An invitation to inner model theory

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Young Set Theory Meeting

How it all started

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3. Gödel proved his result by constructing L, the smallest *inner model* of set theory.
4. L is defined as follows.
   1. $L_0 = \emptyset$.
   2. $L_{\alpha+1} = \{A \subseteq L_\alpha : A$ is first order definable over $\langle L_\alpha, \in \rangle$ with parameters $\}$.
   3. $L_\lambda = \cup_{\alpha<\lambda} L_\alpha$.
   4. $L = \cup_{\alpha \in Ord} L_\alpha$.

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   4. $L = \cup_{\alpha \in Ord} L_\alpha$.
5. (Gödel) $L \vDash ZFC + GCH$.

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Theorem (Gödel)

$\mathbb{R}^L$ is Σ^1_2.

Theorem (Shoenfield)

For $x \in \mathbb{R}$, $x \in L$ iff x is Δ^1_2 in a countable ordinal.

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3. Jensen's *fine structure*: A detailed analysis of how sets get into L.
4. Consequences of fine structure: L has rich combinatorial structure. Things like $\Box$ and $\Diamond$ hold in it.

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Proof.

Suppose not. Thus, we have $V = L$. Let κ be the least measurable cardinal. Then let U be a normal κ-complete ultrafilter on κ. Let $M = Ult(L, U)$. Then $M = L$. Let $j_U : L \to L$. We must have that $j_U(\kappa) > \kappa$ and by elementarity, $L \vDash j_U(\kappa)$ is the least measurable cardinal. Contradiction! □

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2. Solovay showed that $0^\#$ is a Π^1_2 singleton.
3. Jensen showed that $0^\#$ exists iff *covering* fails.Covering says that for any set of ordinals X there is $Y \in L$ such that $X \subseteq Y$ and $|X| = |Y| \cdot \omega_1$.
4. Thus, if there is a measurable cardinal, or if $0^\#$ exists, then V is very far from L and moreover, there is a canonical object, namely $0^\#$, which is not in L.

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Motivation: Is there then a canonical model of *ZFC* just like *L* that contains or absorbs all the complexity and the canonicity present in the universe in situations when *L* provably does not? Or is it the case that large cardinals are too complicated to coexist with such a canonical hierarchy?

A philosophical point: both canonicity and complexity in models of set theory are either a consequence or a trace of large cardinals, just like in the case of $0^{\#}$. Thus, to capture canonicity present in the universe it should be enough to capture the large cardinals present in the universe.

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"Canonical" is completely undefined.

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2. An extender E is a coherent sequence of ultrafilters. It is best to think of them as just ultrafilters that code bigger embeddings.
3. More precisely, given $j : V \to M$ such that $crit(j) = \kappa$ and given λ such that $\lambda \leq j(\kappa)$ one can define the (κ, λ)-extender E derived from j by

$$(a, X) \in E \leftrightarrow a \in \lambda^{<\omega}, X \subseteq \kappa^{|a|} \text{ and } a \in j(X).$$

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1. For $a \in \lambda^{<\omega}$, $E_a = \{X : (a, X) \in E\}$ is an ultrafilter.
2. Moreover, there is a way of taking an ultrapower of V by E so that M is "essentially" the ultrapower.

The idea continued: An example of a large cardinal axiom.

Definition

κ is superstrong iff there is λ such that there is a (κ, λ) extender E such that if $j_E : V \to M$ is the ultrapower embedding then $j_E(\kappa) = \lambda$.

The idea.

To absorb the large cardinal structure of the universe, it is natural to construct models of the form $L[\vec{E}]$ where $\vec{E}$ is a sequence of extenders.

Mouse

This idea led to the notion of a mouse. The terminology is due to Jensen and the modern notion barely resembles the original one.

Premouse

To define mice we need to define premice.

Definition

A premouse is a structure of the form $L_\alpha[\vec{E}]$ where $\vec{E}$ is a sequence of extenders. Premice usually have fine structure and to emphasize this we write $\mathcal{J}_\alpha^{\vec{E}}$.

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2. Iterability is a fancy way of saying that all the ways of taking ultrapowers and direct limits produce well-founded models. More precisely, look at the picture.

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2. An iteration strategy for $\mathcal{M}$ is a winning strategy for *II* in the iteration game on $\mathcal{M}$. Thus, if *II* plays according to her strategy then all models produced during the game will be well founded.

The inner model problem revisited.

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The core model problem. Construct a mouse that covers V.

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4. The proof that mice are canonical is the comparison lemma. First given two mice $\mathcal{M}$ and $\mathcal{N}$ we write $\mathcal{M} \trianglelefteq \mathcal{N}$ if $\mathcal{M} = \mathcal{J}_\alpha^{\vec{E}}, \mathcal{N} = \mathcal{J}_\beta^{\vec{F}}$ and $\alpha \leq \beta$ and $\vec{E} \trianglelefteq \vec{F}$.

