MORASSES AND FINITE SUPPORT ITERATIONS

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ABSTRACT. We introduce a method of constructing a forcing along a simplified $(\kappa, 1)$-morass such that the forcing satisfies the κ-chain condition. Alternatively, this may be seen as a method to thin out a larger forcing to get a chain condition. As an application, we construct a ccc forcing that adds an ω_2-Suslin tree. Related methods are Shelah's historic forcing and Todorcevic's ρ-functions.

1. INTRODUCTION

There are a number of consistency questions from two-cardinal combinatorics that were answered by Shelah's method of historic forcing or with the help of Todorcevic's ρ-functions: Can there exist a superatomic Boolean algebra with width ω and height ω_2 (Baumgartner and Shelah [1], Martinez [13])? Is it possible, that there is a function $f : \omega_2 \times \omega_2 \to \omega$, such that f is non-constant on any rectangle with infinite sides (Todorcevic [24, 22])? Can one prove in ZFC, that every initially ω_1-compact T_3-space with countable tightness is already compact (Rabus [17], Juhasz and Soukup [10])? Is there consistently a forcing that satisfies ccc and adds a Kurepa tree (Jensen [8], Velickovic [26])? There are many more examples, but we cannot give a comprehensive overview here. In all cases there is a natural forcing with finite conditions that would solve the problem if it preserved cardinals. Since the conditions are finite, the suitable property of the forcing to guarantee cardinal preservation is the countable chain condition (ccc). Therefore one thins out the natural forcing in such a way that the remaining forcing satisfies ccc.

In the following, we will present the simplest case of a morass approach to such questions, i.e. to construct a ccc-forcing of size ω_2. The basic idea is simple: We try to generalize iterated forcing with finite support. Classical iterated forcing with finite support as introduced by Solovay and Tennenbaum [18] works with continuous, commutative systems of complete embeddings of Boolean algebras or partial orders which are indexed along a well-order. The following holds: If every forcing of the system satisfies ccc, then also the direct limit does. So if e.g. all forcings of the system are countable, then its direct limit satisfies ccc. It will, however, have size $\leq \omega_1$ since it is a direct limit, while we want to construct a forcing of size ω_2. To overcome this limitation, we will not consider a linear system indexed along a well-order but a two-dimensional system indexed along a simplified $(\omega_1, 1)$-morass. Since we want to obtain complete embeddings, we have to thin out the natural forcings. The way to do this follows very naturally from our approach. As an example how the thinning out is done, we will construct a ccc forcing that adds an ω_2-Suslin tree. The basic forcing we thin out is Tennenbaum's forcing for adding a

Suslin tree with finite conditions.

Morasses were introduced by R. Jensen in the early 1970's to solve the cardinal transfer problem of model theory in L (see e.g. Devlin [2]). For the proof of the gap-2 transfer theorem a gap-1 morass is used. For higher-gap transfer theorems Jensen has developed so-called higher-gap morasses [9]. In his Ph.D. thesis, the author generalized these to gaps of arbitrary size (see [7, 6, 5]). The theory of morasses is far developed and well examined. In particular, it is known how to construct morasses in L [2, 4, 7, 5] and how to force them [19, 20]. Moreover, D. Velleman has defined so-called simplified morasses, along which morass constructions can be carried out more easily [27, 30, 29]. Their existence is equivalent to the existence of ordinary morasses [3, 15]. The fact that the theory of morasses is so far developed is an advantage of the morass approach compared to historic forcing or ρ-functions. It allows straightforward generalizations to higher cardinals while the conditions of the forcings can be kept finite.

While the general method presented here works for higher cardinals, we can in general not expect that the consistency statements can naively be extended by raising the cardinal parameters. For example, we force an ω_2-Suslin tree along a gap-1 morass. An innocent generalization of the argument that the resulting tree has neither a branch nor an antichain of size ω_2, would yield a tree on ω_3 that has neither a branch nor an antichain of size ω_2, which is of course impossible. The reason why this generalization does not work is that the gap-2 case yields a three-dimensional construction. Therefore, the finite conditions of our forcing have to fit together appropriately in three directions instead of two directions and that is impossible. So if and how a statement generalizes to higher gaps depends heavily on the concrete conditions.

The exact relationship between our approach and the methods of historic forcing and ρ-functions is an open question. The crucial step in our proof that chain conditions are preserved is the definition of the support of a condition. It resembles the definition of the "history" $t^*(\alpha)$ of an ordinal α given by Baumgartner and Shelah [1]. However, there are various ways to set things up and the definition of a FS system given below is just one of them. As far as ρ-functions are concerned, it is possible to directly read off a ρ-function from a simplified gap-1 morass. This is a result of C. Morgan's in [14]. It is, however, unclear how this relates to an approach as below which generalizes finite support iterations.

If $\mathbb{P}$ is the limit of a finite support iteration indexed along α, then we can understand a $\mathbb{P}$-generic extension as being obtained successively in α-many steps. Moreover, there are names for the forcings used in every step. This raises the question if a similar analysis is possible for a forcing which is constructed with our method. It would justify to call them FS iterations along morasses instead of FS systems along a morass, which was the name the author used until the referee pointed out the shortcoming concerning successive extensions.

We should also mention that besides historic forcing and ρ-functions there is another, quite different method to prove consistencies in two-cardinal combinatorics. This is the method of forcing with models as side conditions or with side conditions in morasses. Models as side conditions were introduced by S. Todorcevic [23, 25], which was further developed by P. Koszmider [11] to side conditions in morasses. Unlike the other methods, it produces proper forcings which are usually not ccc.

This is sometimes necessary. For example, Koszmider proved that if CH holds, then there is no ccc forcing that adds a sequence of ω_2 many functions $f : \omega_1 \to \omega_1$ which is ordered by strict domination mod finite. However, he is able to produce a proper forcing which adds such a sequence [11]. More on the method can be found in Morgan's paper [16]. In the context of our approach, this raises the question if it is possible to define something like a countable support iteration along a morass.

2. FS Iterations

Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders. A map $\sigma : \mathbb{P} \to \mathbb{Q}$ is called a complete embedding if

(1) $\forall p, p' \in \mathbb{P}$ $(p' \leq p \to \sigma(p') \leq \sigma(p))$

(2) $\forall p, p' \in \mathbb{P}$ (p and p' are incompatible $\leftrightarrow$ $\sigma(p)$ and $\sigma(p')$ are incompatible)

(3) $\forall q \in \mathbb{Q}$ $\exists p \in \mathbb{P}$ $\forall p' \in \mathbb{P}$ $(p' \leq p \to (\sigma(p') \text{ and } q \text{ are compatible in } \mathbb{Q}))$.

