$, and $f_{\xi\eta}(b) = b$ for all $\xi < \eta < \theta_\alpha$ and all $b \in \mathfrak{G}_{\xi\eta}$.

In addition to the maps $f \in \mathfrak{F}_{\alpha\beta}$, we need maps $\bar{f}$ that are associated to f. For a set of ordinals X, let $ssup(X)$ be the least α such that $X \subseteq \alpha$. And let $\bar{f}(\zeta) = ssup(f[\zeta]) \le f(\zeta)$.
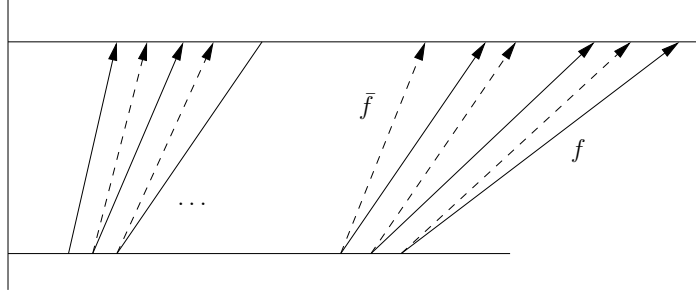
Lemma 5.4

For every $\alpha < \beta \le \kappa$, $f \in \mathfrak{F}_{\alpha\beta}$ and $\zeta \le \theta_\alpha$, there are unique functions $\bar{f}_\zeta : \varphi_\zeta \to \varphi_{\bar{f}(\zeta)}$, $\bar{f}_{\xi\zeta} : \mathfrak{G}_{\xi\zeta} \to \mathfrak{G}_{f(\xi)\bar{f}(\zeta)}$ for all $\xi < \zeta$, and $f^{\#}(\zeta) \in \mathfrak{G}_{\bar{f}(\zeta)f(\zeta)}$ such that:

(1) $f_\zeta = f^{\#}(\zeta) \circ \bar{f}_\zeta$

(2) $\forall \xi < \zeta \ \forall b \in \mathfrak{G}_{\xi\zeta} \ \ f_{\xi\zeta}(b) = f^{\#}(\zeta) \circ \bar{f}_{\xi\zeta}(b)$.
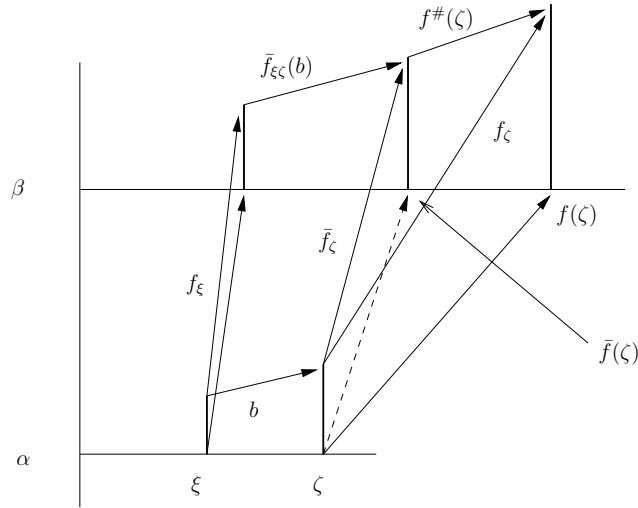
From the previous lemma, we get of course also maps $(\overline{\pi'_{st}})_\zeta$ and $(\overline{\pi'_{st}})_{\xi\zeta}$ for $s \prec' t$ and $\xi < \zeta \le \nu(t)$. To be more explicit, let $s \prec' t$, $s = \langle \alpha, \nu \rangle$ and $t = \langle \beta, \tau \rangle$. Then there exists by the definition of $\prec'$ an $f \in \mathfrak{F}_{\alpha\beta}$ such that $f(\nu) = \tau$. By the previous lemma $f \restriction \nu + 1$, all f_ξ for $\xi \le \nu$ and $f_{\xi\eta}$ for all $\xi < \eta \le \nu$ do not depend on the choice of f. So we may set $(\overline{\pi'_{st}})_\zeta = \bar{f}_\zeta$ and $(\overline{\pi'_{st}})_{\xi\zeta} = \bar{f}_{\xi\zeta}$ for all $\xi < \zeta \le \nu$.

Note, that the existence of functions like in the previous lemma is clear in the case $\beta = \alpha + 1$ by the definition of left-branching embeddings and right-branching ones.

Let us first consider a left-branching embedding $f \in \mathfrak{F}_{\alpha,\alpha+1}$.
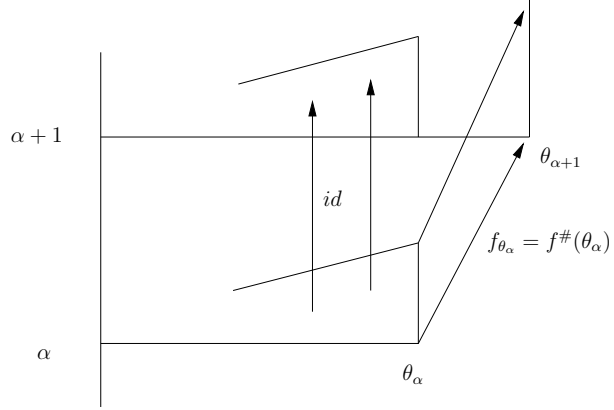
Set $f^{\#}(\zeta) = id$ for all $\zeta < \theta_\alpha$ and $f^{\#}(\theta_\alpha) = f_{\theta_\alpha}$. Set $\bar{f}_\zeta = id$ for all $\zeta \leq \theta_\alpha$. Finally set $\bar{f}_{\xi\zeta} = id$ for all $\xi < \zeta \leq \theta_\alpha$.

Then (1) holds because

$$f_\zeta = id = id \circ id = f^{\#}(\zeta) \circ \bar{f}_\zeta \text{ for all } \zeta < \theta_\alpha$$

and

$$f_{\theta_\alpha} = f^{\#}(\theta_\alpha) \circ id = f^{\#}(\theta_\alpha) \circ \bar{f}_{\theta_\alpha}.$$

Moreover, (2) holds because

$$f_{\xi\zeta}(b) = b = id \circ b = f^{\#}(\zeta) \circ \bar{f}_{\xi\zeta}(b) \text{ for all } \xi < \zeta < \theta_\alpha$$

and

$$f_{\xi\zeta}(b) = f_{\theta_\alpha} \circ b = f^{\#}(\theta_\alpha) \circ \bar{f}_{\xi\zeta}(b) \text{ for all } \xi < \zeta = \theta_\alpha$$

where the first equalitiy holds by the definition of left-branching embedding and the second by the definition of $f^{\#}(\theta_\alpha)$ and $\bar{f}_{\xi\zeta}$.

Let us now consider the right-branching embedding $f \in \mathfrak{F}_{\alpha,\alpha+1}$. Set $\eta := crit(f \upharpoonright \theta_\alpha)$.
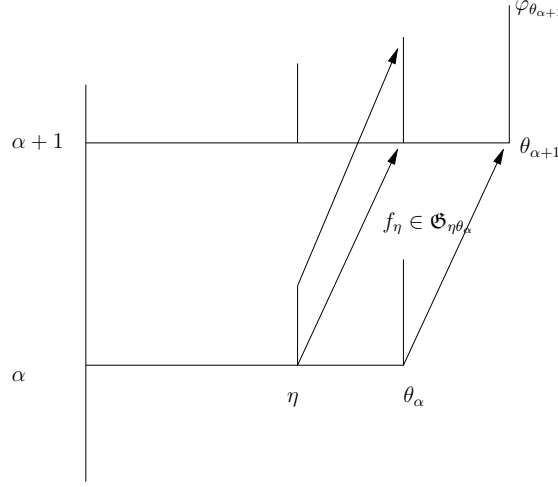
Let $f^{\#}(\zeta) = id$ for all $\eta \neq \zeta \leq \theta_\alpha$ and $f^{\#}(\eta) = f_\eta$. Set $\bar{f}_\zeta = id$ for all $\zeta \leq \eta$ and $\bar{f}_\zeta = f_\zeta$ for all $\eta < \zeta \leq \theta_\alpha$. Finally set $\bar{f}_{\xi\zeta} = id$ for all $\xi < \zeta \leq \eta$ and $\bar{f}_{\xi\zeta} = f_{\xi\zeta}$ for all $\xi < \zeta$, $\eta < \zeta \leq \theta_\alpha$.

Like before, it is easy to check that (1) and (2) hold. We only check (2) for $\xi < \zeta = \eta$. In this case

$$f_{\xi\zeta}(b) = f_{\xi\zeta}(b) \circ id = f_{\xi\zeta}(b) \circ f_\xi = f_\eta \circ b = f^{\#}(\eta) \circ \bar{f}_{\xi\zeta}(b)$$

where the third equality holds by (6) in the definition of embedding.

From the successor case, the general statement of the lemma follows as usual by induction over the levels of the morass. For more details see Velleman's paper.

Lemma 5.5

(3) If $\xi < \bar{f}(\zeta)$ and $b \in \mathfrak{G}_{\xi\bar{f}(\zeta)}$, then $\exists \eta < \zeta \ \exists c \in \mathfrak{G}_{\eta\zeta} \ \exists d \in \mathfrak{G}_{\xi f(\eta)} \quad b = \bar{f}_{\eta\zeta}(c) \circ d$.

(4) $\forall \xi < \zeta \ \forall b \in \mathfrak{G}_{\xi\zeta} \quad \bar{f}_\zeta \circ b = \bar{f}_{\xi\zeta}(b) \circ f_\xi$.

(5) If $\eta < \xi < \zeta$, $b \in \mathfrak{G}_{\xi\zeta}$ and $c \in \mathfrak{G}_{\eta\xi}$, then $\bar{f}_{\eta\zeta}(b \circ c) = \bar{f}_{\xi\zeta}(b) \circ f_{\eta\xi}(c)$.

(6) If $\alpha < \beta < \gamma \leq \kappa$, $f \in \mathfrak{F}_{\beta\gamma}$, $g \in \mathfrak{F}_{\alpha\beta}$ and $\zeta \leq \theta_\alpha$, then
$$\overline{(f \circ g)}_\zeta = \bar{f}_{\bar{g}(\zeta)} \circ \bar{g}_\zeta$$
$$(f \circ g)^{\#}(\zeta) = f_{\bar{g}(\zeta)g(\zeta)}(g^{\#}(\zeta)) \circ f^{\#}(\bar{g}(\zeta)) \text{ and}$$
$$\overline{(f \circ g)}_{\xi\zeta} = \bar{f}_{g(\xi)\bar{g}(\zeta)} \circ \bar{g}_{\xi\zeta} \text{ for all } \xi < \zeta.$$

Historical remarks and references

Practically everything in this section is due to D. Velleman [47, 46].

6

Spread and size of Hausdorff spaces

As an application of three-dimensional forcing, we will prove it to be consistent that there exists a Hausdorff space X with spread $s(X) = \omega_1$ and size $card(X) = 2^{2^{s(X)}}$

As every mathematician knows, a topology τ on a set X is a subset $\tau \subseteq \mathfrak{P}(X)$ such that

(1) $\emptyset, X \in \tau$

(2) $U, V \in \tau \Rightarrow U \cap V \in \tau$

(3) $\{U_i \mid i \in A\} \subseteq \tau \Rightarrow \bigcup\{U_i \mid i \in A\} \in \tau$.

The pair (X, τ) is called a topological space. The $U \in \tau$ are called open sets, their complements are called closed sets. A set is clopen if it is closed and open.

A base B of τ is a subset $B \subseteq \tau$ such that every element of τ is a union of elements of B. A space is 0-dimensional if it has a base which contains only clopen sets.

A space is a Hausdorff space if for all $x \neq y \in X$ there are $x \in U \in \tau$ and $y \in V \in \tau$ such that $U \cap V = \emptyset$. A Hausdorff space is regular if for all closed $C \subseteq X$ and all $x \in X - C$, there exist $C \subseteq U \in \tau$ and $x \in V \in \tau$ such that $U \cap V = \emptyset$.

Every 0-dimensional Hausdorff space (X, τ) is regular. To see this, let B be a clopen base of τ, $C \subseteq X$ be closed and $x \in X - C$. Since C is closed, $X - C$ is open. Hence there is an $U \in B$ such that $x \in u \subseteq X - C$, because B is a base of τ. By the choice of B, U is clopen. So $X - U$ is open, U is open, $C \subseteq X - U$, $x \in U$, $(X - U) \cap U = \emptyset$, and we are done.

A subset $D \subseteq X$ is called discrete if for every $x \in D$ there exists an $U \in \tau$ such that $U \cap D = \{x\}$.

The spread $s(X)$ of X is defined as

$$s(X) = \omega \cdot sup\{card(D) \mid D \text{ is a discrete subset of } X\}.$$

Theorem 6.1 (Hajnal, Juhasz)

Assume that X is a Hausdorff space. Then

$$card(X) \leq 2^{2^{s(X)}}.$$

Proof: Let $\alpha := s(X)$. Assume towards a contradiction that $card(X) \geq (2^{2^{\alpha}})^{+}$. We will define a partition f of $[X]^3$ into 4 pieces. By the Erdös-Rado theorem

$$(2^{2^{\alpha}})^{+} \rightarrow (\alpha^{+})^{3}_{4}.$$

Hence f has a homogeneous set H of size α^{+}. However, from H, we will be able to define a discrete subspace of size α^{+}. This is a contradiction to $\alpha = s(X)$.

We define $f : [X]^3 \rightarrow 4$ as follows: Let $<_X$ be a well-ordering of X. Since X is a Hausdorff space, we can choose for each pair $\{x, y\} \in [X]^2$ with $x <_X y$ disjoint open sets $x \in U(x, y)$ and $y \in V(x, y)$. For $x <_X y <_X z$ set

$f(x, y, z) = 0$ if $x \in U(y, z)$ and $z \in V(x, y)$
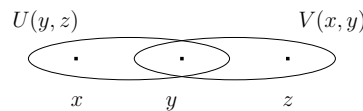
$f(x, y, z) = 0$ if $x \in U(y, z)$ and $z \in V(x, y)$
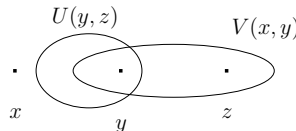
$f(x,y,z) = 0$ if $x \in U(y,z)$ and $z \in V(x,y)$
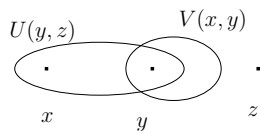
$f(x,y,z) = 0$ if $x \in U(y,z)$ and $z \in V(x,y)$.
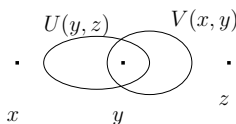
By the Erdös-Rado theorem, there exists $H \subseteq X$ such that $|H| = \alpha^+$ and $|f[[H]^3]| = 1$. Suppose now $y \in H$ and y has both an immediate predecessor x in $<_X$ and an immediate successor z in $<_X$. Then

$$(*) \qquad H \cap U(y,z) \cap V(x,y) = \{y\}.$$

Hence $D = H - \{x \in H \mid x \text{ limit point in } <_X\}$ is a discrete subspace of X of size α^+.

To see $(*)$, assume that $p \in H \cap U(y,z) \cap V(x,y)$ and $p \neq y$. Since $p \neq x$ and $p \neq y$ are obvious, we have either $p <_X x$ or $z <_X p$. Assume first that $p <_X x$. Then $f(p,y,z) \in \{0,2\}$ because $p \in U(y,z)$.
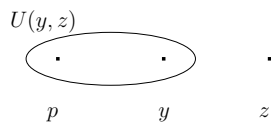
Hence by the homogeneity of H, $f(p,x,y) \in \{0,2\}$. So $p \in U(x,y)$ and thus $p \notin V(x,y)$, which is a contradiction. Now, assume $z <_x p$. Then $f(x,y,p) \in \{0,1\}$ because $p \in V(x,y)$.
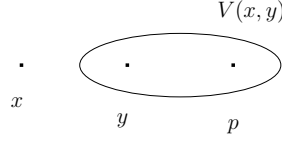
So also $f(y, z, p) \in \{0, 1\}$ by homogeneity of H. Hence $p \in V(y, z)$ and thus $p \notin U(y, z)$. Contradiction. $\square$

This proof is given by I. Juhasz in his book "Cardinal functions in topology" (1971). He continues by asking if the second exponentiation is really necessary. This was answered shortly after by Fedorcuk who constructed a 0-dimensional Hausdorff space with spread ω and size 2^{2^ω} in L. Up to now, it has not been known whether there can be a Hausdorff space with spread ω_1 and size $2^{2^{\omega_1}}$. Given a simplified $(\omega_1, 2)$-morass, we will construct a ccc forcing of size ω_1 which adds such a space that is 0-dimensional.

The natural forcing to add a 0-dimensional Hausdorff space is Cohen forcing. Let

$$P := \{p : x_p \to 2 \mid x_p \subseteq \omega_3 \times \omega_2 \text{ finite}\}.$$

As usual, we set $p \leq q$ iff $q \subseteq p$.

Let G be P-generic. Set $F = \bigcup\{p \mid p \in G\}$. Then $F : \omega_3 \times \omega_2 \to 2$ by a simple density argument. Let τ be the topology on ω_3 which is generated by the sets

$$A_\nu^i := \{\alpha \in \omega_3 \mid F(\alpha, \nu) = i\}.$$

Thus a base for τ is formed by the sets

$$B_\varepsilon := \bigcap\{A_\nu^{\varepsilon(\nu)} \mid \nu \in dom(\varepsilon)\}$$

where $\varepsilon : dom(\varepsilon) \to 2$ is finite and $dom(\varepsilon) \subseteq \omega_2$.

