Amenable actions of the infinite permutation group — Lecture I

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- Lebesque described his integral in terms of invariance under translation and countable additivity (actually, monotone convergence) an asked whether this provided a characterization.
- Banach disproved this by constructing a *finitely* additive, translation invariant measure on the circle that was different from the Lebesque integral in that it is defined on *all* subsets of the circle.
- It was also possible to define such a measure on $\mathbb{R}$ that gives $\mathbb{R}$ finite measure.
- The investigation of such measures led to the Banach-Tarski-Hausdorff Paradox. In his study of this paradox von Neumann introduced the notion of an amenable group.

Definition

A mean *on a discrete group G is a finitely additive probability measure on G. For $X \subseteq G$ and $g \in G$ define $gX = \{gx \mid x \in X\}$. A mean μ is said to be left invariant if $\mu(gX) = \mu(X)$ for all $g \in G$ and $X \subseteq G$.*

Means on locally compact groups can be defined in a similar spirit.

Definition

A discrete group is called amenable *if there exists a left invariant mean on it.*

Example

Finite groups are amenable.

Example

$\mathbb{Z}$ is amenable.

A naive approach would be to construct a mean on $\mathbb{Z}$ in the same way that ultrafilters on $\mathbb{N}$ are constructed. While this is possible, the details are considerably more involved than the ultrafilter construction. Note that a mean can never be two valued.

To construct a mean on $\mathbb{Z}$ it is useful to identify means on a discrete group G as elements of $\ell^*_\infty(G)$. Given a mean μ on G define $\mathfrak{m}_\mu : \ell_\infty(G) \to \mathbb{R}$ by

$$\mathfrak{m}_\mu(f) = \int f(g)d\mu(g)$$

taking care about the lack of countable additivity of μ: Note that $\mathfrak{m}_\mu(gf) = \mathfrak{m}_\mu(f)$ if μ is left invariant. (Here $gf(h) = f(g^{-1}h)$.)

On the other hand, if $\mathfrak{m} \in \ell^*_\infty(G)$ is left invariant as above, then defining $\mu_{\mathfrak{m}}(A) = \mathfrak{m}(\chi_A)$ yields a left invariant mean.

Recall that $\ell_1(G)^* = \ell_\infty(G)$ with $f(h) = \sum_{g\in G} f(g)h(g)$ where $h \in \ell_1(G)$ and $f \in \ell_\infty(G)$. For $k \geq 1$ let $m_k \in \ell_1(\mathbb{Z})$ be defined by

$$m_k(j) = \begin{cases} 1/(2k+1) & \text{if } |j| \leq k \\ 0 & \text{otherwise} \end{cases}$$

and note that $\|m_k\|_1 = 1$. Hence the m_k can be identified with elements of unit ball of $\ell_\infty^*(\mathbb{Z})$ and so they have a weak* complete accumulation point $\mathfrak{m}$ in the unit ball of $\ell_\infty^*(\mathbb{Z})$ — in other words, $\mu_{\mathfrak{m}}(\mathbb{Z}) = 1$.

It suffices to show that $\mathfrak{m}(n+f) = \mathfrak{m}(f)$ for $n \in \mathbb{Z}$ and $f \in \ell_\infty(\mathbb{Z})$.

To see this note that

$$\mathfrak{m}(f) = \lim_k f(m_k) = \sum_{j\in\mathbb{Z}} f(j)m_k(j) = \frac{1}{2k+1}\sum_{j=-k}^{k} f(j)$$

while

$$\mathfrak{m}(n+f) = \lim_k (n+f)(m_k) = \sum_{j\in\mathbb{Z}} f(j-n)m_k(j) = \frac{1}{2k+1}\sum_{j=-k}^{k} f(j-n)$$

and note that $|\sum_{j=-k}^{k} f(j) - \sum_{j=-k}^{k} f(j-n)| \leq n\|f\|_\infty$ and hence

$$\mathfrak{m}(f) - \mathfrak{m}(n+f) = \lim_k \frac{n\|f\|_\infty}{2k+1} = 0$$

Example

$\mathbb{F}_2$ is not amenable.