In (3), we call p a reduction of q to $\mathbb{P}$ with respect to σ.

If only (1) and (2) hold, we say that σ is an embedding. If $\mathbb{P} \subseteq \mathbb{Q}$ such that the identity is an embedding, then we write $\mathbb{P} \subseteq_\perp \mathbb{Q}$.

We say that $\mathbb{P} \subseteq \mathbb{Q}$ is completely contained in $\mathbb{Q}$ if $id \upharpoonright \mathbb{P} : \mathbb{P} \to \mathbb{Q}$ is a complete embedding.

Let $\alpha \in Lim$. A finite support (FS) iteration is a sequence $\langle \mathbb{P}_\xi \mid \xi \leq \alpha \rangle$ of partial orders together with a commutative system $\langle \sigma_{\xi\eta} \mid \xi < \eta \leq \alpha \rangle$ of complete embeddings $\sigma_{\xi\eta} : \mathbb{P}_\xi \to \mathbb{P}_\eta$ such that $\bigcup \{ \sigma_{\xi\eta}[\mathbb{P}_\xi] \mid \xi < \eta \} = \mathbb{P}_\eta$ for limit η.

This is the original definition by Solovay and Tennenbaum in [18], except that they use Boolean algebras instead of partial orders. Moreover, it is well known that if $\sigma : \mathbb{P}_1 \to \mathbb{P}_2$ is a complete embedding then there is a $\mathbb{P}_1$-name $\dot{\mathbb{Q}}$ such that $\mathbb{P}_2$ and $\mathbb{P}_1 * \dot{\mathbb{Q}}$ are forcing equivalent. This leads to the more common definition of FS iterations where conditions are sequences of names. For the exact relationship between the two approaches see Kunen's textbook [12], chapter VIII §5 and exercise K.

An important property of FS iterations is that they preserve the κ-cc:

Theorem 2.1. *Let $\langle \langle \mathbb{P}_\xi \mid \xi \leq \alpha \rangle, \langle \sigma_{\xi\eta} \mid \xi < \eta \leq \alpha \rangle \rangle$ be a FS iteration. Assume that all $\mathbb{P}_\xi$ with $\xi < \alpha$ satisfy the κ-cc. Then $\mathbb{P}_\alpha$ also satisfies the κ-cc.*

Proof. See the original article by Solovay and Tennenbaum [18] or any standard textbook. □

3. Morasses

A simplified $(\kappa, 1)$-morass is a structure $\mathfrak{M} = \langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathfrak{F}_{\alpha\beta} \mid \alpha < \beta \leq \kappa \rangle \rangle$ satisfying the following conditions:

(P0) (a) $\theta_0 = 1$, $\theta_\kappa = \kappa^+$, $\forall \alpha < \kappa$ $0 < \theta_\alpha < \kappa$.

(b) $\mathfrak{F}_{\alpha\beta}$ is a set of order-preserving functions $f : \theta_\alpha \to \theta_\beta$.

(P1) $|\mathfrak{F}_{\alpha\beta}| < \kappa$ for all $\alpha < \beta < \kappa$.

(P2) If $\alpha < \beta < \gamma$, then $\mathfrak{F}_{\alpha\gamma} = \{ f \circ g \mid f \in \mathfrak{F}_{\beta\gamma}, g \in \mathfrak{F}_{\alpha\beta} \}$.

(P3) If $\alpha < \kappa$, then $\mathfrak{F}_{\alpha,\alpha+1} = \{ id \upharpoonright \theta_\alpha, f_\alpha \}$ where f_α is such that $f_\alpha \upharpoonright \delta = id \upharpoonright \delta$ and $f_\alpha(\delta) \geq \theta_\alpha$ for some $\delta < \theta_\alpha$.

(P4) If $\alpha \leq \kappa$ is a limit ordinal, $\beta_1, \beta_2 < \alpha$ and $f_1 \in \mathfrak{F}_{\beta_1\alpha}$, $f_2 \in \mathfrak{F}_{\beta_2\alpha}$, then there are a $\beta_1, \beta_2 < \gamma < \alpha$, $g \in \mathfrak{F}_{\gamma\alpha}$ and $h_1 \in \mathfrak{F}_{\beta_1\gamma}$, $h_2 \in \mathfrak{F}_{\beta_2\gamma}$ such that $f_1 = g \circ h_1$ and $f_2 = g \circ h_2$.

(P5) For all $\alpha > 0$, $\theta_\alpha = \bigcup\{f[\theta_\beta] \mid \beta < \alpha, f \in \mathfrak{F}_{\beta\alpha}\}$.

Lemma 3.1. *Let $\alpha < \beta \leq \kappa$, $\tau_1, \tau_2 < \theta_\alpha$, $f_1, f_2 \in \mathfrak{F}_{\alpha\beta}$ and $f_1(\tau_1) = f_2(\tau_2)$. Then $\tau_1 = \tau_2$ and $f_1 \restriction \tau_1 = f_2 \restriction \tau_2$.*

Proof by induction over β. The base case of the induction is $\beta = \alpha + 1$. Then the claim follows immediately from (P3). So assume that $\beta = \gamma + 1$. Let, by (P2), $f_i = g_i \circ f_i'$ where $f_i' \in \mathfrak{F}_{\alpha\gamma}$, $g_i \in \mathfrak{F}_{\gamma\beta}$. Let $\tau_i' = f_i'(\tau_i)$. It follows like in the base case that $\tau_1' = \tau_2'$ and $f_1' \restriction \tau_1 = f_2' \restriction \tau_2$. So, by the induction hypothesis, $\tau_1 = \tau_2$ and $f_1' \restriction \tau_1 = f_2' \restriction \tau_2$. Hence $f_1 \restriction \tau_1 = f_2 \restriction \tau_2$.