Hence τ is 0-dimensional because if $B_\varepsilon := \bigcap\{A_\nu^{\varepsilon(\nu)} \mid \nu \in dom(\varepsilon)\}$ then

$$\omega_3 - B_\varepsilon = \bigcup\{A_\nu^{\bar{\varepsilon}(\nu)} \mid \nu \in dom(\varepsilon)\} \in \tau$$

(where $\bar{\varepsilon}(\nu) = 0$ iff $\varepsilon(\nu) = 1$).
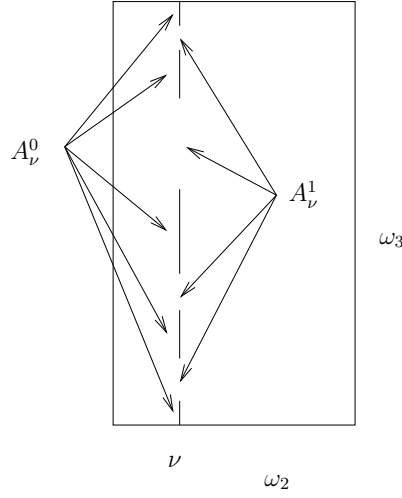
It is also a Hausdorff space because for all $\gamma \neq \delta$ there is by genericity of G some $\mu \in \omega_2$ such that $F(\gamma, \nu) \neq F(\delta, \mu)$. This, however, means that $\gamma \in A_\mu^{F(\gamma,\mu)}$, $\delta \in A_\mu^{F(\delta,\mu)}$ and $A_\mu^{F(\delta,\mu)} \cap A_\mu^{F(\gamma,\mu)} = \emptyset$.

Moreover, (ω_3, τ) has spread $\leq \omega_1$:

Suppose not. Let $\dot{X}, \dot{h}, \dot{B}$ be P-names and $p \in P$ such that

$p \Vdash (\dot{X} \subseteq \omega_3,\ \dot{h} : \omega_2 \to \dot{X}$ is bijective, $\dot{B} : \omega_2 \to V,\ \forall i \in \omega_2\ \dot{B}(i)$ is a basic open set, $\forall i \neq j \in \omega_2\ \dot{h}(i) \in \dot{B}(i) \wedge \dot{h}(i) \notin \dot{B}(j))$.

For every $i \in \omega_2$ let $p_i \leq p$, δ_i and ε_i be such that $p \Vdash \dot{h}(\check{i}) = \check{\delta}_i \wedge \dot{B}(\check{i}) = B_{\check{\varepsilon}_i}$. Assume that all conditions are of the form $p_i : a_{p_i} \times b_{p_i} \to 2$ and that all p_i are isomorpic. That is, all a_{p_i}'s have the same size and all b_{p_i}'s have the same size. And if $\sigma_{ij} : a_{p_i} \to a_{p_j}$ and $\pi_{ij} : b_{p_i} \to b_{p_j}$ is order-preserving, then $p_i(\gamma, \delta) = p_j(\sigma_{ij}(\gamma), \pi_{ij}(\delta))$. Then we can assume by the Δ-system lemma that the a_{p_i}'s and b_{p_i}'s form Δ-systems with roots Δ_a and Δ_b. Moreover, we can assume that Δ_a is always at the same place in a_{p_i}. We can assume the same for the b_{p_i}'s. Furthermore, we can assume that δ_i always has the same position in a_{p_i} and $dom(\varepsilon_i)$ in b_{p_i}.
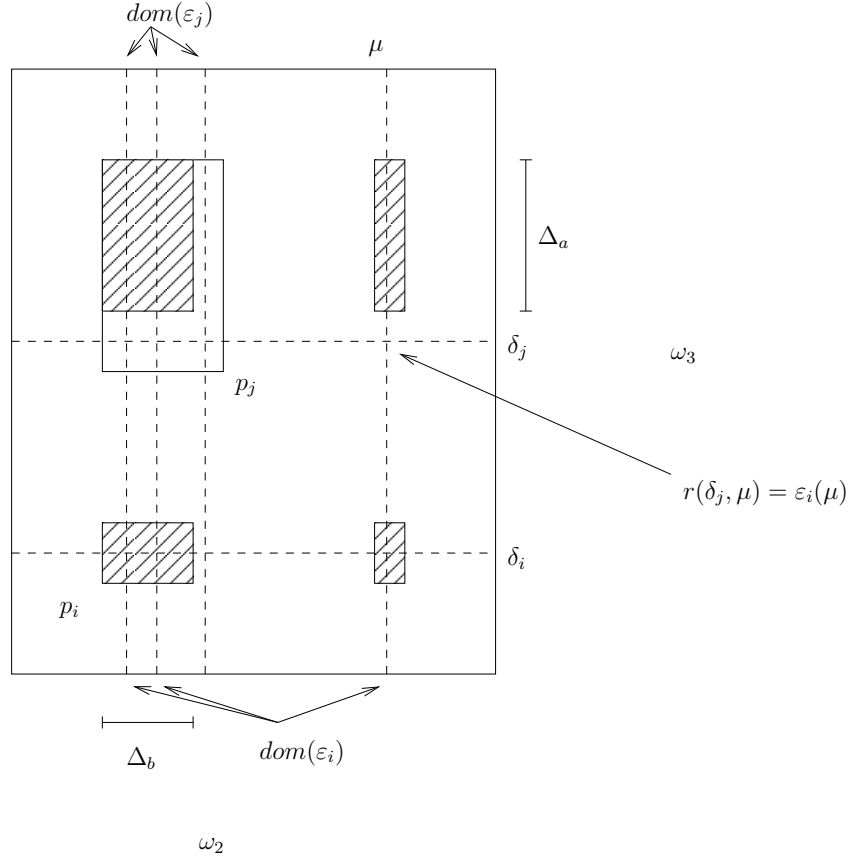
Then for $i, j \in \omega_2$ there exists $r \leq p_i, p_j$ such that $\langle \delta_j, \mu \rangle \in dom(r)$ and $r(\delta_j, \mu) = \varepsilon_i(\mu)$ for all $\mu \in dom(\varepsilon_i)$. Hence $r \Vdash \dot{h}(\check{j}) = \check{\delta}_j \in \dot{B}(\check{i})$ which contradicts the definition of p.

Of course, the Cohen forcing P also satisfies ccc and hence preserves cardinals. However, $P \Vdash 2^\omega \geq \omega_3$ and hence $P \Vdash 2^{2^{\omega_1}} \geq \omega_4$. So (ω_3, τ) will not be as wanted. In the following, we will use our methods to thin out P so that the remaining forcing is equivalent to a ccc forcing of size ω_1. In this case, the usual argument for Cohen forcing shows that GCH is preserved. This argument was given at the end of section 3.

Now, let $\mathfrak{M}$ be a simplified $(\kappa, 2)$-morass like in section 5.

We say that

$$\langle \langle \mathbb{P}_\eta \mid \eta \leq \kappa^{++} \rangle, \langle \sigma_{st} \mid s \prec t \rangle, \langle \sigma'_{st} \mid s \prec' t \rangle, \langle e_\alpha \mid \alpha < \kappa^+ \rangle, \langle e'_\alpha \mid \alpha < \kappa \rangle \rangle$$

is a FS system along $\mathfrak{M}$ if the following conditions hold:

(FS$_2$1) $\langle \langle \mathbb{P}_\eta \mid \eta \leq \kappa^{++} \rangle, \langle \sigma_{st} \mid s \prec t \rangle, \langle e_\alpha \mid \alpha < \kappa^+ \rangle \rangle$ is a FS system along

$$\langle\langle \varphi_\zeta \mid \zeta \le \kappa^+\rangle, \langle \mathfrak{G}_{\zeta\xi} \mid \zeta < \xi \le \kappa^+\rangle\rangle.$$

Let $\mathbb{Q} = \{p^* \restriction supp(p) \mid p \in \mathbb{P}\}$.

Define a partial order $\le$ on $\mathbb{Q}$ by setting $p \le q$ iff $dom(q) \subseteq dom(p)$ and $p(\alpha) \le q(\alpha)$ for all $\alpha \in dom(q)$.

Set $\mathbb{Q}_\gamma := \{p \in \mathbb{Q} \mid dom(p) \subseteq \gamma\}$.

(FS$_2$2) $\langle \sigma'_{st} \mid s \prec' t\rangle$ is a commutative system of injective embeddings $\sigma'_{st} : \mathbb{Q}_{\nu(s)+1} \to \mathbb{Q}_{\nu(t)+1}$ such that if t is a limit point in $\prec'$, then $\mathbb{Q}_{\nu(t)+1} = \bigcup\{\sigma'_{st}[\mathbb{Q}_{\nu(s)+1}] \mid s \prec' t\}$.

(FS$_2$3) $e'_\alpha : \mathbb{Q}_{\theta_{\alpha+1}} \to \mathbb{Q}_{\theta_\alpha}$.

(FS$_2$4) Let $s \prec' t$ and $\pi = \pi'_{st}$. If $\pi(\nu') = \tau'$, $s' = \langle \alpha(s), \nu'\rangle$ and $t' = \langle \alpha(t), \tau'\rangle$, then $\sigma'_{st} : \mathbb{Q}_{\nu(s)+1} \to \mathbb{Q}_{\nu(t)+1}$ extends $\sigma'_{s't'} : \mathbb{Q}_{\nu'+1} \to \mathbb{Q}_{\tau'+1}$.

Hence for $f \in \mathfrak{F}_{\alpha\beta}$, we may define $\sigma_f = \bigcup\{\sigma_{st} \mid s = \langle \alpha, \nu\rangle, t = \langle \beta, f(\nu)\rangle\}$.

(FS$_2$5) If $\pi'_{st} \restriction \nu(s) + 1 = id \restriction \nu(s) + 1$, then $\sigma'_{st} = id \restriction \mathbb{Q}_{\nu(s)+1}$.

(FS$_2$6)(a) If $\alpha < \kappa$, then $\mathbb{Q}_{\theta_\alpha}$ is completely contained in $\mathbb{Q}_{\theta_{\alpha+1}}$ in such a way that $e'_\alpha(p)$ is a reduction of $p \in \mathbb{Q}_{\theta_{\alpha+1}}$.

(b) If $\alpha < \kappa$, then $\sigma'_\alpha := \sigma'_{f_\alpha} : \mathbb{Q}_{\theta_\alpha} \to \mathbb{Q}_{\theta_{\alpha+1}}$ (where f_α is the unique right-branching $f \in \mathfrak{F}_{\alpha,\alpha+1}$) is a complete embedding such that $e'_\alpha(p)$ is a reduction of $p \in \mathbb{Q}_{\theta_{\alpha+1}}$.

(FS$_2$7)(a) If $\alpha < \kappa$ and $p \in \mathbb{Q}_{\theta_\alpha}$, then $e'_\alpha(p) = p$.

(b) If $\alpha < \kappa$ and $p \in rng(\sigma'_\alpha)$, then $e'_\alpha(p) = (\sigma'_\alpha)^{-1}(p)$.

That is,

$$\langle\langle \mathbb{Q}_\eta \mid \eta \le \kappa^+\rangle, \langle \sigma'_{st} \mid s \prec' t\rangle, \langle e'_\alpha \mid \alpha < \kappa\rangle\rangle$$

is a FS system along $\langle\langle \theta_\alpha \mid \alpha \le \kappa\rangle, \langle \mathfrak{F}'_{\alpha\beta} \mid \alpha < \beta \le \kappa\rangle\rangle$.
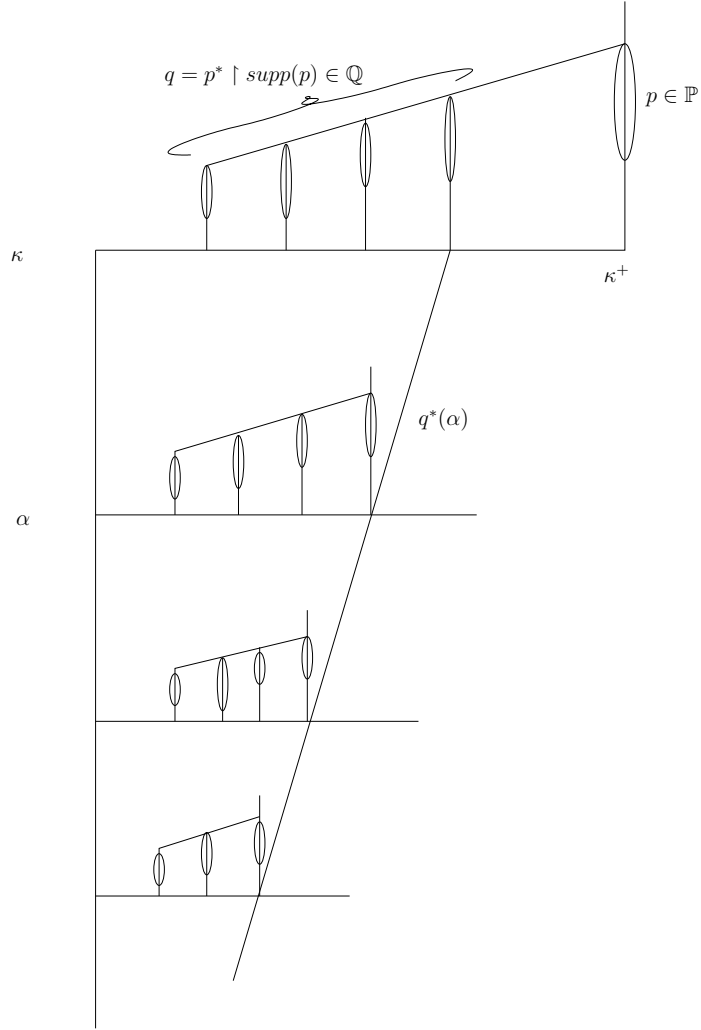
Theorem 6.2

Let $\kappa, \mu > \omega$ be cardinals, κ regular. Let $\langle\langle \mathbb{P}_\eta \mid \eta \leq \kappa^{++}\rangle, \langle \sigma_{st} \mid s \prec t\rangle, \langle \sigma'_{st} \mid s \prec' t\rangle, \langle e_\alpha \mid \alpha < \kappa^+\rangle, \langle e'_\alpha \mid \alpha < \kappa\rangle\rangle$ be a FS system along a $(\kappa, 2)$-morass.

(a) If $\langle \mathbb{Q}, \leq\rangle$ satisfies the μ-cc, then $\mathbb{P}$ also does.

(b) If all $\mathbb{Q}_\eta$ with $\eta < \kappa$ satisfy the μ-cc, then $\mathbb{P}$ also does.

Proof: (a) Exactly like the proof in section 3 that $\mathbb{P}$ satisfies the μ-cc, if all $\mathbb{P}_\eta$ with $\eta < \kappa^+$ do.

(b) Since $\langle\langle \mathbb{Q}_\eta \mid \eta \leq \kappa^+\rangle, \langle \sigma'_{st} \mid s \prec' t\rangle, \langle e'_\alpha \mid \alpha < \kappa\rangle\rangle$ is a FS system along $\langle\langle \theta_\alpha \mid \alpha \leq \kappa\rangle, \langle \mathfrak{F}'_{\alpha\beta} \mid \alpha < \beta \leq \kappa\rangle\rangle$, $\mathbb{Q}$ satisfies the μ-cc by results from section 3. Hence (b) follows from (a). $\square$

As before, we obtain the maps σ_{st} for $s \prec t$ by extending the morass maps π_{st} to our forcing P. Let $\pi : \bar{\theta} \to \theta$ be an order-preserving map. Then $\pi : \bar{\theta} \to \theta$ induces maps $\pi : \bar{\theta} \times \omega_2 \to \theta \times \omega_2$ and $\pi : (\bar{\theta} \times \omega_2) \times 2 \to (\theta \times \omega_2) \times 2$ in the obvious way:

$$\pi : \bar{\theta} \times \omega_2 \to \theta \times \omega_2, \quad \langle \gamma, \delta \rangle \mapsto \langle \pi(\gamma), \delta \rangle$$

$$\pi : (\bar{\theta} \times \omega_2) \times 2 \to (\theta \times \omega_2) \times 2, \quad \langle x, \epsilon \rangle \mapsto \langle \pi(x), \epsilon \rangle.$$

Basically, we will define the maps σ of the FS system by setting $\sigma(p) = \pi[p]$.

Now, we start our construction of $\mathbb{P}$. Let $\mathfrak{M}$ be a simplified $(\omega_1, 2)$-morass. In a first step, we define partial orders $P(\tau)$ for $\tau \leq \omega_3$ and $Q(\tau)$ for $\tau \leq \omega_2$. In a second step, we thin out $P(\tau)$ and $Q(\tau)$ to the $\mathbb{P}_\tau$ and $\mathbb{Q}_\tau$ which form the FS system along the gap-2 morass.

We define $P(\tau)$ by induction on the levels of $\langle\langle \varphi_\zeta \mid \zeta \leq \omega_2 \rangle, \langle \mathfrak{G}_{\zeta\xi} \mid \zeta < \xi \leq \omega_2 \rangle\rangle$ which we enumerate by $\beta \leq \omega_2$.

Base Case: $\beta = 0$

Then we only need to define $P(1)$.

Let $P(1) := \{p \in P \mid x_p \subseteq 1 \times \omega\}$.