To see this suppose that μ is a left invariant probability measure on $\mathbb{F}_2$. Think of $\mathbb{F}_2$ as all reduced words on the two letter alphabet $\{a, b\}$ with identity the empty word $\emptyset$. If B_x denotes all words beginning with $x \in \{a, b, a^{-1}, b^{-1}\}$ then $\mathbb{F}_2 = B_a \cup B_b \cup B_{a^{-1}} \cup B_{b^{-1}} \cup \{\emptyset\}$. Moreover, $aB_{a^{-1}}$ and B_a form a partition of $\mathbb{F}_2$ and so do $bB_{b^{-1}}$ and B_b. Hence, by left invariance

$$1 = \mu(aB_{a^{-1}}) + \mu(B_a) = \mu(B_{a^{-1}}) + \mu(B_a)$$

$$1 = \mu(bB_{b^{-1}}) + \mu(B_b) = \mu(B_{b^{-1}}) + \mu(B_b)$$

yielding a contradiction.

Amenable groups are preserved by subgroups. Why? Let μ be a left invariant probability measure on G and H a subgroup of G. Restricting μ to H works unless $\mu(H) = 0$. Let X be such that $\{Hx\}_{x\in X}$ is a maximal family of right cosets of H. Define $\mu_H(A) = \mu(\bigcup_{x\in X} Ax)$. It is easy to see that μ_H is finitely additive and $\mu_H(H) = 1$. To see that it is left invariant, let $h \in H$. Then

$$\mu_h(A) = \mu(\bigcup_{x\in X} hAx) = \mu(h\bigcup_{x\in X} Ax) = \mu(\bigcup_{x\in X} Ax) = \mu_H(A)$$

Hence $\mathbb{SL}_2(\mathbb{R})$ and $\mathbb{S}(\mathbb{N})$ — the full symmetric group on $\mathbb{N}$ — are not amenable since both contain a copy of $\mathbb{F}_2$. It was a conjecture of von Neumann that the amenable groups could be characterized as precisely those that do not contain a copy of $\mathbb{F}_2$. This was disproved by Olshanskii.

Products of amenable groups are amenable: Hence $\mathbb{Z}^n \times G$ is amenable for any finite group. More generally, extensions of amenable groups by amenable groups are also amenable — in other words, if N is an amenable normal subgroup of G and G/N is amenable, then so is G. (Why? Fubini's Theorem) Quotients of amenable groups are also amenable. (Why? Use the image measure.)

Directed unions of amenable groups are amenable. (Why? This will follow from the Følner Property to be discussed next.) This raises the question of whether the amenable groups are precisely those that can be obtained from finite groups and $\mathbb{Z}$ by subgroups, quotients, extensions and increasing unions. An example of Grigorchuk shows that this is not the case.

For a finite set $X \subseteq G$ and $g \in G$ the number

$$\frac{|gX \Delta X|}{|X|}$$

measures by how much g shifts X away from itself. In the case of $\mathbb{Z}$ this is quite small if X is an interval much larger than g.

Theorem (Følner)

A discrete group G is amenable if and only if for all $\epsilon > 0$ and finite X there is $Y \supseteq X$ such that for all $x \in X$

$$\frac{|xY \Delta Y|}{|Y|} < \epsilon$$

Corollary

Directed unions of amenable groups are amenable.

Corollary

Locally finite groups are amenable. (A group is locally finite if the subgroup generated by any finite set if finite.) More generally, locally amenable groups are amenable.

So, while the full symmetric group on $\mathbb{N}$ is not amenable, the subgroup of all finite permutations is amenable.

Let G be a group acting on a set X.

Definition

The action of G on X is said to be amenable if there is a finitely additive probability measure μ on X such that $\mu(A) = \mu(gA)$ for each $g \in G$ and $A \subseteq X$.

So a discrete group is amenable if and only if its action on itself is amenable.

Moreover, if G is an amenable group acting on X then the action is amenable.

To see this let $x^* \in X$ be arbitrary and let λ be a mean on G. For $A \subseteq X$ define

$$\lambda^*(A) = \lambda(\{g \in G \mid g(x^*) \in A\})$$

and observe that λ^* is a probability measure on X. Moreover, it is G invariant since

$$\lambda^*(hA) = \lambda(\{g \in G \mid g(x^*) \in hA\}) = \lambda(\left\{g \in G \;\middle|\; h^{-1}g(x^*) \in A\right\}) =$$

$$\lambda(\left\{h^{-1}g \in G \;\middle|\; h^{-1}g(x^*) \in A\right\}) = \lambda(\{g \in G \mid g(x^*) \in A\}) = \lambda^*(A)$$

But amenability of G is not needed for the amenability of the action.