Finally, let $\beta \in Lim$. Then there exists by (P4) $\alpha < \gamma < \beta$ and $g \in \mathfrak{F}_{\gamma\beta}$ such that $f_i = g \circ f_i'$, $f_i' \in \mathfrak{F}_{\alpha\gamma}$. So $f_1'(\tau_1) = f_2'(\tau_2)$. Hence $\tau_1 = \tau_2$ and $f_1' \restriction \tau_1 = f_2' \restriction \tau_2$ by the induction hypothesis. Therefore $f_1 \restriction \tau_1 = f_2 \restriction \tau_2$. $\square$

A simplified morass defines a tree $\langle T, \prec \rangle$:

Let $T = \{\langle \alpha, \gamma \rangle \mid \alpha \leq \kappa, \gamma < \theta_\alpha\}$.

For $t = \langle \alpha, \nu \rangle \in T$ set $\alpha(t) = \alpha$ and $\nu(t) = \nu$.

Let $\langle \alpha, \nu \rangle \prec \langle \beta, \tau \rangle$ iff $\alpha < \beta$ and $f(\nu) = \tau$ for some $f \in \mathfrak{F}_{\alpha\beta}$.

If $s := \langle \alpha, \nu \rangle \prec \langle \beta, \tau \rangle =: t$, $f \in \mathfrak{F}_{\alpha\beta}$ and $f(\nu) = \tau$, then $f \restriction (\nu(s) + 1)$ does not depend on f by lemma 3.1. So we may define $\pi_{st} := f \restriction (\nu(s) + 1)$.

Lemma 3.2. *The following hold:*

(a) $\prec$ is a tree, $ht_T(t) = \alpha(t)$.

(b) If $t_0 \prec t_1 \prec t_2$, then $\pi_{t_0 t_1} = \pi_{t_1 t_2} \circ \pi_{t_0 t_1}$.

(c) Let $s \prec t$ and $\pi = \pi_{st}$. If $\pi(\nu') = \tau'$, $s' = \langle \alpha(s), \nu' \rangle$ and $t' = \langle \alpha(t), \tau' \rangle$, then $s' \prec t'$ and $\pi_{s't'} = \pi \restriction (\nu' + 1)$.

(d) Let $\gamma \leq \kappa$, $\gamma \in Lim$. Let $t \in T_\gamma$. Then $\nu(t) + 1 = \bigcup\{rng(\pi_{st}) \mid s \prec t\}$.

Proof. (a) First, we prove that $\prec$ is transitive. Let $\langle \alpha, \nu \rangle \prec \langle \beta, \tau \rangle$ be witnessed by $f \in \mathfrak{F}_{\alpha\beta}$ and $\langle \beta, \tau \rangle \prec \langle \gamma, \eta \rangle$ by $g \in \mathfrak{F}_{\alpha\beta}$. Set $h = g \circ f \in \mathfrak{F}_{\alpha\gamma}$ by (P2). Then $h(\nu) = \eta$. So $\langle \alpha, \nu \rangle \prec \langle \gamma, \eta \rangle$.

Now, let $\langle \alpha, \nu \rangle, \langle \beta, \tau \rangle \prec \langle \gamma, \eta \rangle$ and $\langle \alpha, \nu \rangle \neq \langle \beta, \tau \rangle$. It follows from lemma 3.1 that $\alpha \neq \beta$. Let w.l.o.g. $\alpha < \beta$. Let $\langle \alpha, \nu \rangle \prec \langle \gamma, \eta \rangle$ be witnessed by $f \in \mathfrak{F}_{\alpha\gamma}$. By (P2) choose $g \in \mathfrak{F}_{\beta\gamma}$ and $h \in \mathfrak{F}_{\alpha\beta}$ such that $f = g \circ h$. Then $\langle \alpha, \nu \rangle \prec \langle \beta, h(\nu) \rangle \prec \langle \gamma, \eta \rangle$. However, $h(\nu) = \tau$ by lemma 3.1. Hence $\langle \alpha, \nu \rangle \prec \langle \beta, \tau \rangle$. This proves that $\prec$ is a tree.

Finally, by (P2), for all $t \in T$ there is $s \prec t$ such that $\alpha(s) = \beta$ if $\beta < \alpha(t)$. This shows the second claim.

(b) follows immediately from (a) and the definition.

(c) Let $s \prec t$ be witnessed by $f \in \mathfrak{F}_{\alpha\beta}$. Then $s' \prec t'$ is also witnessed by f and $\pi_{s't'} = \pi \restriction (\nu' + 1)$ holds by definition.

(d) It suffices to prove $\subseteq$. Let $\nu = \nu(t)$ and $\tau < \nu$. By (P5) choose $\alpha_1, \alpha_2 < \gamma$ and $f_i \in \mathfrak{F}_{\alpha_i\gamma}$ such that $\tau \in rng(f_1)$ and $\nu \in rng(f_2)$. By (P4) choose β such that $\alpha_1, \alpha_2 < \beta < \gamma$ and $f_i' \in \mathfrak{F}_{\alpha_i\beta}$, $g \in \mathfrak{F}_{\beta\gamma}$ where $f_i = g \circ f_i'$. Then $\tau, \nu \in rng(g)$. So

let $g(\bar{\tau}) = \tau$ and $g(\bar{\nu}) = \nu$. Hence $\bar{\tau} < \bar{\nu}$, since g is order-preserving. Let $s = \langle \beta, \bar{\nu} \rangle$. Then $s \prec t$ and $\pi_{st}(\bar{\tau}) = \tau$. $\qquad\square$

Lemma 3.3. *Let* $\alpha < \beta \leq \kappa$. *Then* $id \restriction \theta_\alpha \in \mathfrak{F}_{\alpha\beta}$.

Proof by induction on β. The base case of the induction is $\beta = \alpha + 1$. Then the claim is part of (P3). So assume that $\beta = \gamma + 1$. By the induction hypothesis, $id \restriction \theta_\alpha \in \mathfrak{F}_{\alpha\gamma}$. By (P3), $id \restriction \theta_\gamma \in \mathfrak{F}_{\gamma\beta}$. Hence $id \restriction \theta_\alpha = (id \restriction \theta_\gamma) \circ (id \restriction \theta_\alpha) \in \mathfrak{F}_{\alpha\beta}$ by (P2).