Successor Case: $\beta = \alpha + 1$

We first define $P(\varphi_\beta)$. Let it be the set of all $p \in P$ such that

(1) $x_p \subseteq \varphi_\beta \times \omega\beta$

(2) $p \upharpoonright (\varphi_\alpha \times \omega\alpha), f_\alpha^{-1}[p \upharpoonright (\varphi_\beta \times \omega\alpha)] \in P(\varphi_\alpha)$

(3) $p \upharpoonright (\varphi_\alpha \times \omega\alpha)$ and $f_\alpha^{-1}[p \upharpoonright (\varphi_\beta \times \omega\alpha)]$ are compatible in P

where f_α is like in (P3) in the definition of a simplified gap-1 morass.
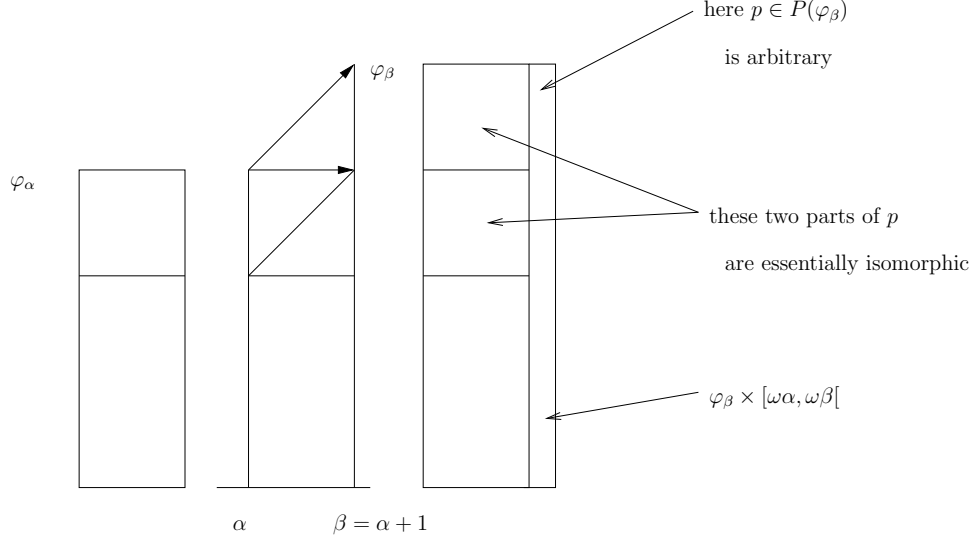
For all $\nu \leq \varphi_\alpha$ $P(\nu)$ is already defined. For $\varphi_\alpha < \nu \leq \varphi_\beta$ set

$$P(\nu) = \{p \in P(\varphi_\beta) \mid x_p \subseteq \nu \times \omega\beta\}.$$

Set

$$\sigma_{st} : P(\nu(s) + 1) \rightarrow P(\nu(t) + 1), p \mapsto \pi_{st}[p].$$

It remains to define e_α. If $p \in rng(\sigma_\alpha)$, then set $e_\alpha(p) = \sigma_\alpha^{-1}(p)$. If $p \in P(\varphi_\alpha)$, then set $e_\alpha(p) = p$. And if $p \notin rng(\sigma_\alpha) \cup P(\varphi_\alpha)$, then set

$$e_\alpha(p) = p \upharpoonright (\varphi_\alpha \times \omega\alpha) \cup f_\alpha^{-1}[p \upharpoonright (\varphi_\beta \times \omega\alpha)].$$

Limit Case: $\beta \in Lim$

For $t \in T_\beta$ set $P(\nu(t)+1) = \bigcup\{\sigma_{st}[P(\nu(s)+1)] \mid s \prec t\}$ and $P(\lambda) = \bigcup\{P(\eta) \mid \eta < \lambda\}$ for $\lambda \in Lim$ where $\sigma_{st} : P(\nu(s) + 1) \rightarrow P(\nu(t) + 1), p \mapsto \pi_{st}[p]$.

Lemma 6.3
$\langle\langle P(\eta) \mid \eta \leq \omega_3\rangle, \langle\sigma_{st} \mid s \prec t\rangle, \langle e_\alpha \mid \alpha < \omega_2\rangle\rangle$ is a FS system along $\langle\langle\varphi_\zeta \mid \zeta \leq \omega_2\rangle, \langle\mathfrak{G}_{\zeta\xi} \mid \zeta < \xi \leq \omega_2\rangle\rangle$.

Proof: Most things are clear. We only prove (FS6). Let $p \in P(\varphi_\beta)$ and $\beta = \alpha + 1$. Let $q := p \upharpoonright (\varphi_\alpha \times \omega\alpha) \cup f_\alpha^{-1}[p \upharpoonright (\varphi_\beta \times \omega\alpha)]$. We have to prove that q is a reduction of p with respect to σ_α and $id \upharpoonright P(\varphi_\alpha)$. To do so, let $r \leq q$. We have to find an $s \leq p, \sigma_\alpha(r), r$ such that $s \in P(\varphi_\beta)$. Define s as

$s := p \cup r \cup f_\alpha[r]$. It is easily seen that s is as wanted. □

By the previous lemma every $p \in P(\omega_3)$ has finite support and we may define p^* for $p \in P(\omega_3)$ like in section 3.

Set $Q = \{p^* \upharpoonright supp(p) \mid p \in P(\omega_3)\}$.

Define a partial order $\leq$ on Q by setting $p \leq q$ iff $dom(q) \subseteq dom(p)$ and $p(\alpha) \leq q(\alpha)$ for all $\alpha \in dom(q)$.

Set $Q(\gamma) = \{p \in Q \mid dom(p) \subseteq \gamma\}$.

Now, we thin out Q along $\langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathfrak{F}'_{\alpha\beta} \mid \alpha < \beta \leq \kappa \rangle \rangle$ to obtain $\mathbb{Q}$. We will, however, not use the maps $f \in \mathfrak{F}_{\alpha\beta}$ but $\bar{f}$ to map $p \in Q(\theta_\alpha)$ to $Q(\theta_\beta)$.

For $f \in \mathfrak{F}_{\alpha\beta}$ and $p \in Q(\theta_\alpha)$ we may define $\bar{f}[p]$ with $dom(\bar{f}[p]) = \bar{f}[dom(p)]$ by setting

$$\bar{f}[p](\bar{f}(\eta)) = \bar{f}_\eta \otimes f[p(\eta)] \quad \text{for all} \quad \eta \in dom(p)$$

where $\bar{f}$, $\bar{f}_\eta$ are as at the end of section 5 and

$$\bar{f}_\eta \otimes f : \varphi_\eta \times \omega\eta \to \varphi_{\bar{f}(\eta)} \times \omega\bar{f}(\eta), \ \langle \gamma, \omega\delta + n \rangle \mapsto \langle \bar{f}_\eta(\gamma), \omega f(\delta) + n \rangle$$

for all $n \in \omega$

$$\bar{f}_\eta \otimes f : (\varphi_\eta \times \eta) \times 2 \to (\varphi_{\bar{f}(\eta)} \times \bar{f}(\eta)) \times 2, \ \langle x, \epsilon \rangle \mapsto \langle \bar{f}_\eta \otimes f(x), \epsilon \rangle.$$

In the same way we may define $\overline{\pi'_{st}}[p]$.
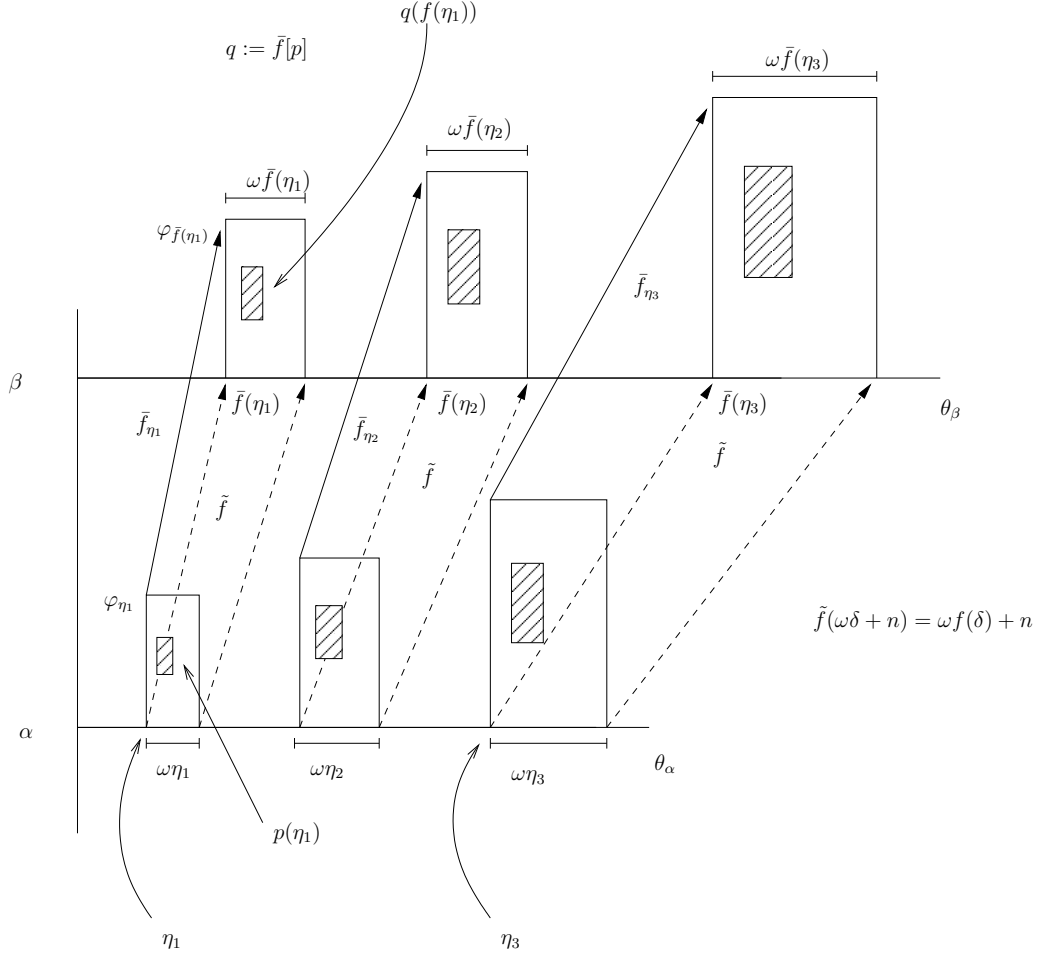
The reason why we use $\bar{f}$ instead of $f \in \mathfrak{F}_{\alpha\beta}$ is that f does not map the support of a condition correctly. For an example, consider the case $\beta = \alpha + 1$ and let $f \in \mathfrak{F}_{\alpha\beta}$ be right-branching. Let δ be the splitting point of f, i.e. $f(\delta) = \theta_\beta$. Assume that $p \in Q(\theta_\alpha)$, $\delta \in dom(p)$ and $dom(p(\delta)) \subseteq \varphi_\delta \times \omega\delta$. Let $f[p]$ be defined by $dom(f[p]) = f[dom(p)]$ and $f[p](f(\eta)) := f_\eta \otimes f[p(\eta)]$ for all $\eta \in dom(p)$. We will show that $f[p] \notin Q(\theta_\beta)$. To do so, notice first that $f_\delta = f^{\#}(\delta) \circ \bar{f}_\delta$ by (1) in the lemma about $\bar{f}$ at the end of section 5. However, $\bar{f}_\delta = id \upharpoonright \varphi_\delta$, because f is right-branching with splitting point δ. So $f_\delta = f^{\#}(\delta)$. Hence $f[p](\theta_\alpha) = f_\delta \otimes f[p(\delta)] = f^{\#}(\delta)[p(\delta)]$ because $dom(p(\delta)) \subseteq \varphi_\delta \times \omega\delta$ and $f \upharpoonright \delta = id \upharpoonright \delta$. However, this contradicts the fact that all $q \in Q(\theta_\beta)$ are of the form $q = r^* \upharpoonright supp(r)$ for some $r \in P(\omega_3)$ because in this case $q(\theta_\alpha) \neq g[\bar{q}]$ for all $g \in \mathfrak{G}_{\gamma\theta_\alpha}$, $\bar{q} \in P(\varphi_\gamma)$ and $\gamma < \theta_\alpha$ by the definition of the support of a condition.
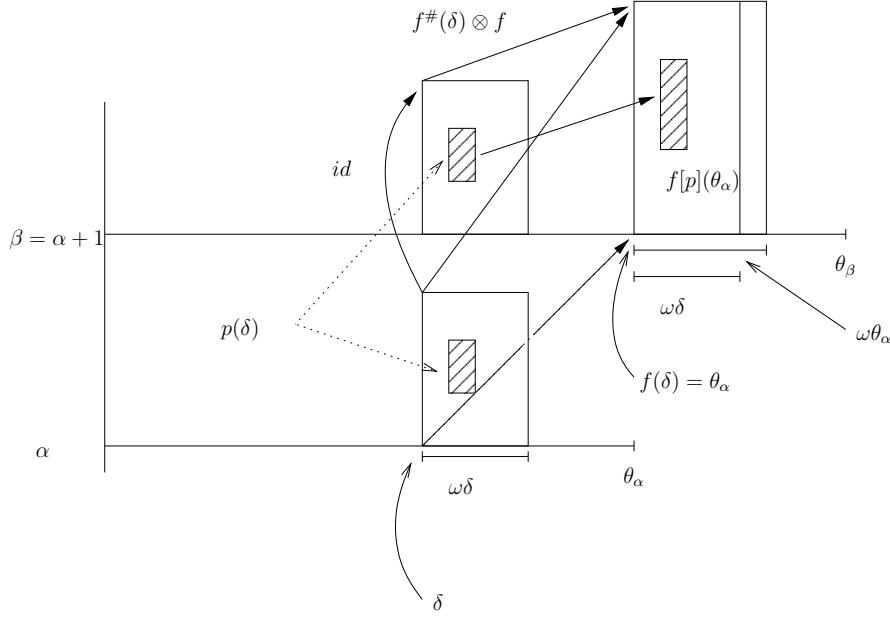
This problem does obviously not occur, if we consider $\bar{f}[p]$.

Lemma 6.4

(a) If $f \in \mathfrak{F}_{\alpha\beta}$ and $p \in Q(\theta_\alpha)$, then $\bar{f}[p] \in Q(\theta_\beta)$.

(b) If $s \prec' t$ and $p \in Q(\nu(s)+1)$, then $\overline{\pi'_{st}}[p] \in Q(\nu(t)+1)$.

Proof: Set $q := \bar{f}[p]$. Let $dom(p) = \{\alpha_1 < \ldots < \alpha_n\}$ and $dom(q) = \{\beta_1 < \ldots < \beta_n\} := \{\bar{f}(\alpha_1) < \ldots < \bar{f}(\alpha_n)\}$. By the definition of the support of a condition, all α_i are successor ordinals. And $f(\alpha_i - 1) = \bar{f}(\alpha_i) - 1$ by the definition of $\bar{f}$. Set $q(\beta_i - 1) = e_{\beta_i - 1}(q(\beta_i))$. Then it suffices to prove that there are functions $g_i \in \mathfrak{G}_{\beta_i, \beta_{i+1}-1}$ such that

(1) $q(\beta_{i+1} - 1) = g_i[q(\beta_i)]$

(2) $q(\beta_i) \notin rng(\sigma_{\beta_i - 1})$, $q(\beta_i) \notin P(\varphi_{\beta_i - 1})$:

Since p is a condition, there are functions $h_i \in \mathfrak{G}_{\alpha_i, \alpha_{i+1}-1}$ such that

$$p(\alpha_{i+1} - 1) = h_i[p(\alpha_i)].$$

So we can set

$$g_i = f_{\alpha_i, \alpha_{i+1}-1}(h_i) \circ f^{\#}(\alpha_i).$$
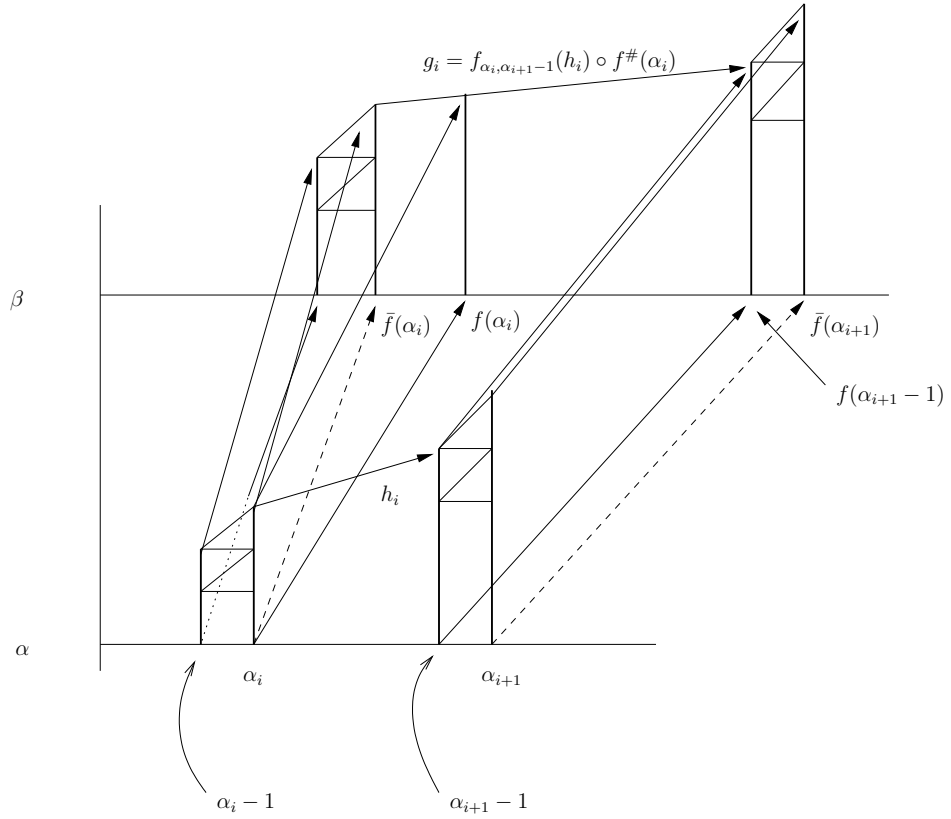
We need to check (1). We first prove that

$$f_{\alpha_{i+1}-1} \otimes f[e_{\alpha_{i+1}-1}(p(\alpha_{i+1}))] = e_{\beta_{i+1}-1}(q(\beta_{i+1})).$$
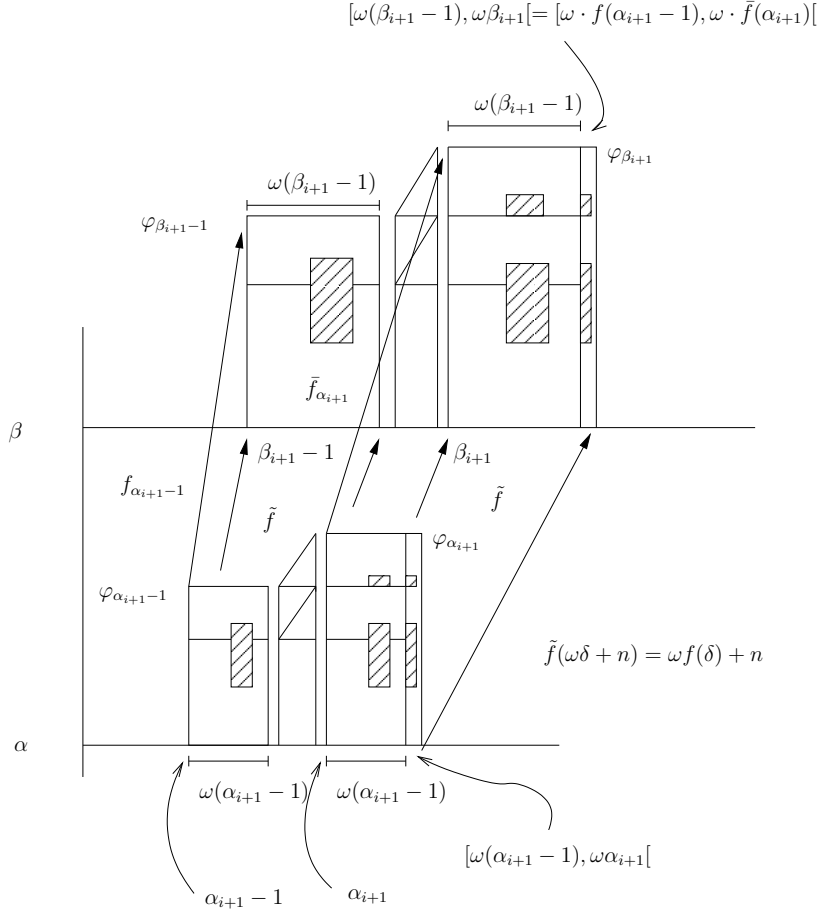
To see this, we use (4) in the lemma on $\bar{f}$ in section 5 which says

$$\forall \xi < \zeta \;\; \forall b \in \mathfrak{G}_{\xi\zeta} \;\;\; \bar{f}_\zeta \circ b = \bar{f}_{\xi\zeta}(b) \circ f_\xi.$$

