LOTS (of) embeddability results

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LOTS and embeddings

A LOTS or linearly ordered topological space is exactly what it says: a linear order equipped with the open interval topology.

A LOTS embedding is an injective continuous function which preserves order.

Embedding quasi-orders

The relation on the class of all LOTS defined by setting $A \preceq B$ if and only if A LOTS-embeds into B is a quasi-order.
There are several aspects of the LOTS embeddability order that we study:

- The top (universal structures or families): can we find a (non-trivial set of) LOTS such that every LOTS in a given class must be LOTS-embeddable into one of them?
- The internal structure of the embeddability order: what are the possible cardinalities of chains and antichains in the LOTS embeddability order for a given class of LOTS?
- The bottom (prime models or bases): can we find a (non-trivial set of) LOTS such that every LOTS must embed one of them?

Countable universal LOTS

Let $(\mathbb{Q}, <, \tau)$ be the LOTS whose underlying linear order is the rationals and which is endowed with the open interval topology τ.

Claim

$(\mathbb{Q}, <, \tau)$ *forms a universal LOTS at* $\aleph_0$*.*

$\mathbb{Q}$ is universal for countable linear orders. Given a countable linear order L there exists $f : L \hookrightarrow \mathbb{Q}$ order-preserving. We may “fix” f so that it is continuous: if f is discontinuous at x then there is a sequence $\bar{x} \subseteq L$ such that x is the limit, but the limit of $f[\bar{x}]$ is not $f(x)$. By removing half-open intervals between $f[\bar{x}]$ and $f(x)$ we can show the resulting order is isomorphic to $\mathbb{Q}$.

Regular uncountable LOTS

Assume that κ is a regular uncountable cardinal such that $\kappa = \kappa^{<\kappa}$.
$\mathbb{Q}(\kappa)$ is the unique linear order of size κ without endpoints with the *κ-density property:*

$$\forall S, T \in [\mathbb{Q}(\kappa)]^{<\kappa}\ [S < T \Rightarrow (\exists x) S < x < T].$$

$\mathbb{Q}(\kappa)$ is universal for linear orders of size κ.

$\bar{\mathbb{Q}}(\kappa)$ is the completion of $\mathbb{Q}(\kappa)$ under $< \kappa$-sequences.

For any infinite limit ordinal $\alpha < \kappa$, the linear orders $\alpha + 1$ and $\alpha + 1 + \alpha^*$, respectively, generate counterexamples to LOTS universality.

Restricting cofinalities and coinitialities

Definition

A LOTS, L, of size κ is κ-*entwined* if for all $x \in L$, $\sup((-\infty, x)) = \inf((x, +\infty)) = x$ implies that both the cofinality and coinitiality of x are equal to κ.

Claim

Assume $\kappa = \kappa^{<\kappa}$ and let $\bar{\mathbb{Q}}(\kappa)$ be the completion of $\mathbb{Q}(\kappa)$ under sequences of length $< \kappa$. Then $\bar{\mathbb{Q}}(\kappa)$ is universal for κ-entwined LOTS.

Note that all countable LOTS are $\aleph_0$-entwined so the fact that $\mathbb{Q}$ is universal is a corollary.

No universal LOTS

The *dominating number* for a quasi-order P is the least cardinality of a subset $Q \subseteq P$ such that for any $p \in P$ there is a $q \in Q$ with $p \leq q$.

Theorem

For any $\kappa \geq 2^{\aleph_0}$ there is no universal LOTS of cardinality κ. Moreover the dominating number for this LOTS embeddability quasi-order is 2^{κ}.

Universality for linear orders below the continuum

- (Shelah) $\neg$CH: The existence of a universal linear order at $\aleph_1$ is independent.
- (Kojman, Shelah) For $\lambda \in (\aleph_1, 2^{\aleph_0})$ regular, there is no universal linear order of size λ.

Countable linear orders and LOTS

Better quasi-ordered is a strengthening of well-quasi ordered which means that there are no infinite antichains and no infinite descending sequences.

Theorem (Laver)

Countable linear orders are better quasi-ordered under order embeddability.

Theorem (Beckmann, Goldstern, Preining)

Countable closed sets of reals are better quasi-ordered under LOTS embeddability.

Uncountable LOTS

Theorem

Let $\kappa \geq 2^{\aleph_0}$. Then there exists:

1. *An antichain $\mathscr{A}$ of size 2^κ consisting of LOTS of size κ (i.e. there is no LOTS embedding from A into B for any two distinct $A, B \in \mathscr{A}$).*
2. *A sequence of length κ^+ of LOTS of size κ, strictly increasing in the LOTS embeddability order.*
3. *For each $\eta < \kappa^+$, a sequence of length η of LOTS of size κ, strictly decreasing in the LOTS embeddability order.*

Antichain proof idea

Claim

Let $\kappa \geq 2^{\aleph_0}$. Then there exists a set $\mathscr{A}$ with $|\mathscr{A}| = 2^{\kappa}$ of LOTS of size κ such that for any $A, B \in \mathscr{A}$ there is no LOTS embedding between A and B.

Proof idea: The intermediate value theorem (IVT): a continuous embedding from a linear continuum, *A*, into a linear order *B* must be surjective onto a convex subset of *B*.

Idea is to find 2 linear continua which do not embed into each other. Concatenate these linear continua together according to the characteristic function for a subset of κ. This will give another linear continuum for each subset of κ. They form an antichain in the LOTS embedding order by the IVT.

Any linear continuum has size $\geq 2^{\aleph_0}$.