Finally, let $\beta \in Lim$. Assume towards a contradiction that $id \restriction \theta_\alpha \notin \mathfrak{F}_{\alpha\beta}$. Let $f \in \mathfrak{F}_{\alpha\beta}$ be such that $sup(f[\theta_\alpha])$ is minimal. Since $f \neq id \restriction \theta_\alpha$, there are $\nu < \tau \in \theta_\beta$ such that $\nu \notin rng(f)$ and $\tau \in rng(f)$. Let $t = \langle \beta, \tau \rangle$. By lemma 3.2 (d), there is an $s \prec t$ such that $\nu \in rng(\pi_{st})$. Let $s = \langle \gamma + 1, \bar{\tau} \rangle$ be the minimal such s. Let $\nu = \pi_{st}(\bar{\nu})$. Furthermore, let $f = f_3 \circ f_2 \circ f_1$ where $f_3 \in \mathfrak{F}_{\gamma+1,\beta}$, $f_2 \in \mathfrak{F}_{\gamma,\gamma+1}$ and $f_1 \in \mathfrak{F}_{\alpha\gamma}$. Then $\pi_{st} = f_3 \restriction \bar{\tau} + 1$. Hence by the minimality of s, $f_2 \neq id \restriction \theta_\gamma$, $\bar{\nu} < \theta_\gamma$ and $\bar{\tau} \geq \theta_\gamma$. Define $g := f_3 \circ (id \restriction \theta_\gamma) \circ f_1$. Then $g \in \mathfrak{F}_{\alpha\gamma}$ by (P2) and $rng(g) \subseteq f_3[\theta_\gamma] \subseteq f_3(\bar{\tau}) = \tau$. Hence $sup(f[\theta_\alpha])$ was not minimal. Contradiction! $\qquad\square$

Theorem 3.4. *(a) If $V = L$, then there is a simplified $(\kappa, 1)$-morass for all regular $\kappa > \omega$.*

(b) If κ is an uncountable regular cardinal such that κ^+ is not inaccessible in L, then there is a simplified $(\kappa, 1)$-morass.

(c) For every regular $\kappa > \omega$, there is a κ-complete (i.e. every decreasing sequence of length $< \kappa$ has a lower bound) forcing $\mathbb{P}$ satisfying κ^+-cc such that $\mathbb{P} \Vdash ($ there is a simplified $(\kappa, 1)$-morass).

Proof. (a) see Devlin [2], VIII 2 and 4 or Velleman [27].

(b) see Devlin [2], VIII 4 and exercise 6, or Velleman [28].

(c) see Velleman [27]. $\qquad\square$

4. FS Systems Along Morasses

Let $\mathfrak{M}$ be a simplified $(\kappa, 1)$-morass. We want to define a generalization of a FS iteration which is not indexed along an ordinal but along $\mathfrak{M}$. One way of doing this is the following definition:

We say that $\langle \langle \mathbb{P}_\eta \mid \eta \leq \kappa^+ \rangle, \langle \sigma_{st} \mid s \prec t \rangle, \langle e_\alpha \mid \alpha < \kappa \rangle \rangle$ is a FS system along $\mathfrak{M}$ if the following conditions hold:

(FS1) $\langle \mathbb{P}_\eta \mid \eta \leq \kappa^+ \rangle$ is a sequence of partial orders such that $\mathbb{P}_\eta \subseteq_\perp \mathbb{P}_\nu$ if $\eta \leq \nu$ and $\mathbb{P}_\lambda = \bigcup \{ \mathbb{P}_\eta \mid \eta < \lambda \}$ for $\lambda \in Lim$.

(FS2) $\langle \sigma_{st} \mid s \prec t \rangle$ is a commutative system of injective embeddings $\sigma_{st} : \mathbb{P}_{\nu(s)+1} \to \mathbb{P}_{\nu(t)+1}$ such that if t is a limit point in $\prec$, then $\mathbb{P}_{\nu(t)+1} = \bigcup \{ \sigma_{st}[\mathbb{P}_{\nu(s)+1}] \mid s \prec t \}$.

(FS3) $e_\alpha : \mathbb{P}_{\theta_{\alpha+1}} \to \mathbb{P}_{\theta_\alpha}$.

(FS4) Let $s \prec t$ and $\pi = \pi_{st}$. If $\pi(\nu') = \tau'$, $s' = \langle \alpha(s), \nu' \rangle$ and $t' = \langle \alpha(t), \tau' \rangle$, then $\sigma_{st} : \mathbb{P}_{\nu(s)+1} \to \mathbb{P}_{\nu(t)+1}$ extends $\sigma_{s't'} : \mathbb{P}_{\nu'+1} \to \mathbb{P}_{\tau'+1}$.

Hence for $f \in \mathfrak{F}_{\alpha\beta}$, we may define $\sigma_f = \bigcup \{ \sigma_{st} \mid s = \langle \alpha, \nu \rangle, t = \langle \beta, f(\nu) \rangle \}$.

(FS5) If $\pi_{st} = id \restriction \nu(s) + 1$, then $\sigma_{st} = id \restriction \mathbb{P}_{\nu(s)+1}$.

(FS6)(a) If $\alpha < \kappa$, then $\mathbb{P}_{\theta_\alpha}$ is completely contained in $\mathbb{P}_{\theta_{\alpha+1}}$ in such a way that

$e_\alpha(p)$ is a reduction of $p \in \mathbb{P}_{\theta_{\alpha+1}}$.

(b) If $\alpha < \kappa$, then $\sigma_\alpha := \sigma_{f_\alpha} : \mathbb{P}_{\theta_\alpha} \to \mathbb{P}_{\theta_{\alpha+1}}$ is a complete embedding such that $e_\alpha(p)$ is a reduction of $p \in \mathbb{P}_{\theta_{\alpha+1}}$.

(FS7)(a) If $\alpha < \kappa$ and $p \in \mathbb{P}_{\theta_\alpha}$, then $e_\alpha(p) = p$.

(b) If $\alpha < \kappa$ and $p \in rng(\sigma_\alpha)$, then $e_\alpha(p) = \sigma_\alpha^{-1}(p)$.

To simplify notation, set $\mathbb{P} := \mathbb{P}_{\kappa^+}$.

Unlike in the case of FS iterations, it is unclear how a generic extension with respect to $\mathbb{P}_{\kappa^+}$ can be viewed as being obtained by successive extensions. This would justify to call a FS system along $\mathfrak{M}$ a FS *iteration* along $\mathfrak{M}$.

However, like in the case of FS iterations it is sometimes more convenient to represent $\mathbb{P}$ as a set of functions $p^* : \kappa \to V$ such that $p^*(\alpha) \in \mathbb{P}_{\theta_\alpha}$ for all $\alpha < \kappa$.