Applying it for $\xi = \alpha_{i+1} - 1$, $\zeta = \alpha_{i+1}$ and $b = id \upharpoonright \varphi_{\alpha_{i+1}-1}$, we get

$$q(\beta_{i+1}) \upharpoonright (\varphi_{\beta_{i+1}-1} \times \omega(\beta_{i+1}-1)) = \bar{f}_{\alpha_{i+1}} \otimes f[p(\alpha_{i+1})] \upharpoonright (\varphi_{\beta_{i+1}-1} \times \omega(\beta_{i+1}-1)) =$$

$$= f_{\alpha_{i+1}-1} \otimes f[p(\alpha_{i+1}) \upharpoonright (\varphi_{\alpha_{i+1}-1} \times \omega(\alpha_{i+1}-1))]$$

where the first equality holds by the definition of $q = \bar{f}[p]$.

Applying it for $\xi = \alpha_{i+1} - 1$, $\zeta = \alpha_{i+1}$ and the splitting map b of $\mathfrak{G}_{\alpha_{i+1}-1,\alpha_{i+1}}$, we obtain

$$\bar{f}_{\xi\zeta}(b)^{-1}[q(\beta_{i+1}) \upharpoonright (\varphi_{\beta_{i+1}} \times \omega(\beta_{i+1}-1))] =$$

$$= \bar{f}_{\xi\zeta}(b)^{-1}[\bar{f}_{\alpha_{i+1}} \otimes f[p(\alpha_{i+1})] \restriction (\varphi_{\beta_{i+1}} \times \omega(\beta_{i+1} - 1))] =$$

$$(\bar{f}_{\xi\zeta}(b)^{-1} \circ \bar{f}_{\alpha_{i+1}}) \otimes f[p(\alpha_{i+1})] \restriction (\varphi_{\alpha_{i+1}} \times \omega(\alpha_{i+1} - 1))] =$$

$$= (f_\xi \circ b^{-1}) \otimes f[p(\alpha_{i+1})] \restriction (\varphi_{\alpha_{i+1}} \times \omega(\alpha_{i+1} - 1))] =$$

$$= (f_\xi \otimes f)[b^{-1}[p(\alpha_{i+1})] \restriction (\varphi_{\alpha_{i+1}} \times \omega(\alpha_{i+1} - 1))].$$

However, by definition

$$e_{\beta_{i+1}-1}(q(\beta_{i+1})) =$$

$$q(\beta_{i+1}) \restriction (\varphi_{\beta_{i+1}-1} \times \omega(\beta_{i+1} - 1)) \quad \cup \quad \bar{f}_{\xi\zeta}(b)^{-1}[q(\beta_{i+1}) \restriction (\varphi_{\beta_{i+1}} \times \omega(\beta_{i+1} - 1))]$$

and

$$e_{\alpha_{i+1}-1}(p(\alpha_{i+1})) =$$

$$p(\alpha_{i+1}) \restriction (\varphi_{\alpha_{i+1}-1} \times \omega(\alpha_{i+1} - 1)) \quad \cup \quad b^{-1}[p(\alpha_{i+1}) \restriction (\varphi_{\alpha_{i+1}} \times \omega(\alpha_{i+1} - 1))].$$

This proves that

$$f_{\alpha_{i+1}-1} \otimes f[e_{\alpha_{i+1}-1}(p(\alpha_{i+1}))] = e_{\beta_{i+1}-1}(q(\beta_{i+1})).$$

Hence

$$q(\beta_{i+1} - 1) = e_{\beta_{i+1}-1}(q(\beta_{i+1})) = f_{\alpha_{i+1}-1} \otimes f[e_{\alpha_{i+1}-1}(p(\alpha_{i+1}))] =$$

$$= f_{\alpha_{i+1}-1} \otimes f[h_i(p(\alpha_i))] = (f_{\alpha_{i+1}-1} \circ h_i) \otimes f[p(\alpha_i)] =$$

$$= (f_{\alpha_i,\alpha_{i+1}-1}(h_i) \circ f_{\alpha_i}) \otimes \bar{f}[p(\alpha_i)]$$

by (6) in the definition of embeddings. However, $f_{\alpha_i} = f^{\#}(\alpha_i) \circ \bar{f}_{\alpha_i}$ by the lemma on $\bar{f}$ in section 5.. So

$$(f_{\alpha_i,\alpha_{i+1}-1}(h_i) \circ f_{\alpha_i}) \otimes f[p(\alpha_i)] =$$

$$= (f_{\alpha_i,\alpha_{i+1}-1}(h_i) \circ f^{\#}(\alpha_i) \circ \bar{f}_{\alpha_i}) \otimes f[p(\alpha_i)] =$$

$$= f_{\alpha_i,\alpha_{i+1}-1}(h_i) \circ f^{\#}(\alpha_i)[\bar{f}_{\alpha_i} \otimes f[p(\alpha_i)]] =$$

$$= f_{\alpha_i,\alpha_{i+1}-1}(h_i) \circ f^{\#}(\alpha_i)[q(\beta_i)]$$

and we are done.

To see (2), notice that by the definition of the support of a condition $p(\alpha_i) \notin rng(\sigma_{\alpha_i-1})$ and $p(\alpha_i) \notin P(\varphi_{\alpha_i-1})$. Now, we can use (4) from the lemma on $\bar{f}$ in section 5 to obtain that $q(\beta_i) \notin rng(\sigma_{\beta_i-1})$ and $q(\alpha_i) \notin P(\varphi_{\beta_i-1})$. The argument is very similar to the one we used to prove

$$f_{\alpha_{i+1}-1} \otimes f[e_{\alpha_{i+1}-1}(p(\alpha_{i+1}))] = e_{\beta_{i+1}-1}(q(\beta_{i+1})). \quad \square$$

In the following we thin out $Q(\gamma)$ to $\mathbb{Q}_\gamma$ to obtain a FS system along our gap-2 morass.

We define $\mathbb{Q}_\gamma$ by induction on the levels of $\langle\langle\theta_\alpha \mid \alpha \leq \omega_1\rangle, \langle\mathfrak{F}'_{\alpha\beta} \mid \alpha < \beta \leq \omega_1\rangle\rangle$.

Base Case: $\beta = 0$

Then we only need to define $\mathbb{Q}_1$.
Let $\mathbb{Q}_1 = Q(1)$.

Successor Case: $\beta = \alpha + 1$

We first define $\mathbb{Q}_{\theta_\beta}$. To do so, let $\mathbb{P}_{\varphi_{\theta_\beta}}$ be the set of all $p \in P(\varphi_{\theta_\beta})$ such that

(1) $(\bar{f}_{\theta_\alpha} \otimes f)^{-1}[p] \in \mathbb{P}_{\varphi_{\theta_\alpha}}$
(2) $p^*(\theta_\alpha)$ and $(\bar{f}_{\theta_\alpha} \otimes f)^{-1}[p]$ are compatible

where f is the unique right-branching embedding of $\mathfrak{F}_{\alpha\beta}$.
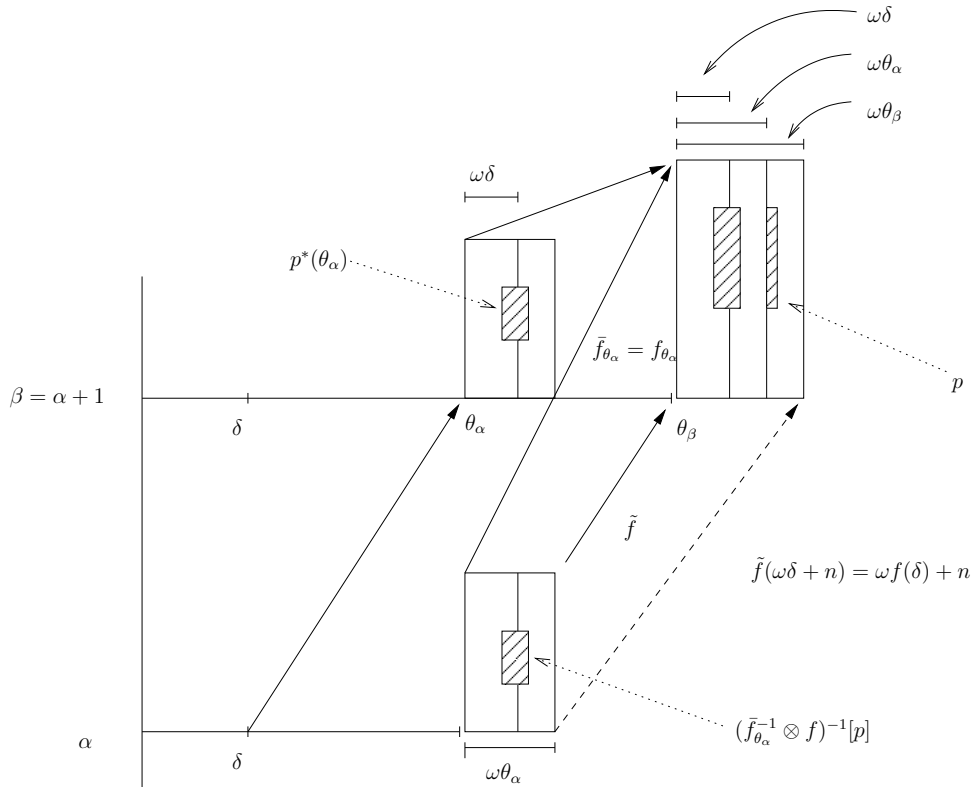
Set

$$\mathbb{Q}_{\theta_\beta} = \{p^* \restriction (supp(p) \cap \theta_\beta) \mid p \in \mathbb{P}_{\varphi_{\theta_\beta}}\}.$$

For $t \in T'_\beta$ set $\mathbb{Q}_{\nu(t)+1} = \{p \in \mathbb{Q}_{\theta_\beta} \mid dom(p) \subseteq \nu(t) + 1\}$ and $\mathbb{Q}_\lambda = \bigcup\{\mathbb{Q}_\eta \mid \eta < \lambda\}$ for $\lambda \in Lim$.

Set

$$\sigma'_{st} : \mathbb{Q}_{\nu(s)+1} \to \mathbb{Q}_{\nu(t)+1}, \ p \mapsto \overline{\pi'_{st}}[p].$$

It remains to define e'_α. If $p \in rng(\sigma'_\alpha)$, then set $e'_\alpha(p) = \sigma'^{-1}_\alpha(p)$. If $p \in \mathbb{Q}_{\theta_\alpha}$, then set $e'_\alpha(p) = p$. And if $p \notin rng(\sigma'_\alpha) \cup \mathbb{Q}_{\theta_\alpha}$, then choose a $r \in \mathbb{P}_{\varphi_{\theta_\beta}}$ with $p = r^* \restriction supp(r)$ and set

$$q := r^*(\theta_\alpha) \cup (\bar{f}_{\theta_\alpha} \otimes f)^{-1}[r]$$

Set $e'_\alpha(p) = q^* \restriction (supp(q) \cap \theta_\alpha)$.

Limit Case: $\beta \in Lim$

For $t \in T'_\beta$ set $\mathbb{Q}_{\nu(t)+1} = \bigcup\{\sigma'_{st}[\mathbb{Q}_{\nu(s)+1}] \mid s \prec' t\}$ and $\mathbb{Q}_\lambda = \bigcup\{\mathbb{Q}_\eta \mid \eta < \lambda\}$ for $\lambda \in Lim$ where $\sigma'_{st} : \mathbb{Q}_{\nu(s)+1} \to \mathbb{Q}_{\nu(t)+1}, \ p \mapsto \overline{\pi'_{st}}[p]$.

Finally, set $\mathbb{P}_\eta = \{p \in P(\eta) \mid p^* \restriction supp(p) \in \mathbb{Q}_{\omega_2}\}$ and $\mathbb{P} := \mathbb{P}_{\omega_3}$.

Remark 1:

We postpone the proof that this definies indeed an FS system along our gap-2 morass $\mathfrak{M}$. However, we check the crucial condition (FS$_2$6) already here. To do so, let $p \in \mathbb{Q}_{\theta_\beta}$ and $\beta = \alpha + 1$. Let $r \in \mathbb{P}_{\varphi_{\theta_\beta}}$ be such that $p = r^* \restriction supp(r)$ and

$$q := r^*(\theta_\alpha) \cup (\bar{f}_{\theta_\alpha} \otimes f)^{-1}[r]$$

where f is the right-branching embedding of $\mathfrak{F}_{\alpha\beta}$.

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$$p = r^* \restriction supp(r)$$

$$r$$

$$r^*(\theta_\alpha)$$

$$\beta = \alpha + 1$$

$$\theta_\alpha \qquad \theta_\beta$$

$$t \leq s$$

$$q = r^*(\theta_\alpha) \cup (\bar{f}_{\theta_\alpha} \otimes f)^{-1}[r]$$

$$\alpha \qquad \theta_\alpha$$

$$\delta \qquad \nu \qquad t(\nu)$$
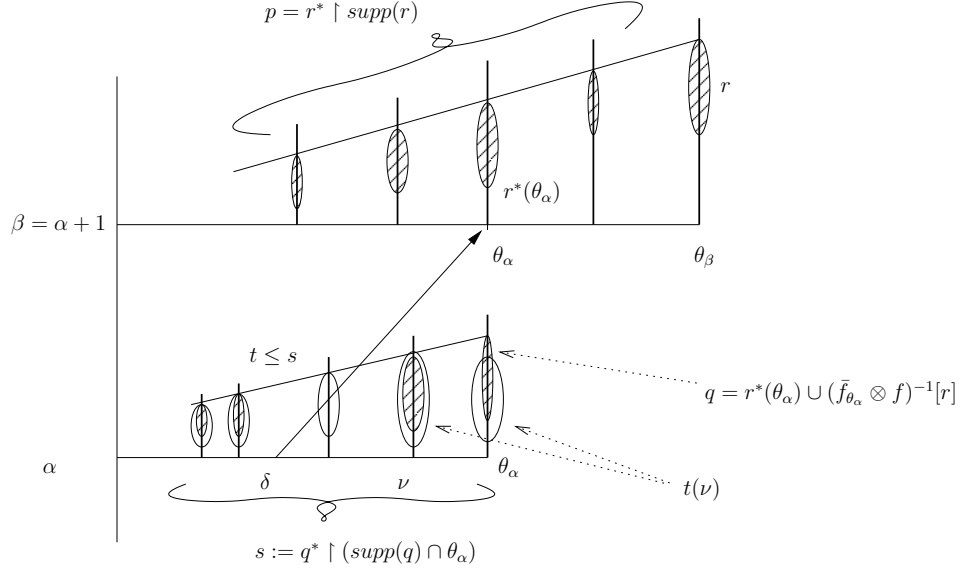
$$s := q^* \restriction (supp(q) \cap \theta_\alpha)$$

We have to prove that $s := q^* \restriction (supp(p) \cap \theta_\alpha) \in \mathbb{Q}_{\theta_\alpha}$ is a reduction of p with respect to σ'_α and $id \restriction \mathbb{Q}_{\theta_\alpha}$. To do so, let $t \in \mathbb{Q}_{\theta_\alpha}$ with $t \leq s$. We have to find an $u \in \mathbb{Q}_{\theta_\beta}$ such that $u \leq p, \sigma'_\alpha(t), t$. Notice first that by results from section 3,

$$s \leq r^* \restriction (supp(r) \cap \theta_\alpha)$$

and

$$s \leq (\bar{f}_{\theta_\alpha} \otimes f)^{-1}[r]^* \restriction supp((\bar{f}_{\theta_\alpha} \otimes f)^{-1}[r]).$$

Hence

$$t \leq r^* \restriction (supp(r) \cap \theta_\alpha)$$

and

$$t \leq (\bar{f}_{\theta_\alpha} \otimes f)^{-1}[r]^* \restriction supp((\bar{f}_{\theta_\alpha} \otimes f)^{-1}[r]).$$

Let $\nu = max(dom(t))$. Again by results from section 3, $t(\nu)$ and q are compatible. Set $\nu = q \cup t(\nu) \in \mathbb{P}_{\varphi_{\theta_\alpha}}$ and $w = r \cup v \cup (\bar{f}_{\theta_\alpha} \otimes f)[v]$. Then $w \leq r, t(\nu), (\bar{f}_{\theta_\alpha} \otimes f)[t(\nu)]$. Hence $u := w^* \restriction supp(w) \leq p$ by results from section 3 because $w \leq r$. Moreover, $u \leq t, \sigma'_\alpha(t)$. This is proved from $w \leq t(\nu), (\bar{f}_{\theta_\alpha} \otimes f)[t(\nu)]$ like in the proof of

$$p \in \mathbb{Q}_{\theta_\alpha} \quad \wedge \quad f \in \mathfrak{F}_{\alpha\beta} \quad \Rightarrow \quad \bar{f}[p] \in \mathbb{Q}_{\theta_\beta}.$$

`Remark 2:`

Suppose $p \in \mathbb{P}$ is given. Let G be any generic filter with $p \in G$. Let $F = \bigcup\{p \mid p \in G\}$. Then by (2) in the successor step of the construction, F is not only already determined on $dom(p)$, but a lot more of F is already determined. Set

$$D = \{n \in \omega \mid \exists \delta, \gamma \; \langle \gamma, \omega\delta + n \rangle \in dom(p)\}.$$

Then it will turn out that F is at least not yet determined on

$$\omega_3 \times \{\omega\delta + n \mid n \in \omega - D, \delta \in \omega_2\}.$$

Hence we can show with the same argument we used above for the forcing P also for $\mathbb{P}$ that it adds a Hausdorff space.