Example

Let $\mathcal{J}$ be any maximal ideal on the set X and let $G_{\mathcal{J}}$ be the group of all permutations θ of X such that $A \in \mathcal{J}$ if and only if $\theta(A) \in \mathcal{J}$. Let $\mu_{\mathcal{J}}$ be the $\{0,1\}$-valued measure on X defined by $\mu_{\mathcal{J}}(A) = 0$ if and only if $A \in \mathcal{J}$.

Then the natural action of $G_{\mathcal{J}}$ on X is amenable and this is witnessed by $\mu_{\mathcal{J}}$.

If $\mathcal{J}$ contains $[X]^{<|X|}$ then $\mu_{\mathcal{J}}$ is unique. To see this, suppose that ν is some other measure on X. Since $\mathcal{J}$ is maximal and $\{0,1\}$-valued there must be some $A \in \mathcal{J}$ such that $\nu(A) > 0$. Then $|X \setminus A| = |X| = \kappa$ and it is possible to choose $B \in \mathcal{J}$ such that $A \subseteq B$ and $|B \setminus A| = |A|$.

Let $k > 1/\nu(A)$ and let $\{\alpha_\xi\}_{\xi\in\kappa}$ enumerate A and let $\{\beta_{\xi,j}\}_{\xi\in\kappa,j\in k}$ enumerate B. Then the permutation θ defined by

$$\theta(x) = \begin{cases} \beta_\xi, 0 & \text{if } x = \alpha_\xi \\ \beta_\xi, j & \text{if } j < k-1 \text{ and } x = \beta_{\xi,j-1} \\ \alpha_\xi & \text{if } x = \beta_{\xi,k-1} \\ x & \text{otherwise} \end{cases}$$

belongs to $G_{\mathcal{J}}$.

- The mean $\mu_{\mathcal{J}}$ is $\{0,1\}$-valued and this is impossible for means of groups acting on themselves.
- The mean $\mu_{\mathcal{J}}$ is unique and this is also impossible for means of infinite (non-compact) groups acting on themselves.
- The group $G_{\mathcal{J}}$ is not amenable itself for non-trivial ideals.

Joe Rosenblatt asked whether there is an *amenable* group acting on a set with a unique mean. Matt Foreman's answer[1] to this question will be the subject of the next lectures.

[1] Matthew Foreman, *Amenable Groups and Invariant Measures* Journal of Functional Analysis **126** 7-25, 1994

In particular, it will be shown to be independent of set theory that there is a locally finite group of permutations of $\mathbb{N}$ whose natural action on $\mathbb{N}$ has a unique $\{0,1\}$-valued invariant mean. Before proceeding with this it is worth remarking that a group of permutations of $\mathbb{N}$ whose natural action on $\mathbb{N}$ has a unique $\{0,1\}$-valued invariant mean can not have a simple definition.

It will be shown in Lecture III that if the natural action of G on $\mathbb{N}$ has a *unique* invariant mean μ then this mean is defined by $\mu(A) < r$ for any rational r if and only if

$$(\exists Z \in [G]^{<\aleph_0})(\forall k \in \mathbb{N})\frac{|\{z \in Z \mid zk \in A\}|}{|Z|} < r$$

In the case of a $\{0,1\}$-valued invariant mean μ this yields that $\{A \subseteq \mathbb{N} \mid \mu(A) = 1\}$ is an ultrafilter. The preceding definition shows that if the definition of G is simple, then so is the quantifier "$\exists Z \in [G]^{<\aleph_0}$". This ultrafilter would then have to be analytic.

Overview of next three lecture

- The second lecture will present the construction, assuming some hypotheses on cardinal invariants, of a locally finite group of permutations acting on $\mathbb{N}$ with a unique invariant mean.
- The third lecture will look at the definition used in establishing that a group like the one described in the second lecture can not be analytic. This will be used to show that in the Cohen model there are no locally finite groups of permutation of $\mathbb{N}$ acting on $\mathbb{N}$ with a unique invariant mean.
- The final lecture will look at some extension to non discrete groups, make some remarks about groups that are not locally finite and state open questions and conjectures.