To define such a function p^* from $p \in \mathbb{P}$ set recursively

$p_0 = p$

$\nu_n(p) = min\{\eta \mid p_n \in \mathbb{P}_{\eta+1}\}$

$t_n(p) = \langle \kappa, \nu_n(p) \rangle$

$p^{(n)}(\alpha) = \sigma_{st}^{-1}(p_n)$ if $s \in T_\alpha$, $s \prec t_n(p)$ and $p_n \in rng(\sigma_{st})$.

Note that, by lemma 3.2 (a), s is uniquely determined by α and $t_n(p)$. Hence we really define a function. Set

$\gamma_n(p) = min(dom(p^{(n)}))$.

By (FS2), $\gamma_n(p)$ is a successor ordinal or 0. Hence, if $\gamma_n(p) \neq 0$, we may define

$p_{n+1} = e_{\gamma_n(p)-1}(p^{(n)}(\gamma_n(p)))$.

If $\gamma_n(p) = 0$, we let p_{n+1} be undefined.

Finally, set $p^* = \bigcup\{p^{(n)} \restriction [\gamma_n(p), \gamma_{n-1}(p)[\mid n \in \omega\}$ where $\gamma_{-1}(p) = \kappa$.

Note: If $n > 0$ and $\alpha \in [\gamma_n(p), \gamma_{n-1}(p)[$, then $p^*(\alpha) = \sigma_{s\bar{t}}^{-1}(p_n)$ where $\bar{t} = \langle \gamma_n(p) - 1, \nu_n(p) \rangle$ because $p^*(\alpha) = p^{(n)}(\alpha) = \sigma_{st}^{-1}(p_n) = (\sigma_{\bar{t}t} \circ \sigma_{s\bar{t}})^{-1}(p_n) = \sigma_{s\bar{t}}(p_n)$ where the first two equalities are just the definitions of p^* and $p^{(n)}$. For the third equality note that $\bar{t} \prec t$ since $id \restriction \theta_\alpha \in \mathfrak{F}_{\alpha\beta}$ for all $\alpha < \beta \leq \kappa$ by lemma 3.3. So the equality follows from the commutativity of $\langle \sigma_{st} \mid s \prec t \rangle$. The last equality holds by (FS5).

It follows from the previous observation that $\langle \gamma_n(p) \mid n \in \omega \rangle$ is decreasing. So the recursive definition above breaks down at some point, i.e. $\gamma_n(p) = 0$ for some $n \in \omega$. However, that is good news because of the following.

The support of p is defined by $supp(p) = \{\gamma_n(p) \mid n \in \omega\}$. Hence $supp(p)$ is finite.

Lemma 4.1. *If $p^*(\alpha)$ and $q^*(\alpha)$ are compatible for $\alpha = max(supp(p) \cap supp(q))$, then p and q are compatible.*

Proof. Suppose that p and q are incompatible. Without loss of generality let $\nu := min\{\eta \mid p \in \mathbb{P}_{\eta+1}\} \leq min\{\eta \mid q \in \mathbb{P}_{\eta+1}\} =: \tau$. Set $s = \langle \kappa, \nu \rangle$ and $t = \langle \kappa, \tau \rangle$. Let $t' \prec t$ be minimal such that $\nu \in rng(\pi_{t't})$ and $p, q \in rng(\sigma_{t't})$. By (FS2), $t' \in T_{\alpha_0+1}$ for some $\alpha < \kappa$. Let $\pi_{t't}(\nu') = \nu$ and $s' = \langle \alpha + 1, \nu' \rangle$. Let $\bar{s}, \bar{t}$ be the direct predecessors of s' and t' in $\prec$. Set $p' = \sigma_{s's}^{-1}(p)$, $q' = \sigma_{t't}^{-1}(q)$. Then $p' = p^*(\alpha_0 + 1)$, $q' = q^*(\alpha_0 + 1)$ by the definition of p^*. Moreover, p' and q' are not compatible, because if $r \leq p', q'$, then $\sigma_{t't}(r) \leq p, q$ by (FS2). Now, we consider

several cases.

Case 1: $\nu' \notin rng(\pi_{\bar{t}t'})$

Then $\pi_{\bar{s}s'} = id \restriction \nu(\bar{s})+1$ and $\sigma_{\bar{s}s'} = id \restriction \mathbb{P}_{\nu(\bar{s})+1}$ by the minimality of α_0. Moreover, $\bar{p} := p'$ and $\bar{q} := e_\alpha(q')$ are not compatible, because if $r \leq p', e_\alpha(q')$, then there is $u \leq r, q', p'$ by (FS6)(a). There is no difference between compatibility in $\mathbb{P}_{\theta_{\alpha+1}}$ and in $\mathbb{P}_{\nu(t')+1}$ by (FS1). Finally, note that $\bar{p} = p^*(\alpha_0)$ and $\bar{q} = q^*(\alpha_0)$ by the definition of p^* and (FS7).

Case 2: $\nu' \in rng(\pi_{\bar{t}t'})$ and $\pi_{\bar{s}s'} = id \restriction \nu(\bar{s}) + 1$

Then $\pi_{\bar{t}t'} \neq id \restriction \nu(\bar{t})+1$ by the minimality of α_0 and $\bar{p} := p'$ and $\bar{q} := e_\alpha(q')$ are not compatible (like in case 1). However, $\bar{p} = p^*(\alpha_0)$ and $\bar{q} = q^*(\alpha_0)$ by the definition of p^* and (FS7).

Case 3: $\nu' \in rng(\pi_{\bar{t}t'})$, $\pi_{\bar{s}s'} \neq id \restriction \nu(\bar{s}) + 1$ and $\alpha_0 + 1 \notin supp(p)$

Then $\pi_{\bar{t}t'} \neq id \restriction \nu(\bar{t}) + 1$ by the minimality of α_0. Set $\bar{p} := \sigma_{\bar{s}s'}^{-1}(p')$ and $\bar{q} = e_\alpha(q')$. Then $\bar{p}$ and $\bar{q}$ are not compatible, because if $r \leq \bar{p}, \bar{q}$, then there is $u \leq \sigma_\alpha(r), q', p'$ by (FS6)(b). However, $\bar{p} = p^*(\alpha_0)$ and $\bar{q} = q^*(\alpha_0)$ by the definition of p^* and (FS7).