`Remark 3:`

Assume that $\beta = \alpha + 1$ and that f is the right-branching embedding of $\mathfrak{F}_{\alpha\beta}$. Let $p_1, p_2 \in \mathbb{P}_{\varphi_{\theta_\alpha}}$ be compatible and $g \in \mathfrak{G}_{\theta_\alpha\theta_\beta}$. Then also $g[p_1]$ and $\bar{f}_{\theta_\alpha} \otimes f[p_2]$ are compatible, i.e. $g[p_1]$ and $\bar{f}_{\theta_\alpha} \otimes f[p_2]$ agree on the common part of their domains. To prove this, let

$$\langle \gamma, \eta \rangle \in dom(g[p_1]) \cap dom(\bar{f}_{\theta_\alpha} \otimes f[p_2])$$

$$g(\langle \gamma_1, \eta_1 \rangle) = \langle \gamma, \eta \rangle \qquad \bar{f}_{\theta_\alpha} \otimes f(\langle \gamma_2, \eta_2 \rangle) = \langle \gamma, \eta \rangle.$$

Since f is right-branching, $\bar{f}_{\theta_\alpha} = f_{\theta_\alpha}$. Let δ be the critical point of $f \restriction \theta_\alpha$. Then $\eta < \omega\delta$ and therefore $\eta = \eta_1 = \eta_2$.
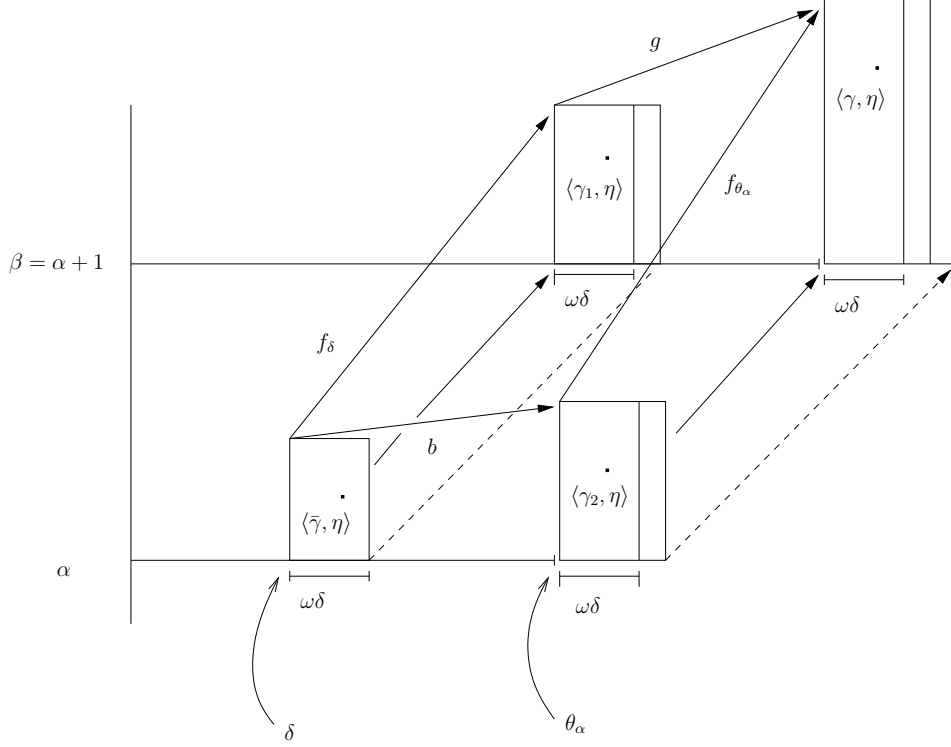
By (6) in the definition of right-branching, there exists a $b \in \mathfrak{G}_{\delta\theta_\alpha}$ such that $f_{\delta\theta_\alpha}(b) = g$. Hence, by (6) in the definition of embedding,

$$f_{\theta_\alpha} \circ b = g \circ f_\delta.$$

So there exists $\langle \bar{\gamma}, \eta \rangle \in \varphi_\delta \times \omega\delta$ such that

$$f_{\theta_\alpha} \circ b(\langle \bar{\gamma}, \eta \rangle) = g \circ f_\delta(\langle \bar{\gamma}, \eta \rangle) = \langle \gamma, \eta \rangle$$

$$f_\delta(\langle \bar{\gamma}, \eta \rangle) = \langle \gamma_1, \eta \rangle \qquad b(\langle \bar{\gamma}, \eta \rangle) = \langle \gamma_2, \eta \rangle.$$

By (5) in the definition of right-branching embedding, $f_\delta \in \mathfrak{G}_{\delta\theta_\alpha}$. Hence $p_1(\gamma_1, \eta) = p_1^*(\delta)(\bar{\gamma}, \eta)$. Moreover, $p_2(\gamma_2, \eta) = p_2^*(\delta)(\bar{\gamma}, \eta)$ because $b \in \mathfrak{G}_{\delta\theta_\alpha}$. However, p_1 and p_2 are compatible. Therefore, also $p_1^*(\delta)$ and $p_2^*(\delta)$ are compatible. So $p_1^*(\delta)(\bar{\gamma}, \eta) = p_2^*(\delta)(\bar{\gamma}, \eta)$. This in turn implies $p_1(\gamma_1, \eta) = p_2(\gamma_2, \eta)$. Hence $g[p_1](\gamma, \eta) = \bar{f}_{\theta_\alpha} \otimes f[p_2](\gamma, \eta)$. That's what we wanted to show.

The same argument shows for all $p \in \mathbb{P}_{\varphi_{\theta_\alpha}}$ and all $g \in \mathfrak{G}_{\theta_\alpha\theta_\beta}$ that $g[p] \in \mathbb{P}_{\varphi_{\theta_\beta}}$,

$\bar{f}_{\theta_\alpha} \otimes f[p] \in \mathbb{P}_{\varphi_{\theta_\beta}}$ and $g[p] \cup (\bar{f}_{\theta_\alpha} \otimes f)[p] \in \mathbb{P}_{\varphi_{\theta_\beta}}$.

For arbitrary $\alpha < \beta \le \omega_1$ and $f \in \mathfrak{F}_{\alpha\beta}$ define

$$f_{\theta_\alpha} \otimes f : \varphi_{\theta_\alpha} \times \omega\theta_\alpha \to \varphi_{\theta_\beta} \times \omega\theta_\beta, \quad \langle \gamma, \omega\delta + n \rangle \mapsto \langle f_{\theta_\alpha}(\gamma), \omega f(\delta) + n \rangle$$

for all $n \in \omega$ and

$$f_{\theta_\alpha} \otimes f : (\varphi_{\theta_\alpha} \times \omega\theta_\alpha) \times 2 \to (\varphi_{\theta_\beta} \times \omega\theta_\beta) \times 2, \quad \langle x, \epsilon \rangle \mapsto \langle f_{\theta_\alpha} \otimes f(x), \epsilon \rangle.$$

If $\beta = \alpha + 1$, then $\mathfrak{F}_{\alpha\beta}$ is an amalgamation by (3) in the definition of a simplified gap-2 morass. Hence $f \in \mathfrak{F}_{\alpha\beta}$ is either left-branching or right-branching. Let $p \in \mathbb{P}_{\varphi_{\theta_\alpha}}$ and assume that f is right-branching. Then $f_{\theta_\alpha} \otimes f[p] = \bar{f}_{\theta_\alpha} \otimes f[p]$ because $\bar{f}_{\theta_\alpha} = f_{\theta_\alpha}$. If f is left-branching, then $f_{\theta_\alpha} \in \mathfrak{G}_{\theta_\alpha\theta_\beta}$ and $f \upharpoonright \theta_\alpha = id \upharpoonright \theta_\alpha$. Hence $f_{\theta_\alpha} \otimes f[p] = f_{\theta_\alpha}[p]$. So in both cases

$$f_{\theta_\alpha} \otimes f[p] \in \mathbb{P}_{\varphi_{\theta_\beta}}.$$

By induction, this is also true if $\beta = \alpha + n$ for some $n \in \omega$. What does happen at limit levels?

Lemma 6.5

For all $\beta \in Lim$, $\mathbb{P}_{\theta_\beta} = \bigcup \{ f_{\theta_\alpha} \otimes f[\mathbb{P}_{\varphi_{\theta_\alpha}}] \mid f \in \mathfrak{F}_{\alpha\beta}, \alpha < \beta \}$.

Proof: We first prove $\supseteq$. Let $\alpha < \beta$, $p \in \mathbb{P}_{\varphi_{\theta_\alpha}}$ and $f \in \mathfrak{F}_{\alpha\beta}$. We have to prove that $r := f_{\theta_\alpha} \otimes f[p] \in \mathbb{P}_{\varphi_{\theta_\beta}}$. That is, we have to show that $r^* \upharpoonright supp(r) \in \mathbb{Q}_{\theta_\beta}$. But by the argument of lemma 6.4, $r^* \upharpoonright supp(r) = \bar{f}[q]$ where $q := p^* \upharpoonright supp(p) \in \mathbb{Q}_{\theta_\alpha}$. Hence $\bar{f}[q] = r^* \upharpoonright supp(r)$ by the definition of $\mathbb{Q}_{\theta_\beta}$.

For the converse, let $p \in \mathbb{P}_{\varphi_{\theta_\beta}}$. Hence $r := p \restriction supp(p) \in \mathbb{Q}_{\theta_\beta}$ by the definition of $\mathbb{P}_{\varphi_{\theta_\beta}}$.
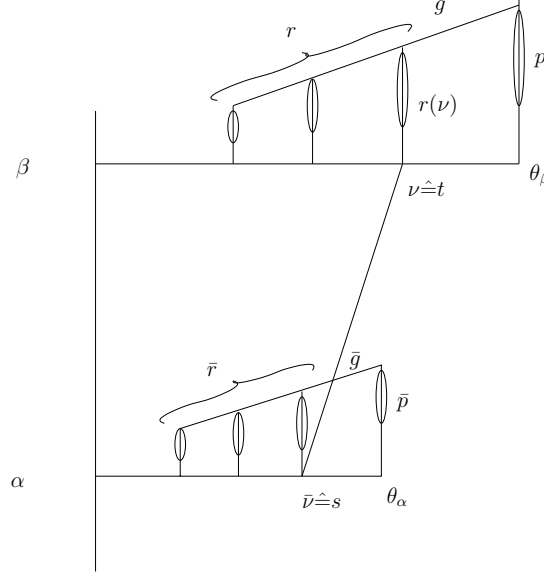
Set $\nu := max(dom(r))$ and $t := \langle \beta, \nu \rangle$. Moreover, let $g \in \mathfrak{G}_{\nu\theta_\beta}$ be such that $g[r(\nu)] = p$. Let, by the definition of $\mathbb{Q}_{\nu+1}$, $s \prec' t$ be such that $r = \sigma'_{st}(\bar{r})$ for some $\bar{r}$ in $\mathbb{Q}_{\nu(s)+1}$. Hence $r = \bar{f}[\bar{r}]$ for some $f \in \mathfrak{F}_{\alpha\beta}$ such that $s := \langle \alpha, \bar{\nu} \rangle$ and $f(\bar{\nu}) = \nu$. In particular, also $\bar{f}(\bar{\nu}) = \nu$. That is, if we set $\nu = \xi + 1$, then $\xi \in rng(f)$. Hence $\bar{f}_{\bar{\nu}} = f_{\bar{\nu}}$ and $r(\nu) = f_{\bar{\nu}} \otimes f[r(\bar{\nu})]$. Moreover, by (5)(c) in the definition of a simplified gap-2 morass, we may assume that $g = f_{\bar{\nu}\theta_\alpha}(\bar{g})$ for some $\bar{g} \in \mathfrak{G}_{\bar{\nu}\theta_\alpha}$. But then $p = f_{\theta_\alpha} \otimes f[\bar{p}]$ where $\bar{p} = \bar{g}[\bar{r}(\bar{\nu})]$ by (6) in the definition of embedding. $\square$

Let G be $\mathbb{P}$-generic and $F = \bigcup\{p \mid p \in G\}$. By the usual density argument, the following lemma shows that $F : \omega_3 \times \omega_2 \to 2$.

Lemma 6.6

$$\langle \langle \mathbb{P}_\eta \mid \eta \leq \kappa^{++} \rangle, \langle \sigma_{st} \mid s \prec t \rangle, \langle \sigma'_{st} \mid s \prec' t \rangle, \langle e_\alpha \mid \alpha < \kappa^+ \rangle, \langle e'_\alpha \mid \alpha < \kappa \rangle \rangle$$

is a FS system along $\mathfrak{M}$.

Proof: (FS$_2$2), (FS$_2$3), (FS$_2$4), (FS$_2$5) and (FS$_2$7) are clear from the construction. (FS$_2$6) was proved in remark 1. So we are only left with (FS$_2$1). That is, we have to prove that

$$\langle \langle \mathbb{P}_\eta \mid \eta \leq \kappa^{++} \rangle, \langle \sigma_{st} \mid s \prec t \rangle, \langle e_\alpha \mid \alpha < \kappa^+ \rangle \rangle$$

is a FS system along $\langle\langle\varphi_\zeta \mid \zeta \leq \omega_2\rangle, \langle\mathfrak{G}_{\zeta\xi} \mid \zeta < \xi \leq \omega_2\rangle\rangle$. We know that

$$\langle\langle P(\eta) \mid \eta \leq \omega_3\rangle, \langle\sigma_{st} \mid s \prec t\rangle, \langle e_\alpha \mid \alpha < \omega_2\rangle\rangle$$

is a FS system along $\langle\langle\varphi_\zeta \mid \zeta \leq \omega_2\rangle, \langle\mathfrak{G}_{\zeta\xi} \mid \zeta < \xi \leq \omega_2\rangle\rangle$. From this it follows immediately that (FS4), (FS5) and (FS7) also hold for

$$\langle\langle\mathbb{P}_\eta \mid \eta \leq \kappa^{++}\rangle, \langle\sigma_{st} \mid s \prec t\rangle, \langle e_\alpha \mid \alpha < \kappa^+\rangle\rangle.$$

Moreover, (FS1) holds, because

$$(*) \qquad \mathbb{P}_\eta = \{p \in \mathbb{P} \mid p \in P(\eta)\}$$

and for $P(\eta)$ we know (FS1) already. By $(*)$, one has to prove for (FS2), (FS3) and (FS6) that certain conditions are elements of $\mathbb{P}$. In the case of (FS2), for example, one has to show that $\sigma_{st}(p) \in \mathbb{P}_{\nu(t)+1}$ for all $p \in \mathbb{P}_{\nu(s)+1}$. In all three cases that's not difficult. $\square$

Lemma 6.7

Let $p \in \mathbb{P}$ and $\langle\gamma, \delta\rangle \in \omega_3 \times \omega_2$ such that $\langle\gamma, \delta\rangle \notin dom(p)$. Then there exists a $q \leq p$ such that $\langle\gamma, \delta\rangle \in dom(q)$.

Proof: We prove by induction over the levels of the gap-2 morass, which we enumerate by $\beta \leq \omega_1$, the following
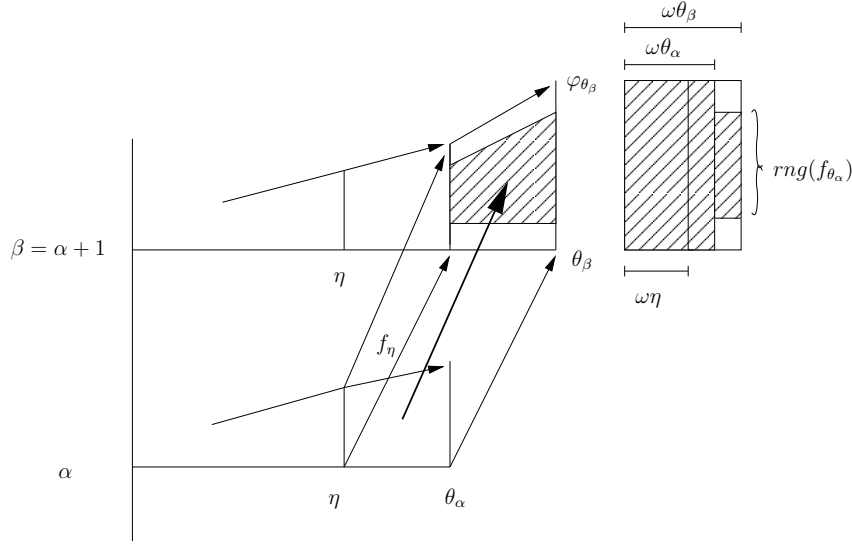
Claim: Let $p \in \mathbb{P}_{\varphi_{\theta_\beta}}$ and $\langle\gamma, \delta\rangle \in \varphi_{\theta_\beta} \times \omega\theta_\beta$ such that $\langle\gamma, \delta\rangle \notin dom(p)$. Then there exists a $q \leq p$ such that $\langle\gamma, \delta\rangle \in dom(q)$.