Dense embeddings

Definition

Let A and B be linear orders and $f : A \to B$ a linear order embedding. Then we call f a *dense embedding* if there is a convex subset $C \subseteq B$ such that $f[A]$ is a dense subset of C.

Fact

Let A, B be linear orders and suppose $f : A \to B$ is a dense embedding. Then f is continuous.

Embedding structure for separable linear orders

Theorem

- *(Cantor) The reals* $(\mathbb{R}, <)$ *form a universal separable linear order.*
- *(Baumgartner) Under PFA there is a unique* $\aleph_1$*-dense set of reals up to isomorphism.*

The reals then also form a universal separable LOTS.

As with order embeddings, Baumgartner's result implies that this set of reals forms a universal and prime model for separable LOTS of size $\aleph_1$.

The embedding q.o. under PFA is thus totally flat: all separable LOTS of size $\aleph_1$ are bi-embeddable.

Basis basics

Reminder: A basis $\mathscr{B}$ for $\mathscr{A}$ is a collection of structures in $\mathscr{A}$ such that for every $A \in \mathscr{A}$ there is $B \in \mathscr{B}$ such that $B \preceq A$.

ω and ω^* form a basis for the infinite linear orders and infinite LOTS.

Fact

Under CH there is no countable basis for the uncountable linear orders or LOTS.

Linear order basis under PFA

Theorem (Moore)

Assuming PFA there is a five-element basis for the uncountable linear orders: every uncountable linear order will embed (at least) one of these five elements. The basis elements are as follows:

- ω_1 *and* ω_1^*
- X *a fixed* $\aleph_1$*-dense set of reals*
- C *a fixed* $\aleph_1$*-dense non-stationary Countryman line, and* C^* *its inverse*

5 elements for a linear order basis is minimal

Why is this minimal?

ω_1 and ω_1^* are always minimal order types.

Separable orders only (order) embed separable orders.

There are uncountable orders which do not embed ω_1 or ω_1^* and do not contain a separable suborder. These are the Aronszajn lines.

Countryman lines

Definition

A *Countryman line*, C, is a linear order of size $\aleph_1$ such that the product $C \times C$ is the union of countably many chains (in the product order).

Shelah proved that Countryman lines exist in ZFC.

Every Countryman line is also an Aronszajn line.

$\aleph_1$-dense Countryman lines with a particular property, called non-stationarity, are unique up to isomorphism / reverse isomorphism under PFA.

Basis for Aronszajn lines

Let C be a fixed $\aleph_1$-dense non-stationary Countryman line, and C^* its inverse.

Theorem (Moore)

Assuming PFA, every Aronszajn line contains one of C or C^*.

Lemma

If a linear order D embeds into both C and C^*, then D must be countable.

Basis for Aronszajn LOTS

Lemma

(PFA) Given an Aronszajn line A, one of either $C \times \mathbb{Z}$ or $C^* \times \mathbb{Z}$ must embed into A.

No limit points!

Basis for separable LOTS

However, for the $\aleph_1$-dense set of reals, X, avoiding all limit points does not work: if $|B| \geq 2$ then $X \times B \not\hookrightarrow X$, for any linear order B.

Lemma

Let A be an uncountable linear order such that there exists a linear order embedding $f : X \to A$. Then we can find a LOTS embedding $f' : X \times B \to A$ for some $B \in \{1, 2, \omega, \omega^*, \mathbb{Z}\}$.

Basis for LOTS which order embed ω_1 or ω_1^*

Here again we build the appropriate enhancements of ω_1 and ω_1^* which do not have limit points:

Start with ω_1 and replace each limit ordinal with a copy of ω^*.
Call this $\omega_1 \bar{\times} \omega^*$.

Start with ω_1^* and replace each limit ordinal with a copy of ω.
Call this $\omega_1^* \bar{\times} \omega$.

Lemma

For an uncountable linear order A if ω_1 order embeds into A then either ω_1 LOTS-embeds into A or $\omega_1 \bar{\times} \omega^*$ LOTS-embeds into A. If ω_1^* order embeds into A then either ω_1^* LOTS-embeds into A or $\omega_1^* \bar{\times} \omega$ LOTS-embeds into A.

Eleven element basis for uncountable LOTS

Theorem

(PFA) The set:

$$\{X, X \times 2, X \times \omega, X \times \omega^*, X \times \mathbb{Z},$$
$$C \times \mathbb{Z}, C^* \times \mathbb{Z},$$
$$\omega_1, \omega_1 \bar{\times} \omega^*,$$
$$\omega_1^*, \omega_1^* \bar{\times} \omega\}$$

forms a minimal basis for uncountable LOTS.

Note that this finite basis relies on there being linear orders with no limit points.

Basis for dense LOTS

A linear order L is dense iff for every point $x \in L$ there is a there are increasing and decreasing sequences in L such that x is the supremum, respectively infimum of these sequences.

A finite basis for the dense linear orders must be restricted to those with points of limited cofinality and coinitiality.

Theorem

(PFA) There is a six element basis for those uncountable dense LOTS in which all points have cofinality and coinitiality ω. The basis is $\{X, X \times \mathbb{Q}, C \times \mathbb{Q}, C^* \times \mathbb{Q}, \omega_1 \times \mathbb{Q}, \omega_1^* \times \mathbb{Q}\}$.

Uncountable universal LOTS: open questions

- Is it consistent that there is a universal LOTS at $\aleph_1 < 2^{\aleph_0}$?
- Moore proves that there is a universal Aronszajn line assuming PFA. This is not a universal LOTS. Can it be shown that there is no universal Aronszajn LOTS in any model of ZFC?
- Are the countable LOTS well-quasi ordered under embeddability?