Case 4: $\nu' \in rng(\pi_{\bar{t}t'})$, $\pi_{\bar{s}s'} \neq id \restriction \nu(\bar{s}) + 1$ and $\alpha_0 + 1 \notin supp(q)$

Then $\pi_{\bar{t}t'} \neq id \restriction \nu(\bar{t}) + 1$. Set $\bar{q} := \sigma_{\bar{s}s'}^{-1}(q')$ and $\bar{p} = e_\alpha(p')$. Then $\bar{q}$ and $\bar{p}$ are not compatible, because if $r \leq \bar{p}, \bar{q}$, then there is $u \leq \sigma_\alpha(r), p', q'$ by (FS6)(b).

Case 5: $\alpha_0 + 1 \in supp(p) \cap supp(q)$

Then $\alpha_0 + 1 = max(supp(p) \cap supp(q))$, since $\alpha_0 + 1 \geq max(supp(q))$ because by definition $q \in rng(\sigma_{rt})$ where $r \prec t$ and $r \in T_{max(supp(q))}$. However, $p' = p^*(\alpha_0+1)$, $q' = q^*(\alpha_0 + 1)$ are not compatible. Contradiction.

So in case 5 we are finished. If we are in cases 1 - 4, we define recursively α_{n+1} from $p^*(\alpha_n)$ and $q^*(\alpha_n)$ in the same way as we defined α_0 from p and q. Like in the previous proof that $\langle \gamma_n(p) \mid n \in \omega \rangle$ is decreasing, we see that $\langle \alpha_n \mid n \in \omega \rangle$ is decreasing. Hence the recursion breaks off, we end up in case 5 and get the desired contradiction. $\square$

Theorem 4.2. *Let $\mu, \kappa > \omega$ be cardinals, κ regular. Let $\langle \langle \mathbb{P}_\eta \mid \eta \leq \kappa^+ \rangle, \langle \sigma_{st} \mid s \prec t \rangle, \langle e_\alpha \mid \alpha < \kappa \rangle \rangle$ be a FS system along a $(\kappa, 1)$-morass $\mathfrak{M}$. Assume that all $\mathbb{P}_\eta$ with $\eta < \kappa$ satisfy the μ-cc. Then $\mathbb{P}_{\kappa^+}$ also does.*

Proof. Let $A \subseteq \mathbb{P}_{\kappa^+}$ be a set of size μ. Assume by the Δ-system lemma that $\{supp(p) \mid p \in A\}$ forms a Δ-system with root Δ. Set $\alpha = max(\Delta)$. Then $\mathbb{P}_{\theta_\alpha}$ satisfies the μ-cc by the hypothesis of the lemma. So there are $p \neq q \in A$ such that $p^*(\alpha)$ and $q^*(\alpha)$ are compatible. Hence p and q are compatible by the previous lemma. $\square$

5. A CCC Forcing That Adds An ω_2-Suslin Tree

As an application, we construct along an $(\omega_1, 1)$-morass a ccc forcing $\mathbb{P}$ that adds an ω_2-Suslin tree.

The natural forcing to do this with finite conditions is Tennenbaum's forcing (see [21]): Define $P(\theta)$ as the set of all finite trees $p = \langle x_p, <_p \rangle$, $x_p \subseteq \theta$, such that $\alpha < \beta$ if $\alpha <_p \beta$.

Set $p \leq q$ iff $x_p \supset x_q$ and $<_q = <_p \cap x_q^2$.

For $\theta = \omega_1$, $P(\theta)$ is Tennenbaum's forcing to add an ω_1-Suslin tree which satisfies ccc.

However, if $\theta > \omega_1 + 1$, then

$$A = \{p \in P(\theta) \mid x_p = \{\alpha, \alpha + 1, \alpha + 2, \omega_1, \omega_1 + 1\}, \alpha < \omega_1,$$

$$\alpha <_p \alpha + 1 <_p \omega_1, \alpha <_p \alpha + 2 <_p \omega_1 + 1, \alpha + 1 \not<_p \alpha + 2\}$$

is an antichain of size ω_1.

So $P(\theta)$ does not satisfy the ccc and in order to thin it out so that it obtains ccc, we have to restrict the possible values of the infima in our trees.

Let $\pi : \bar{\theta} \to \theta$ be a order-preserving map. Then $\pi : \bar{\theta} \to \theta$ induces maps $\pi : \bar{\theta}^2 \to \theta^2$ and $\pi : P(\bar{\theta}) \to P(\theta)$ in the obvious way:

$$\pi : \bar{\theta}^2 \to \theta^2, \quad \langle \alpha, \beta \rangle \mapsto \langle \pi(\alpha), \pi(\beta) \rangle$$

$$\pi : P(\bar{\theta}) \to P(\theta), \quad \langle x_p, <_p \rangle \mapsto \langle \pi[x_p], \pi[<_p] \rangle.$$

If $p \in P(\theta)$, then set

$$\pi^{-1}[p] := \langle \pi^{-1}[x_p \cap rng(\pi)], \pi^{-1}[<_p \cap rng(\pi)] \rangle.$$

It is easily seen that then $\pi^{-1}[p] \in P(\bar{\theta})$.

Now, let us assume that we restrict the allowed values of the infimum $i_p(\alpha, \beta)$ of $\alpha, \beta \in x_p$ in the tree $p \in P(\theta)$ to a set $F(\alpha, \beta)$. For $\delta < \theta$, we want to find a reduction of $p \in P(\theta)$ with respect to $id \restriction \delta$. Let us look for example at a p with $x_p = \{\alpha, \beta\}$ and $<_p = \{\langle \alpha, \beta \rangle\}$ such that $\alpha < \delta < \beta$. Then we cannot just take $(id \restriction \delta)^{-1}$ as reduction because we could extend it to a condition q such that $i_q(\alpha, \gamma)$ exists for some $\gamma \in \delta$. However, $i_q(\alpha, \gamma)$ could have any value in $F(\alpha, \gamma)$, while in a common extension r of p and q we have $i_r(\gamma, \beta) = i_q(\alpha, \gamma)$ and $i_r(\alpha, \beta)$ has to be an element of $F(\alpha, \beta)$. We can solve this problem by taking s with $x_s = \{\alpha, \beta'\}$ and $<_s = \{\langle \alpha, \beta' \rangle\}$ as reduction for some β' with $F(\alpha, \beta') = F(\alpha, \beta)$. This leads to the following definition in which the $F(\alpha, \beta)$ are not needed anymore. But they could be introduced as the ranges of the morass maps. A similar problem arises in Baumgartner's and Shelah's forcing to add a a thin-very tall superatomic Boolean algebra [1]. They explicitly define a function F like above, which they obtain by historic forcing.