Base case: $\beta = 0$

Trivial.

Successor case: $\beta = \alpha + 1$

Let f be the right-branching embedding of $\mathfrak{F}_{\alpha\beta}$ and $\eta := crit(f \upharpoonright \theta_\alpha)$.

We consider three cases.

Case 1: $\gamma \in rng(f_{\theta_\alpha})$, $\delta \in [0, \omega\eta[\cup [\omega\theta_\alpha, \omega\theta_\beta[$

Let $\bar{p} = (f_{\theta_\alpha} \otimes f)^{-1}[p] \cup p^*(\theta_\alpha)$, $f_{\theta_\alpha}(\bar{\gamma}) = \gamma$ and $\tilde{f}(\bar{\delta}) = \delta$ where $\tilde{f}(\omega\tau + n) = \omega f(\tau) + n$ for all $n \in \omega$. Then by the induction hypothesis, there exists in $\mathbb{P}_{\varphi_{\theta_\alpha}}$ a $\bar{q} \leq \bar{p}$ such that $\langle \bar{\gamma}, \bar{\delta} \rangle \in dom(\bar{q})$. Set

$$q := p \cup (f_{\theta_\alpha} \otimes f)[\bar{q}].$$

Then $q \leq p$, $\langle \gamma, \delta \rangle \in dom(q)$ and $q \in \mathbb{P}_{\varphi_{\theta_\beta}}$ by remark 3.

Case 2: $\delta \in [0, \omega\theta_\alpha[$

Let $\bar{\gamma} \prec \gamma$, $\gamma \in T_{\theta_\alpha}$. Then there exists a $g \in \mathfrak{F}_{\theta_\alpha\theta_\beta}$ such that $g(\bar{\gamma}) = \gamma$. Pick such a g. By the induction hypothesis, there exists a $\bar{q} \leq p^*(\theta_\alpha) \cup (f_{\theta_\alpha} \otimes f)^{-1}[p]$ such that $\langle \bar{\gamma}, \delta \rangle \in dom(\bar{q})$. Set

$$q := p \cup g[\bar{q}].$$

Then $q \leq p$, $\langle \gamma, \delta \rangle \in dom(q)$ and $q \in \mathbb{P}_{\varphi_{\theta_\beta}}$ by remark 3.

`Case 3:` $\delta \in [\omega\theta_\alpha, \omega\theta_\beta[,\ \gamma \notin rng(f_{\theta_\alpha})$

Then set $q = p \cup \{\langle\langle\gamma,\delta\rangle,1\rangle\}$. According to the case which we are in, $q^*(\theta_\alpha) = p^*(\theta_\alpha)$ and $(f_{\theta_\alpha} \otimes f)^{-1}[q] = (f_{\theta_\alpha} \otimes f)^{-1}[p]$. Hence $q^*(\theta_\alpha)$ and $(f_{\theta_\alpha} \otimes f)^{-1}[q]$ are compatible, because $p^*(\theta_\alpha)$ and $(f_{\theta_\alpha} \otimes f)^{-1}[p]$ are compatible. So $q \in \mathbb{P}_{\varphi_{\theta_\beta}}$. Obviously, $q \leq p$ and $\langle\gamma,\delta\rangle \in dom(q)$.

Limit case: $\beta \in Lim$

By the previous lemma, $\mathbb{P}_{\varphi_{\theta_\beta}} = \bigcup\{(f_{\theta_\alpha} \otimes f)[\mathbb{P}_{\varphi_{\theta_\alpha}}] \mid \alpha < \beta, f \in \mathfrak{F}_{\alpha\beta}\}$. By (5) in the definition of a simplified gap-2 morass, $\varphi_{\theta_\beta} = \bigcup\{f_{\theta_\alpha}[\varphi_{\theta_\alpha}] \mid \alpha < \beta, f \in \mathfrak{F}_{\alpha\beta}\}$ and $\theta_\beta = \bigcup\{f[\theta_\alpha] \mid \alpha < \beta, f \in \mathfrak{F}_{\alpha\beta}\}$. Hence by (4) in the definition of a simplified gap-2 morass, we can pick $\alpha < \beta$, $f \in \mathfrak{F}_{\alpha\beta}$, $\bar{p} \in \mathbb{P}_{\varphi_{\theta_\alpha}}$, $\bar{\gamma} \in \varphi_{\theta_\alpha}$ and $\bar{\delta} \in \omega\theta_\alpha$ such that $f_{\theta_\alpha} \otimes f[\bar{p}] = p$, $f_{\theta_\alpha}(\bar{\gamma}) = \gamma$ and $\tilde{f}(\bar{\delta}) = \delta$ where $\tilde{f}(\omega\tau + n) = \omega f(\tau) + n$ for all $n \in \omega$. By the induction hypothesis, there exists $\bar{q} \leq \bar{p}$ such that $\langle\bar{\gamma},\bar{\delta}\rangle \in dom(\bar{q})$. Set $q := f_{\theta_\alpha} \otimes f[\bar{q}]$. Then q is as wanted. $\square$

At the beginning of the section, we proved that the Cohen forcing P adds a Hausdorff space. Using the next lemma, the same proof works for $\mathbb{P}$.

Lemma 6.8

Let $p \in \mathbb{P}$ and $\gamma \neq \delta \in \omega_3$. Then there is $q \leq p$ in $\mathbb{P}$ and $\mu \in \omega_3$ such that $q(\gamma,\mu) \neq q(\delta,\mu)$.

Proof: We prove by induction over the levels of the gap-2 morass, which we enumerate by $\beta \leq \omega_1$, the following

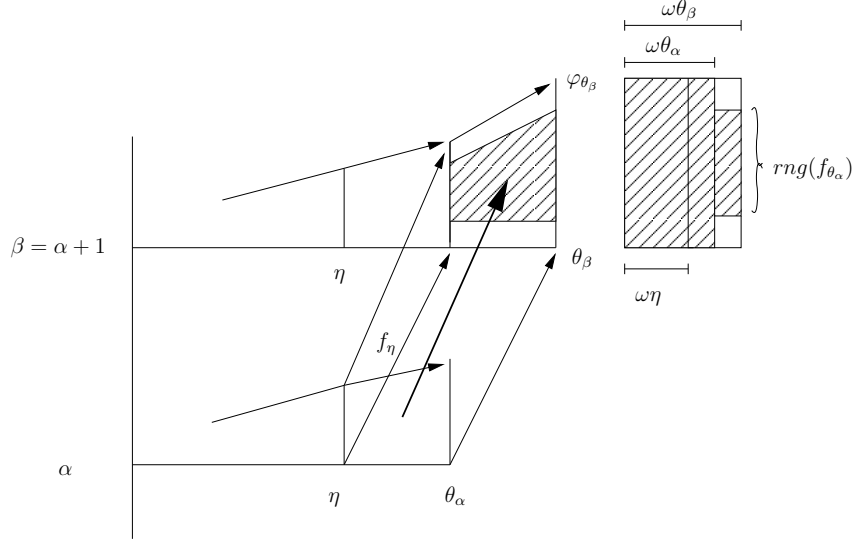*Claim:* Let $p \in \mathbb{P}_{\varphi_{\theta_\beta}}$ and $\gamma \neq \delta \in \varphi_{\theta_\beta}$. Then there is $q \leq p$ in $\mathbb{P}_{\varphi_{\theta_\beta}}$ and $\mu \in \omega\theta_\beta$ such that $q(\gamma,\mu) \neq q(\delta,\mu)$.

Base Case: $\beta = 0$

Trivial.

Successor Case: $\beta = \alpha + 1$

Let f be the right-branching embedding of $\mathfrak{F}_{\alpha\beta}$.

We consider four cases.

Case 1: $\gamma, \delta \in rng(f_{\theta_\alpha})$

Let $p \in \mathbb{P}_{\varphi_{\theta_\beta}}$ be given, $f_{\theta_\alpha}(\bar{\gamma}) = \gamma$ and $f_{\theta_\alpha}(\bar{\delta}) = \delta$. Set $\bar{p} = (f_{\theta_\alpha} \otimes f)^{-1}[p] \cup p^*(\theta_\alpha)$. By the induction hypothesis, there exists a $\bar{q} \in \mathbb{P}_{\varphi_{\theta_\alpha}}$ and a $\bar{\mu} = \omega\bar{\tau} + n \in \omega\theta_\alpha$ $(n \in \omega)$ such that $\bar{q} \leq \bar{p}$ and $\bar{q}(\bar{\gamma}, \bar{\mu}) \neq \bar{q}(\bar{\delta}, \bar{\mu})$. Set

$$q = p \cup (f_{\theta_\alpha} \otimes f)[\bar{q}]$$

and $\mu = \omega f(\bar{\tau}) + n$. Then $q \in \mathbb{P}_{\varphi_{\theta_\beta}}$ by remark 3, $q \leq p$ and $q(\gamma, \mu) = \bar{q}(\bar{\gamma}, \bar{\mu}) \neq \bar{q}(\bar{\delta}, \bar{\mu}) = q(\delta, \mu)$.

Case 2: $\gamma, \delta \notin rng(f_{\theta_\alpha})$

We consider two subcases. Assume first that $\theta_\beta \notin Lim$. Then choose some $\mu \in [\omega(\theta_\beta - 1), \omega\theta_\beta[$ such that $\mu \notin \{\tau_2 \mid \exists\tau_1 \ \langle\tau_1, \tau_2\rangle \in dom(p)\}$. Set

$$q = p \cup \{\langle\langle\gamma, \mu\rangle, 0\rangle, \langle\langle\delta, \mu\rangle, 1\rangle\}.$$

By the choice of μ, $q \in P(\varphi_{\theta_\beta})$. According to the case which we are in, $q^*(\theta_\alpha) = p^*(\theta_\alpha)$ and $(f_{\theta_\alpha} \otimes f)^{-1}[q] = (f_{\theta_\alpha} \otimes f)^{-1}[p]$. Hence q^* and $(f_{\theta_\alpha} \otimes f)^{-1}[q]$ are compatible because q^* and $(f_{\theta_\alpha} \otimes f)^{-1}[q]$ are compatible. So $q \in \mathbb{P}_{\varphi_{\theta_\beta}}$ and it is obviously as wanted.

Now, suppose that $\theta_\beta \in Lim$. Assume w.l.o.g. that $\gamma < \delta$. Set $t = \langle\theta_\beta, \delta\rangle$.

Let $s \prec t$ be minimal such that $\gamma \in rng(\pi_{st})$. Let $s \in T_\eta$. Pick $\mu \in [\omega \cdot max(\theta_\alpha, \eta), \omega\theta_\beta[$ such that $\mu \notin \{\tau_2 \mid \exists \tau_1 \langle \tau_1, \tau_2 \rangle \in dom(p)\}$. Set

$$q = p \cup \{\langle\langle\gamma, \mu\rangle, 0\rangle, \langle\langle\delta, \mu\rangle, 1\rangle\}.$$

Like in the first subcase, $q \in P(\varphi_{\theta_\beta})$ by the choice of μ. Also like in the first subcase, we can see that $q \in \mathbb{P}_{\varphi_{\theta_\beta}}$. Hence q is as wanted.

`Case 3:` $\gamma \in rng(f_{\theta_\alpha})$, $\delta \notin rng(f_{\theta_\alpha})$

Again, we consider two subcases. Assume first that $\theta_\beta \notin Lim$. Then choose $\mu \in [\omega(\theta_\beta - 1), \omega\theta_\beta[$ such that $\mu \notin \{\tau_2 \mid \exists \tau_1 \langle \tau_1, \tau_2 \rangle \in dom(p)\}$. Let $f_{\theta_\alpha}(\bar\gamma) = \gamma$ and $\tilde{f}(\bar\mu) = \mu$ where $\tilde{f}(\omega\tau + n)0\omega f(\tau) + n$.
Let

$$\bar{p} = p^*(\theta_\alpha) \cup (f_{\theta_\alpha} \otimes f)^{-1}[p].$$

Then there exists by the previous lemma in $\mathbb{P}_{\varphi_{\theta_\alpha}}$ a $\bar{q} \le \bar{p}$ such that $\langle \bar\gamma, \bar\mu \rangle \in dom(\bar{q})$. Set

$$r = p \cup (f_{\theta_\alpha} \otimes f)[\bar{q}]$$

and

$$q = r \cup \{\langle\langle\delta, \mu\rangle, \epsilon\rangle\}$$

where $f_{\theta_\alpha} \otimes f[q](\gamma, \eta) \ne \epsilon \in 2$.
By the choice of μ, $q \in P(\varphi_{\theta_\beta})$. By remark 3, $r \in \mathbb{P}_{\varphi_{\theta_\beta}}$. Hence $r^*(\theta_\alpha)$ and $(f_{\theta_\alpha} \otimes f)^{-1}[r]$ are compatible. According to the case which we are in, $q^*(\theta_\alpha) = r^*(\theta_\alpha)$ and $(f_{\theta_\alpha} \otimes f)^{-1}[q] = (f_{\theta_\alpha} \otimes f)^{-1}[r]$. So also $q \in \mathbb{P}_{\varphi_{\theta_\beta}}$. It is also as wanted.

Now, suppose that $\theta_\beta \in Lim$. Assume w.l.o.g. that $\gamma < \delta$. Set $t = \langle \theta_\beta, \delta \rangle$. Let $s \prec t$ be minimal such that $\gamma \in rng(\pi_{st})$. Let $s \in T_\eta$. Pick $\mu \in [\omega \cdot max(\theta_\alpha, \eta), \omega\theta_\beta[$ such that $\mu \notin \{\tau_2 \mid \exists \tau_1 \langle \tau_1, \tau_2 \rangle \in dom(p)\}$. Let $f_{\theta_\alpha}(\bar\gamma) = \gamma$ and $\tilde{f}(\bar\mu) = \mu$ where $\tilde{f}(\omega\tau + n) = \omega f(\tau) + n$ for all $n \in \omega$. Let

$$\bar{p} = p^*(\theta_\alpha) \cup (f_{\theta_\alpha} \otimes f)^{-1}[p].$$

From now on, proceed exactly like in the first subcase.

`Case 4:` $\gamma \notin rng(f_{\theta_\alpha})$, $\delta \in rng(f_{\theta_\alpha})$
Like case 4.

Limit Case: $\beta \in Lim$
By a previous lemma, $\mathbb{P}_{\varphi_{\theta_\beta}} = \bigcup\{(f_{\theta_\alpha} \otimes f)[\mathbb{P}_{\varphi_{\theta_\alpha}}] \mid \alpha < \beta, f \in \mathfrak{F}_{\alpha\beta}\}$. By (5) in the definition of a simplified gap-2 morass, $\varphi_{\theta_\beta} = \bigcup\{f_{\theta_\alpha}[\varphi_{\theta_\alpha}] \mid \alpha < \beta, f \in$

$\mathfrak{F}_{\alpha\beta}\}$ and $\theta_\beta = \bigcup\{f[\theta_\alpha] \mid \alpha < \beta, f \in \mathfrak{F}_{\alpha\beta}\}$. Hence by (4) in the definition of a simplified gap-2 morass, we can pick $\alpha < \beta$, $f \in \mathfrak{F}_{\alpha\beta}$, $\bar{p} \in \mathbb{P}_{\varphi_{\theta_\alpha}}$, $\bar{\gamma} \in \varphi_{\theta_\alpha}$ and $\bar{\delta} \in \omega\theta_\alpha$ such that $f_{\theta_\alpha} \otimes f[\bar{p}] = p$, $f_{\theta_\alpha}(\bar{\gamma}) = \gamma$ and $\tilde{f}(\bar{\delta}) = \delta$ where $\tilde{f}(\omega\tau + n) = \omega f(\tau) + n$ for all $n \in \omega$. By the induction hypothesis, there exists $\bar{q} \leq \bar{p}$ such that $\bar{q}(\bar{\gamma}, \bar{\mu}) \neq \bar{q}(\bar{\delta}, \bar{\mu})$. Set $q := f_{\theta_\alpha} \otimes f[\bar{q}]$. Then q is as wanted. $\square$

Finally, we prove the appropriate lemma for spread $< \omega_2$.

Lemma 6.9

Let $\langle p_i \mid i \in \omega_2 \rangle$ be a sequence of conditions $p_i \in \mathbb{P}$ such that $p_i \neq p_j$ if $i \neq j$. Let $\langle \delta_i \mid i \in \omega_2 \rangle$ be a sequence of ordinals $\delta_i \in \omega_3$ such that $\delta_i \in rng(x_{p_i})$ for all $i \in \omega_2$. Then there exist $i \neq j$ and $p \in \mathbb{P}$ such that $p \leq p_i, p_j$, $\langle \delta_i, \mu \rangle \in x_p$ and $p(\delta_i, \mu) = p(\delta_j, \mu)$ for all $\mu \in rng(x_{p_j})$.

Proof: We can assume by the Δ-system lemma that all x_{p_i} are isomorphic relative to the order of the ordinals, that $p_i \cong p_j$ for all $i, j \in \omega_2$, that $\pi(\delta_i) = \delta_j$ if $\pi : dom(x_{p_i}) \cong dom(x_{p_j})$, that $\{rng(x_{p_i}) \mid i \in \omega_2\}$ forms a Δ-system with root Δ, and that $\pi \restriction \Delta = id \restriction \Delta$ if $\pi : rng(x_{p_i}) \cong rng(x_{p_j})$. To prove the lemma, we consider two cases.

`Case 1:` $rng(x_{p_i}) = \Delta$ for all $i \in \omega_2$

Then we set $\eta = max(\Delta)$. Since there are ω_2-many p_i while $\mathbb{P}_{\varphi_{\eta+1}}$ has only ω_1-many elements, there exist p_i and p_j with $i \neq j$ such that $p_i^*(\eta+1) = p_j^*(\eta+1)$. Hence by the usual arguments p_i and p_j are compatible. Set $p = p_i \cup p_j$. Then p is as wanted, because $p_i \cong p_j$ and $\pi(\delta_i) = \delta_j$ if $\pi : dom(x_{p_i}) \cong dom(x_{p_j})$.
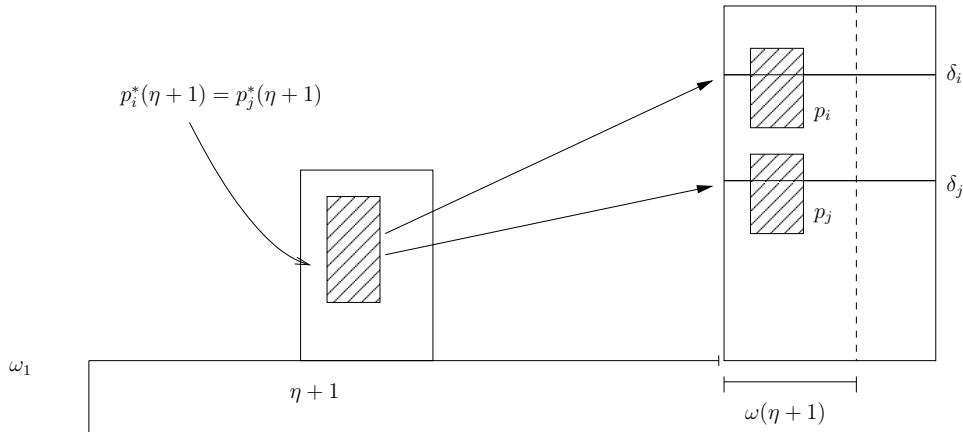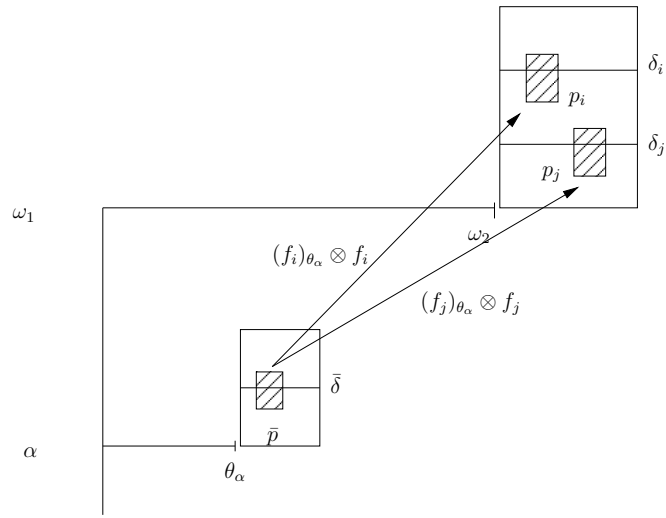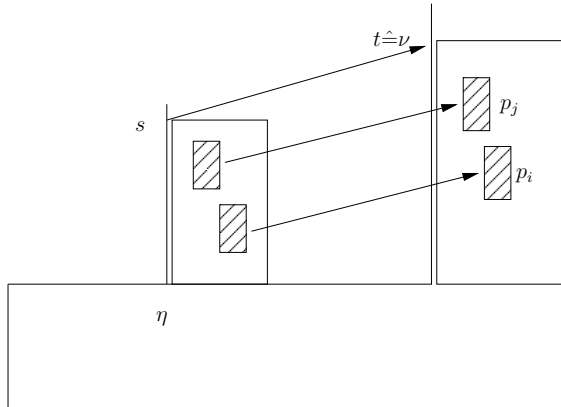
Case 2: $rng(x_{p_i}) \neq \Delta$ for all $i \in \omega_2$

Then $\{min(rng(x_{p_i}) - \Delta) \mid i \in \omega_2\}$ is unbounded in ω_2. For every $i \in \omega_2$ choose $\alpha_i < \omega_1$, $f_i \in \mathfrak{F}_{\alpha_i \omega_1}$, $\bar{\delta}_i \in \varphi_{\theta_{\alpha_i}}$ and $\bar{p}_i \in \mathbb{P}_{\varphi_{\theta_{\alpha_i}}}$ such that

$$p_i = (f_i)_{\theta_{\alpha_i}} \otimes f_i[\bar{p}_i] \quad \text{and} \quad \delta_i = (f_i)_{\theta_{\alpha_i}}(\bar{\delta}_i).$$
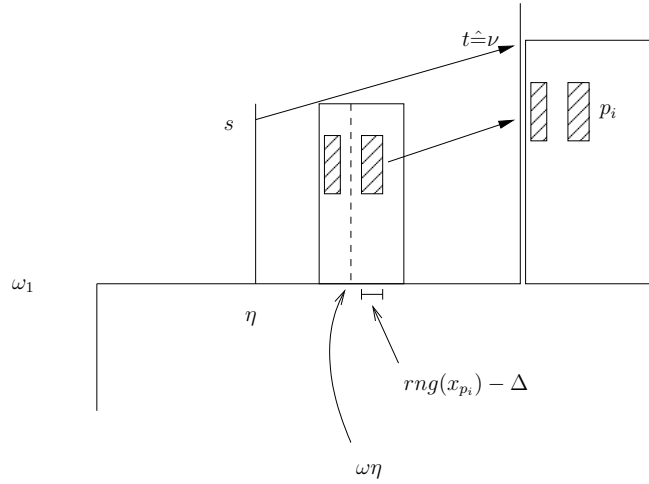
Since there are ω_2-many δ_i and p_i but only ω_1-many possible $\bar{\delta}_i$ and $\bar{p}_i$, we can assume that $\alpha_i = \alpha_j$, $\bar{\delta}_i = \bar{\delta}_j$ and $\bar{p}_i = \bar{p}_j$ for all $i, j \in \omega_2$. Set $\bar{p} = \bar{p}_i$, $\alpha = \alpha_i$ and $\bar{\delta} = \bar{\delta}_i$.

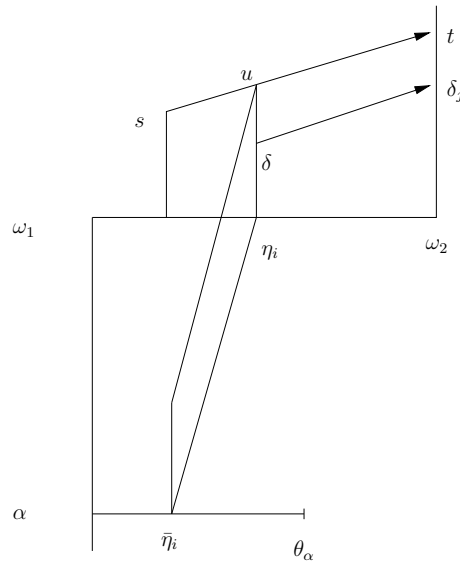Let $\nu \in \omega_3$ be such that $p_i \in \mathbb{P}_\nu$ for all $i \in \omega_3$. Let $t = \langle \omega_2, \nu \rangle$. Let $s \prec t$ such that $p_i \in rng(\sigma_{st})$ for ω_1-many $i \in \omega_2$.

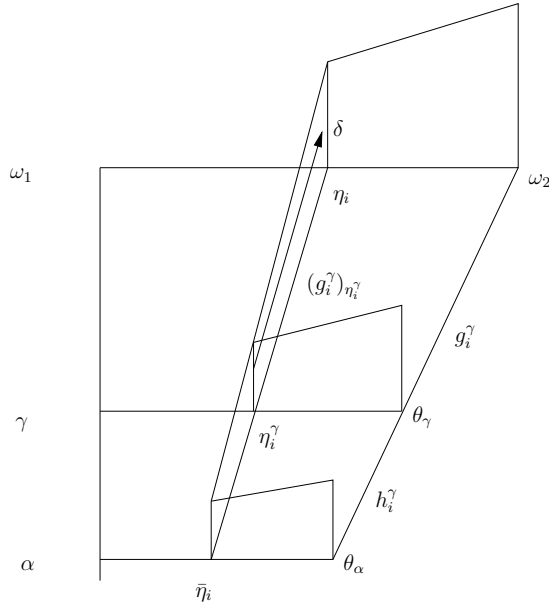Let $s \in T_\eta$. Pick p_i such that $min(rng(x_{p_i}) - \Delta) > \omega\eta$.

Let $\eta_i = min(rng(x_{p_i}) - \Delta)$. Then by the choice of f_i, $\eta_i \in rng(f_i \restriction \theta_\alpha)$. Let $u \prec t$ be such that $u \in T_{\eta_i}$. Let $f_i(\bar\eta_i) = \eta_i$.

Since there are ω_1-many $j \in \omega_2$ such that $p_j \in rng(\sigma_{st})$, there are also ω_1-many $j \in \omega_2$ such that $p_j \in rng(\sigma_{ut})$. On the other hand, $rng((f_i)_{\bar\eta_i})$ is countable. So we can pick a $j \in \omega_2$ such that $\delta \notin rng((f_i)_{\bar\eta_i})$, $\pi_{ut}(\delta) = \delta_j$ and $p_j \in rng(\sigma_{ut})$. In the following we will show that there exists $p \leq p_i, p_j$
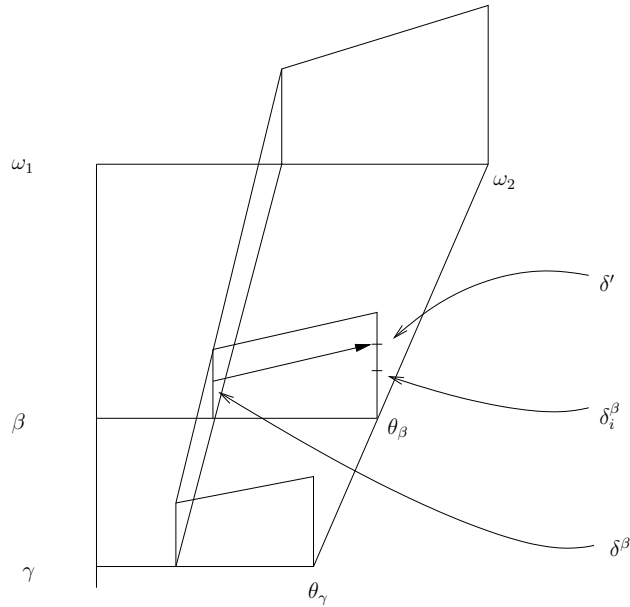
such that $\langle \delta_j, \mu \rangle \in x_p$ and $p(\delta_i, \mu) = p(\delta_j, \mu)$ for all $\mu \in rng(x_{p_i})$.

For $\alpha < \beta \leq \omega_1$, let $f_i = g_i^\beta \circ h_i^\beta$ where $g_i^\beta \in \mathfrak{F}_{\alpha\beta}$ and $h_i^\beta \in \mathfrak{F}_{\beta\omega_1}$. Let $g_i^\beta(\eta_i^\beta) = \eta_i$ and γ be minimal such that $\delta \in rng((g_i^\gamma)_{\eta_i^\gamma})$.
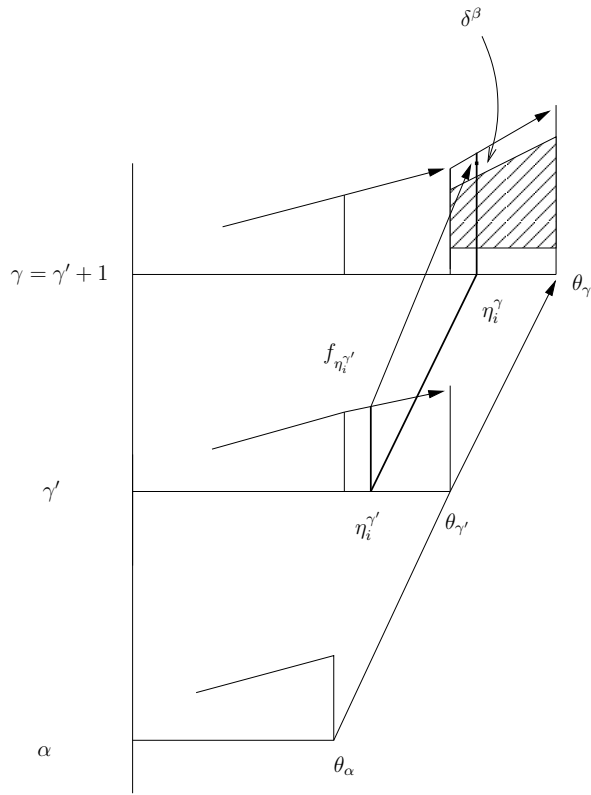

For $\gamma \leq \beta \leq \omega_1$, let $(g_i^\beta)_{\eta_i^\beta}(\delta^\beta) = \delta$, $p_i^\beta = (h_i^\beta)_{\theta_\alpha} \otimes h_i^\beta[\bar{p}]$, $g_i^\beta[\Delta_\beta] = \Delta$ and $\delta_i^\beta = (h_i^\beta)_{\theta_\alpha}(\bar{\delta})$. We prove by induction over $\gamma \leq \beta \leq \omega_1$ the following

Claim 1: If $\langle \eta_i^\beta, \delta^\beta \rangle \prec \langle \theta_\beta, \delta' \rangle$, then there exists $p^\beta \leq p_i^\beta$ such that $\langle \delta', \mu \rangle \in x_{p^\beta}$ and $p^\beta(\delta_i^\beta, \mu) = p^\beta(\delta', \mu)$ for all $\mu \in rng(x_{p_i^\beta}) - \Delta_\beta$.

Base case: $\beta = \gamma$

By the definition of γ and (5) in the definition of a simplified gap-2 morass, γ is a successor ordinal. Let $\gamma = \gamma' + 1$. Moreover, $\theta_{\gamma'} \leq \eta_i^\gamma$. Hence $p_i^\gamma = f_{\theta_{\gamma'}} \otimes f[p_i^{\gamma'}]$ where f is the right-branching embedding of $\mathfrak{F}_{\gamma'\gamma}$.

We first notice, that $\delta' \notin rng(f_{\theta_{\gamma'}})$. Assume that this was not the case. Then pick a $\pi \in \mathfrak{G}_{\eta_i^\gamma \theta_\gamma}$ such that $\pi(\delta^\beta) = \delta'$. By (6) in the definition of right-branching, there is a $\bar\pi \in \mathfrak{G}_{\eta_i^{\gamma'} \theta_{\gamma'}}$ such that $f_{\eta_i^{\gamma'} \theta_{\gamma'}}(\bar\pi) = \pi$. Let $f_{\theta_{\gamma'}}(\bar\delta') = \delta'$. Let $\langle \eta_i^{\gamma'}, \rho \rangle \prec \langle \theta_{\gamma'}, \bar\delta' \rangle$.
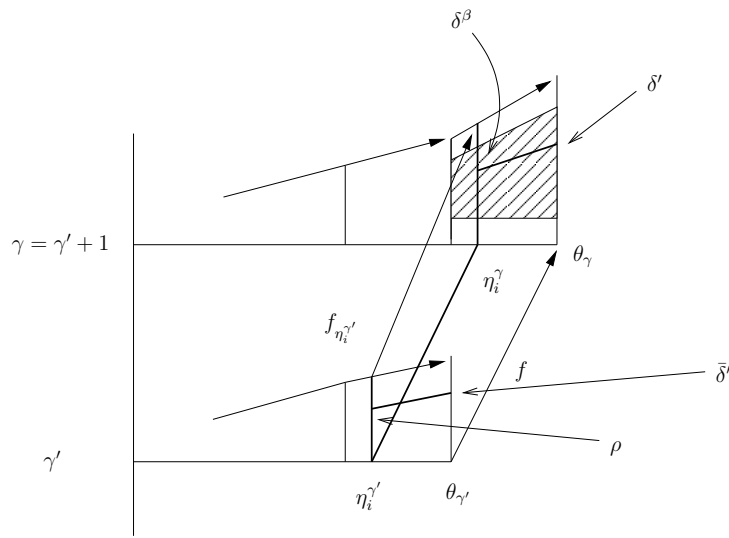
By (6) in the definition of embedding,
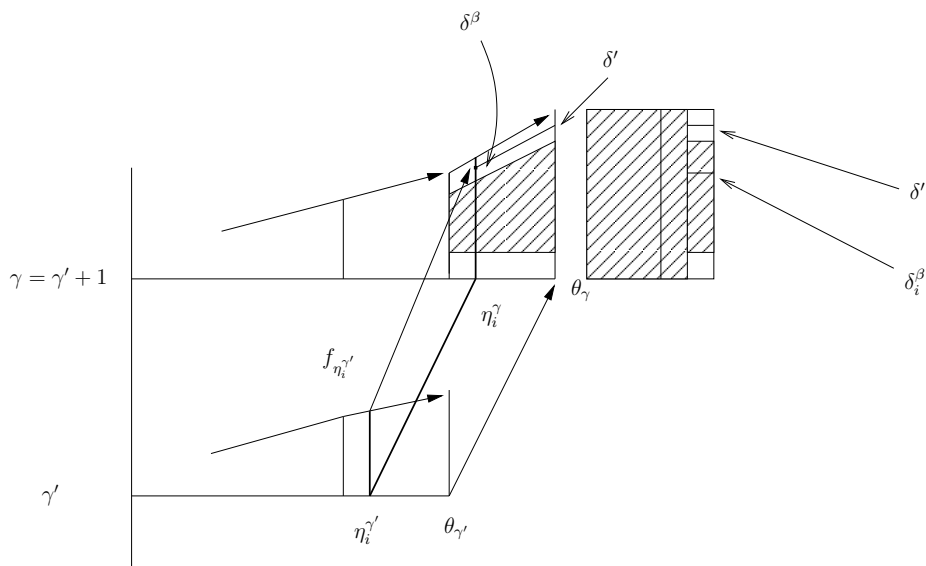
$$f_{\eta_i^{\gamma'}} \circ \bar{\pi} = \pi \circ f_{\theta\gamma'}.$$

Hence $f_{\eta_i^{\gamma'}}(\rho) = \delta^\beta$, which contradicts the definition of γ.