We define our FS system by induction on $\beta \leq \omega_1$.

Base Case: $\beta = 0$

Then we need to define only $\mathbb{P}_1$. Set $\mathbb{P}_1 := P(1)$.

Successor Case: $\beta = \alpha + 1$

We first define $\mathbb{P}_{\theta_\beta}$. To do so, let

$$\mathbb{P}'_{\theta_\beta} := \{\langle x_p \cup x_{f_\alpha(p)}, <_p \cup <_{f_\alpha(p)} \rangle \mid p \in \mathbb{P}_{\theta_\alpha}\}$$

$$\cup \{\langle x_p \cup x_{f_\alpha(p)}, tc(<_p \cup <_{f_\alpha(p)} \cup \{\langle \eta, min\{\gamma \in [\theta_\alpha, \theta_{\alpha+1}[\mid \gamma \leq_{f_\alpha(p)} f_\alpha(\eta)\}\rangle\})\rangle$$

$$\mid p \in \mathbb{P}_{\theta_\alpha}, \eta \in x_p, \eta < f_\alpha(\eta)\}.$$

In this definition, $tc(x)$ denotes the transitive closure of the binary relation x. As we know from Tennenbaum's original proof, every element of $\mathbb{P}'_{\theta_\beta}$ is an element of $P(\theta_\beta)$ which extends p and $f_\alpha(p)$. This is easily seen.

Now, define
$$\mathbb{P}_{\theta_\beta} := \{p \in P(\theta_\beta) \mid r \le p \text{ for some } r \in \mathbb{P}'_{\theta_\beta}\}.$$
For $t \in T_\beta$ set $\mathbb{P}_{\nu(t)+1} = P(\nu(t)+1) \cap \mathbb{P}_{\theta_\beta}$ and $\mathbb{P}_\lambda = \bigcup\{\mathbb{P}_\eta \mid \eta < \lambda\}$ for $\lambda \in Lim$.
Let $\sigma_{st} : \mathbb{P}_{\nu(s)+1} \to \mathbb{P}_{\nu(t)+1}, p \mapsto \pi_{st}(p)$.

We still need to define e_α. If $p \in rng(\sigma_\alpha)$, then set $e_\alpha(p) = \sigma_\alpha^{-1}(p)$. If $p \in \mathbb{P}_{\theta_\alpha}$, then set $e_\alpha(p) = p$. Finally, if $p \notin rng(\sigma_\alpha) \cup \mathbb{P}_{\theta_\alpha}$, then pick an $r \in \mathbb{P}'_{\theta_\beta}$ such that $r \le p$ and set $e_\alpha(p) = f_\alpha^{-1}[r]$.

Limit Case: $\beta \in Lim$

Then everything is already uniquely determined by (FS1) and (FS2). That is, for $t \in T_\beta$ set $\mathbb{P}_{\nu(t)+1} = \bigcup\{\sigma_{st}[\mathbb{P}_{\nu(s)+1}] \mid s \prec t\}$ and $\mathbb{P}_\lambda = \bigcup\{\mathbb{P}_\eta \mid \eta < \lambda\}$ for $\lambda \in Lim$.
Let $\sigma_{st} : \mathbb{P}_{\nu(s)+1} \to \mathbb{P}_{\nu(t)+1}, p \mapsto \pi_{st}(p)$.

Lemma 5.1. $\mathbb{P}$ *satisfies the ccc.*

Proof. Since all $P(\theta)$ for $\theta < \omega_1$ have size $\le \omega$, it suffices by theorem 4.2 to show that $\langle\langle\mathbb{P}_\eta \mid \eta \le \kappa^+\rangle, \langle\sigma_{st} \mid s \prec t\rangle, \langle e_\alpha \mid \alpha < \kappa\rangle\rangle$ is a FS system along the morass.
Most conditions of the definition of a FS system are clear. We only prove (FS6). Let $p \in \mathbb{P}_{\theta_\beta}$ and $\beta = \alpha + 1$. We may assume that $p \in \mathbb{P}'_{\theta_\beta}$, because by definition $\mathbb{P}_{\theta_\beta}$ is dense in $\mathbb{P}'_{\theta_\beta}$. We have to prove that $\sigma_\alpha^{-1}[p]$ is a reduction of p with respect to σ_α and $id \upharpoonright \mathbb{P}_{\theta_\alpha}$. To do so for σ_α, let $q \le \sigma_\alpha^{-1}[p] =: s$. We have to find an $r \le p, \sigma_\alpha(q)$ such that $r \in \mathbb{P}_{\theta_\beta}$. We consider two cases. If p is of the form $\langle x_s \cup x_{f_\alpha(s)}, <_s \cup <_{f_\alpha(s)}\rangle$, then define $r := \langle x_q \cup x_{f_\alpha(q)}, <_p \cup <_{f_\alpha(q)}\rangle$. It is easily seen that this is an extension of p and $\sigma_\alpha(q)$. If p is of the form
$$\langle x_s \cup x_{f_\alpha(s)}, tc(<_s \cup <_{f_\alpha(s)} \cup \{\langle \eta, min\{\gamma \in [\theta_\alpha, \theta_{\alpha+1}[\mid \gamma \le_{f_\alpha(s)} f_\alpha(\eta)\}\rangle\})\rangle$$
for some $\eta \in x_s$, then define r as
$$\langle x_q \cup x_{f_\alpha(q)}, tc(<_q \cup <_{f_\alpha(q)} \cup \{\langle \eta, min\{\gamma \in [\theta_\alpha, \theta_{\alpha+1}[\mid \gamma \le_{f_\alpha(q)} f_\alpha(\eta)\}\rangle\})\rangle.$$
Again, it is easily seen that this is an extension of p and $\sigma_\alpha(q)$. That proves that $\sigma_\alpha^{-1}[p]$ is a reduction of p with respect to σ_α. The proof that $\sigma_\alpha^{-1}[p]$ is a reduction of p with respect to $id \upharpoonright \mathbb{P}_{\theta_\alpha}$ is completely analogous. $\square$