Summarizing what we know so far the picture looks as follows:

Hence we can define a condition $p^\beta \leq p_i^\beta$, $p^\beta \in \mathbb{P}$ by setting

$$p^\beta = p_i^\beta \cup \{\langle\langle\delta', \mu\rangle, p_i^\beta(\delta_i^\beta, \mu)\rangle \mid \mu \in rng(x_{p_i^\beta}) - \theta_{\gamma'}\}.$$

This p^β is as wanted.

Successor step: $\beta = \rho + 1$

We consider two cases:

`Case 1`: $p_i^\beta = g[p_i^\rho]$ for some $g \in \mathfrak{G}_{\theta_\rho \theta_\beta}$

In this case $\eta_i^\rho = \eta_i^\beta < \theta_\rho$. Let $\langle\eta_i^\beta, \delta^\beta\rangle \prec \langle\theta_\rho, \delta''\rangle \prec \langle\theta_\beta, \delta'\rangle$. Let $\pi \in \mathfrak{G}_{\theta_\rho \theta_\beta}$ such that $\pi(\delta'') = \delta'$. Then by the induction hypothesis, there exists $p' \leq p_i^\rho$ such that

$$p'(\delta'', \mu) = p'(\delta_i^\rho, \mu)$$
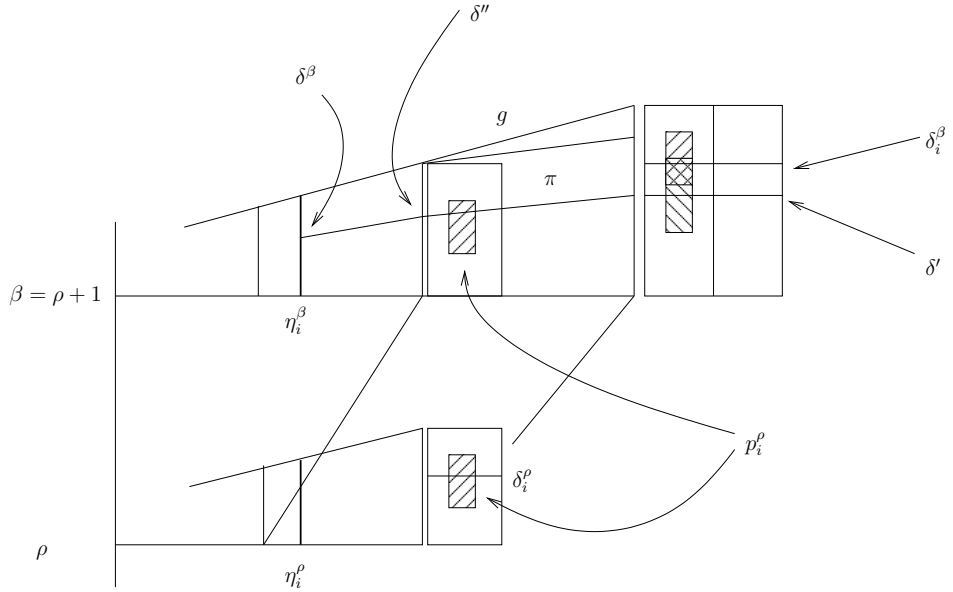
for all $\mu \in rng(x_{p_i^\rho}) - \Delta_\rho$. Set

$$p^\beta = \pi[p'] \cup g[p'].$$

Then by remark 3, $p^\beta \in \mathbb{P}$ and

$$p^\beta(\delta', \mu) = p'(\delta'', \mu) = p'(\delta_i^\rho, \mu) = p_i^\beta(\delta_i^\beta, \mu)$$

for all $\mu \in rng(x_{p_i^\rho}) - \Delta_\rho = rng(x_{p_i^\beta}) - \Delta_\beta$. Hence p^β is as wanted.
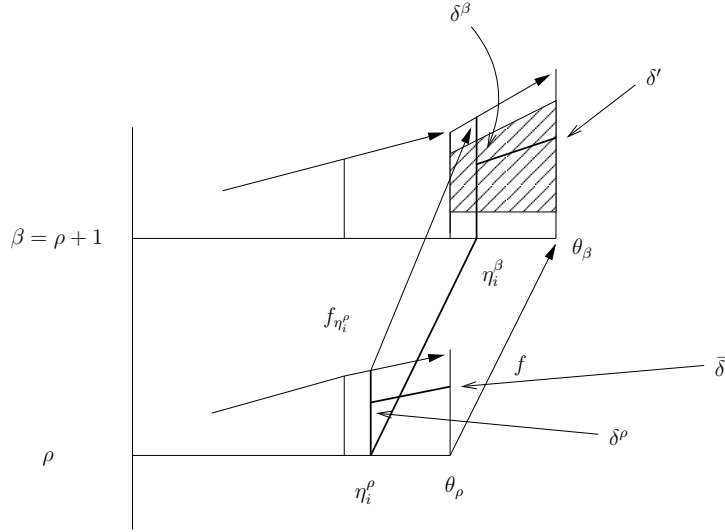
Case 2: $p_i^\beta = f_{\theta_\rho}[p_i^\rho]$ where f is the right-branching embedding of $\mathfrak{F}_{\rho\beta}$

We consider three subcases.

Subcase 1: $\delta' \in rng(f_{\theta_\rho})$

Let $f_{\theta_\rho}(\bar{\delta}') = \delta'$. Then by (6) in the definition of embedding, $\delta^\rho \prec \bar{\delta}'$.

Hence by the induction hypothesis, there exists $p^\rho \leq p_i^\rho$ such that $\langle \bar{\delta}', \mu \rangle \in x_{p^\rho}$ and $p^\rho(\delta_i^\rho, \mu) = p^\rho(\bar{\delta}', \mu)$ for all $\mu \in rng(x_{p_i^\rho}) - \Delta_\rho$. Set

$$p^\beta = f_{\theta_\rho} \otimes f[p^\rho].$$

Then p^β is as wanted.

Subcase 2: $\delta' \notin rng(f_{\theta_\rho})$ and $\theta_\rho \leq \eta_i^\beta$

Exactly like the base case of the induction.

Subcase 3: $\delta' \notin rng(f_{\theta_\rho})$ and $\eta_i^\beta < \theta_\rho$.

This case is a combination of the base case of the induction and of case 1. Let $\langle \eta_i^\beta, \delta^\beta \rangle \prec \langle \theta_\rho, \delta'' \rangle \prec \langle \theta_\beta, \delta' \rangle$. Let $\pi \in \mathfrak{G}_{\theta_\rho \theta_\beta}$ such that $\pi(\delta'') = \delta'$. Then by the induction hypothesis, there exists $p^\rho \leq p_i^\rho$ such that $\langle \bar{\delta}', \mu \rangle \in x_{p^\rho}$ and

$$p^\rho(\delta_i^\rho, \mu) = p^\rho(\delta'', \mu)$$

for all $\mu \in rng(x_{p_i^\rho}) - \Delta_\rho$.
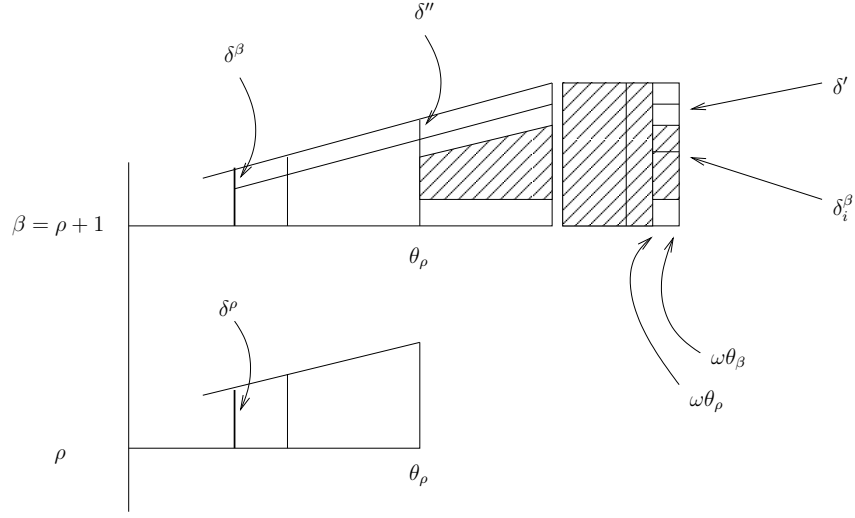
Set

$$p^\beta = \pi[p^\rho] \cup (f_{\theta_\rho} \otimes f)[p^\rho] \cup \{\langle\langle\delta', \mu\rangle, p_i^\beta(\delta_i^\beta, \mu)\rangle \mid \mu \in rng(x_{p_i^\beta}) - \theta_\rho\}.$$

By remark 3, $p^\beta \in \mathbb{P}$. We claim that p^β is as wanted. For $\mu \in rng(x_{p_i^\beta}) - \theta_\rho$,

$$p^\beta(\delta_i^\beta, \mu) = p^\beta(\delta', \mu)$$

holds by definition. For $\mu \in rng(x_{p_i^\beta}) \cap \theta_\rho = rng(x_{p_i^\rho}) \cap \theta_\rho$, we have

$$p^\beta(\delta', \mu) = p^\rho(\delta'', \mu) = p^\rho(\delta_i^\rho, \mu) = p_i^\beta(\delta_i^\beta, \mu).$$

This finishes the proof of the successor step.

Limit case: $\beta \in Lim$

By lemma 6.5 and by (4) and (5) in the definition of a simplified gap-2 morass, we can pick a $\rho < \beta$ and a $f \in \mathfrak{F}_{\rho\beta}$ such that $\delta' \in rng(f_{\theta_\rho})$ and $f_{\theta_\rho} \otimes f[p_i^\rho] = p_i^\beta$. Let $f_{\theta_\rho}(\bar{\delta}') = \delta'$. Then by (6) in the definition of embedding, $\langle\eta_i^\rho, \delta^\rho\rangle \prec \langle\theta_\rho, \bar{\delta}'\rangle$.
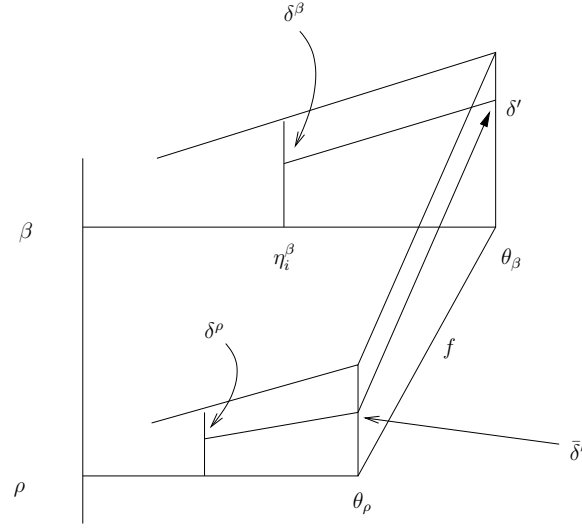
Hence we can pick by the induction hypothesis a $p^\rho \le p_i^\rho$ such that $\langle \bar\delta', \mu \rangle \in x_{p^\rho}$ and

$$p^\rho(\delta_i^\rho, \mu) = p^\rho(\bar\delta', \mu)$$

for all $\mu \in rng(x_{p_i^\rho}) - \Delta_\rho$. Set

$$p^\beta = f_{\theta_\rho} \otimes f[p^\rho].$$

Then p^β is obviously as wanted. This finishes the proof of claim 1.

Finally, we can prove by induction over $\alpha < \beta \le \omega_1$

Claim 2: For $\alpha \le \beta < \gamma$, set $p^\beta := p_i^\beta$. For $\gamma \le \beta < \omega_1$, let p^β be like in claim 1. Then there exists for all $\alpha \le \beta < \omega_1$ a $p \in \mathbb{P}$ such that $p \le p^\beta, p_j^\beta$.

Base case: $\beta = \alpha$

Trivial.

Successor case: $\beta = \rho + 1$

We consider four cases.

Case 1: $p_i^\beta = g[p_i^\rho]$ and $p_j^\beta = h[p_j^\rho]$ for some $g, h \in \mathfrak{G}_{\theta_\rho \theta_\beta}$

By the induction hypothesis, there exists a $\bar p \le p^\rho, p_j^\rho$. Set

$$p = g[\bar p] \cup h[\bar p] \cup p^\beta.$$

It is not difficult to see that $p \in \mathbb{P}$ in all the different cases which occur in the definition of p^β.

Case 2: $p_i^\beta = f_{\theta_\rho} \otimes f[p_i^\rho]$ and $p_j^\beta = g[p_j^\rho]$ where $g \in \mathfrak{G}_{\theta_\rho \theta_\beta}$ and f is the right-branching embedding of $\mathfrak{F}_{\rho\beta}$

By the induction hypothesis, there exists a $\bar{p} \le p^\rho, p_j^\rho$. Set

$$p = g[\bar{p}] \cup (f_{\theta_\rho} \otimes f)[\bar{p}] \cup p^\beta.$$

It is not difficult to see that $p \in \mathbb{P}$ in all the different cases which occur in the definition of p^β.

Case 3: $p_j^\beta = f_{\theta_\rho} \otimes f[p_j^\rho]$ and $p_i^\beta = g[p_i^\rho]$ where $g \in \mathfrak{G}_{\theta_\rho \theta_\beta}$ and f is the right-branching embedding of $\mathfrak{F}_{\rho\beta}$

Like case 2.

Case 4: $p_i^\beta = f_{\theta_\rho} \otimes f[p_i^\rho]$ and $p_j^\beta = f_{\theta_\rho} \otimes f[p_j^\rho]$ where f is the right-branching embedding of $\mathfrak{F}_{\rho\beta}$

By the induction hypothesis, there exists a $\bar{p} \le p^\rho, p_j^\rho$. Set

$$p = (f_{\theta_\rho} \otimes f)[\bar{p}] \cup p^\beta.$$

It is not difficult to see that $p \in \mathbb{P}$ in all the different cases which occur in the definition of p^β.

Limit case: $\beta \in Lim$

This is proved very similar to the limit step in claim 1.

This finishes claim 2 and proves the lemma, if we set $\beta = \omega_1$ and $\delta' = \delta_j$. $\square$

Lemma 6.10
(a) $i : \mathbb{P}_{\omega_3} \to \mathbb{Q}_{\omega_2}, p \mapsto p^* \restriction supp(p)$ is a dense embedding.
(b) There is a ccc-forcing $\bar{\mathbb{P}}$ of size ω_1 such that $\mathbb{Q}_{\omega_2}$ embedds densely into $\bar{\mathbb{P}}$.

Proof: (a) By results from section 3.
(b) Note, that $\langle\langle\mathbb{Q}_\eta \mid \eta \le \omega_3\rangle, \langle\sigma'_{st} \mid s \prec' t\rangle, \langle e'_\alpha \mid \alpha < \omega_1\rangle\rangle$ is a FS iteration along $\langle\langle\theta_\alpha \mid \alpha \le \omega_1\rangle, \langle\mathfrak{F}'_{\alpha\beta} \mid \alpha < \beta \le \omega_1\rangle\rangle$. Hence we can define $\bar{\mathbb{P}}$ from $\mathbb{Q}_{\omega_2}$ like we defined $\mathbb{Q}_{\omega_2}$ from $\mathbb{P}_{\omega_3}$. That $\mathbb{Q}_{\omega_2}$ embedds densely into $\bar{\mathbb{P}}$ is proved like before. $\square$

Theorem 6.11
If there is a simplified $(\omega_1, 2)$-morass, then there is a ccc-forcing $\bar{\mathbb{P}}$ of size ω_1 that adds a 0-dimensional Hausdorff topology on ω_3 which has spread ω_1.

Proof: By the previous lemmas, most of the claim is clear. We only prove that τ has spread ω_1. Assume not. Let $\dot{X}$, $\dot{h}$ and $\dot{B}$ be names and $p \in \mathbb{P}$ a condition such that

$p \Vdash (\dot{X} \subseteq \omega_3, \dot{h} : \omega_2 \to \dot{X}$ is bijective, $\dot{B} : \omega_2 \to V$, $\forall i \in \omega_2$ $\dot{B}(i)$ is a basic open set, $\forall i \neq j \in \omega_2$ $\dot{h}(i) \in \dot{B}(i) \wedge \dot{h}(j) \notin \dot{B}(i))$.

For every $i \in \omega_2$ let $p_i \leq p$ and δ_i, ε_i be such that $p_i \Vdash \dot{h}(\check{i}) = \check{\delta}_i \wedge \dot{B}(i) = B_{\check{\varepsilon}_i}$. By the previous lemma, there are $i \neq j$ and $r \in \mathbb{P}$ such that $r \leq p_i, p_j$, $\langle \delta_i, \mu \rangle \in x_r$ and $r(\delta_i, \mu) = r(\delta_j, \mu)$ for all $\mu \in rng(x_{p_j})$. Hence $r \Vdash \dot{h}(j) = \check{\delta}_j \in \dot{B}(i)$ which contradicts the definition of p. $\square$

Historical remarks and references

Theorem 6.1 was first proved for regular spaces by J. de Groot [4]. Then this was generalized to arbitrary Hausdorff spaces by A. Hajnal and I. Juhasz [14]. The proof given here is from Juhasz's book [25]. Fedorcuk's construction of a 0-dimensional Hausdorff space with spread ω and size 2^{2^ω} in L is given in [11].

The construction which we presented in this chapter was first described in Irrgang [17].

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