Lemma 5.2. *If $\gamma_0(p) = \gamma_0(q)$, $p^*(\gamma_0(p)) = q^*(\gamma_0(q))$, $\pi : p \cong q$ and $\alpha \le \pi(\alpha)$, then there exists an $r \le p, q$ such that $\langle\alpha, \pi(\alpha)\rangle \in \le_r$.*

Proof. Let p and q be as in the hypothesis of the lemma. We prove by induction over $\eta \in [\gamma_0(p), \omega_1]$ that if $\pi : p^*(\eta) \cong q^*(\eta)$ (where $p^*(\omega_1) := p$) and $\alpha \le \pi(\alpha)$, then there exists an $r \le p^*(\eta), q^*(\eta)$ such that $\langle\alpha, \pi(\alpha)\rangle \in \le_r$.
Base Case: $\eta = \gamma_0(p) = \gamma_0(q)$
In this case the claim is trivial because $p^*(\eta) = q^*(\eta)$.
Successor Case: $\eta = \gamma + 1$
Let $\pi : p^*(\eta) \cong q^*(\eta)$ and $\alpha \le \pi(\alpha)$. Let $\sigma_p : p^*(\gamma) \cong p^*(\eta)$, $\sigma_q : q^*(\gamma) \cong q^*(\eta)$ and $\sigma_p(\bar{\alpha}_p) = \alpha$, $\sigma_q(\bar{\alpha}_q) = \pi(\alpha)$. By the induction hypothesis, there is an $s \le p^*(\gamma), q^*(\gamma)$ such that $\langle\bar{\alpha}_q, \bar{\alpha}_p\rangle \in \le_s$ or $\langle\bar{\alpha}_p, \bar{\alpha}_q\rangle \in \le_s$. Let $\bar{\alpha} := max\{\bar{\alpha}_p, \bar{\alpha}_q\}$. Now, we consider two cases. If $\bar{\alpha} < f_\gamma(\bar{\alpha})$, we define r as
$$\langle x_s \cup x_{f_\alpha(s)}, tc(<_s \cup <_{f_\alpha(s)} \cup \{\langle \bar{\alpha}, min\{\beta \in [\theta_\gamma, \theta_{\gamma+1}[\mid \beta \le_{f_\alpha(s)} f_\gamma(\bar{\alpha})\}\rangle\})\rangle.$$
If $\bar{\alpha} = f_\gamma(\bar{\alpha})$, then we define
$$r := \langle x_s \cup x_{f_\alpha(s)}, <_s \cup <_{f_\alpha(s)}\rangle.$$

In both cases, it is easily seen that $r \leq p^*(\eta), q^*(\eta)$ and $\langle \alpha, \pi(\alpha) \rangle \in \leq_r$.

Limit Case: $\eta \in Lim$

By (FS1) and (FS2), there are a $t \in T_\eta$ and an $s \prec t$ such that $p^*(\eta), q^*(\eta) \in rng(\sigma_{st})$. Let $s \in T_\gamma$, $\sigma_{st}(\bar{\alpha}) = \alpha$ and $\sigma_{st} \circ \bar{\pi} = \pi \circ \sigma_{st}$. Then $\sigma_{st}(p^*(\eta)) = p^*(\gamma)$ and $\sigma_{st}(q^*(\eta)) = q^*(\gamma)$. Moreover, by the induction hypothesis, there is a $\bar{r} \leq p^*(\gamma), q^*(\gamma)$ such that $\langle \bar{\alpha}, \bar{\pi}(\bar{\alpha}) \rangle \in \leq_{\bar{r}}$. Set $r := \sigma_{st}(\bar{r})$. Then r is as desired.   □

Lemma 5.3. *If* $\gamma_0(p) = \gamma_0(q)$, $p^*(\gamma_0(p)) = q^*(\gamma_0(q))$, $\pi : p \cong q$ *and* $\alpha \leq \pi(\alpha)$, *then there exists an* $r \leq p, q$ *such that* $\langle \alpha, \pi(\alpha) \rangle \notin <_r$.

Proof. Basically the proof proceeds like the proof of lemma 5.3. However, in the successor case, we always use common extensions of the form $\langle x_p \cup x_{f_\gamma(p)}, <_p \cup <_{f_\gamma(p)} \rangle$.   □

Theorem 5.4. *If there is a simplified* $(\omega_1, 1)$*-morass, then there is a ccc forcing that adds an* ω_2*-Suslin tree.*

Proof. We show that $\mathbb{P}$ forces an ω_2-Suslin tree. To do so, we prove that the generic tree has neither an antichain nor a chain of size ω_2. First, assume towards a contradiction that there is an antichain of size ω_2. Then there is a $p \in \mathbb{P}$ and by ccc of $\mathbb{P}$ a sequence $\langle \dot{x}_i \mid i \in \omega_2 \rangle$ such that $p \Vdash (\{\dot{x}_i \mid i \in \omega_2\}$ is an antichain). Let $\langle \alpha_i \mid i \in \omega_2 \rangle$ and $\langle p_i \mid i \in \omega_2 \rangle$ be such that $p_i \leq p$ for all $i \in \omega_2$ and $p_i \Vdash (\dot{x}_i = \check{\alpha}_i \wedge \dot{x}_i \in \check{x}_{p_i})$. Since $card(\mathbb{P}_{\omega_1}) = \omega_1$, there is $q \in \mathbb{P}_{\omega_1}$, $\eta \in \omega_1$ and a subset $X \subseteq \omega_2$ of size ω_2 such that $\gamma_0(p_i) = \eta$ and $p_i^*(\gamma_0(p_i)) = q$ for all $i \in X$. Hence all p_i with $i \in X$ are isomorphic. Since x_q is finite, there are $i \neq j \in X$ such that $\pi(\alpha_i) = \alpha_j$ and $\alpha_i \leq \alpha_j$ where $\pi : p_i \cong p_j$. By lemma 5.2, there exists an $r \leq p_i, p_j$ such that $\langle \alpha_i, \alpha_j \rangle \in \leq_r$. Hence $r \Vdash (\alpha_i$ and α_j are comparable). That contradicts the definition of p. The proof that there is no chain of size ω_2 works the same using lemma 5.3 instead of lemma 5.2.   □

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