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5. (Comparison) Given two mice $\mathcal{M}$ and $\mathcal{N}$ with iteration strategies Σ and Λ there are a Σ-iterate $\mathcal{P}$ of $\mathcal{M}$ and a Λ-iterate $\mathcal{Q}$ of $\mathcal{N}$ such that either $\mathcal{P} \trianglelefteq \mathcal{Q}$ or $\mathcal{Q} \trianglelefteq \mathcal{P}$.

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6. Thus $\mathbb{R}^{\mathcal{M}}$ is compatible with $\mathbb{R}^{\mathcal{N}}$.

Mice have *L*-like properties.

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2. Mice satisfy *GCH*.
3. $\Box$'s and $\Diamond$'s hold in almost all mice and hence, mice have rich combinatorial structure.
4. Mice have various degree of correctness. For instance, if ϕ is Σ^1_4 then $\phi \leftrightarrow \mathcal{M}_2 \vDash \phi$. Here $\mathcal{M}_2$ is the minimal proper class mouse with 2 Woodin cardinals.

The motivational problem.

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Can we find another motivation?

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1. Calibration of consistency strengths.
2. Proofs of determinacy.
3. Analysis of models of determinacy.

Theorem (Steel)

PFA, in fact the failure of square at a singular strong limit cardinal, implies that AD holds in $L(\mathbb{R})$ *and hence, there is an inner model with* ω *Woodins.*

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Theorem (Steel)

$AD^{L(\mathbb{R})}$ implies that all regular cardinals below Θ are measurable.

A new motivation.

One of the main open problems in set theory is the following conjecture.

Conjecture

PFA is equiconsistent with a supercompact cardinal.

As it is already known that one can force *PFA* from supercompact cardinals, the direction that is open is whether one can produce a model of supercompactness from a model of *PFA*. We know essentially one method of doing such things and that is via solving the inner model problem for large cardinals.

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A lot of current research is motivated by this conjecture and a new approach, that is triggered towards its resolution, has recently emerged. The approach is via developing two things at the same time.

1. Develop tools for proving determinacy from hypothesis such as *PFA*.
2. Develop tools for proving equiconsistencies between determinacy hypothesis and large cardinals.

But what kind of determinacy hypothesis?

It turns out that there is a hierarchy of determinacy axioms called *the Solovay hierarchy*. The definition is technical so buckle up.

The Solovay Sequence

First, recall that assuming AD,

$$\Theta = \sup\{\alpha : \text{there is a surjection } f : \mathbb{R} \to \alpha\}.$$

Then, again assuming AD, the Solovay sequence is a closed sequence of ordinals $\langle \theta_\alpha : \alpha \leq \Omega \rangle$ defined by:

1. $\theta_0 = \sup\{\alpha : \text{there is an ordinal definable surjection } f : \mathcal{P}(\omega) \to \alpha\}$,
2. If $\theta_\alpha < \Theta$ then $\theta_{\alpha+1} = \sup\{\beta : \text{there is a ordinal definable surjection } f : \mathcal{P}(\theta_\alpha) \to \beta\}$,
3. $\theta_\lambda = \sup_{\alpha<\lambda} \theta_\alpha$.

The Solovay hierarchy

$$AD^{+} + \Theta = \theta_0 <_{con} AD^{+} + \Theta = \theta_1 <_{con} ... AD^{+} + \Theta = \theta_\omega <_{con}$$
$$... AD^{+} + \Theta = \theta_{\omega_1} <_{con} AD^{+} + \Theta = \theta_{\omega_1+1} <_{con} \cdots$$

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$AD_{\mathbb{R}}$ + “Θ is regular” is a natural limit point of the hierarchy and is quite strong.

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2. Next, we develop tools that allow us to go back and forth between large cardinal hierarchy and the Solovay hierarchy. More precisely, given S from the Solovay hierarchy we can find a corresponding large cardinal axiom ϕ and show that S and ϕ are equiconsistent. This step is usually done by proving instances of the *Mouse Set Conjecture* and showing that HOD's of models of determinacy are essentially mice that carry large cardinals.

The mouse set conjecture.

Conjecture (The mouse set conjecture)

Assume AD^+ and that there is no inner model with superstrong cardinal. Then for any two real x and y, x is OD from y iff x is in a y-mouse.

Instance of MSC.

1. (Kripke) $x \in \Delta^1_1(y)$ iff $x \in L_{\omega_1^{CK}(y)}[y]$.
2. (Shoenfield) x is $\Delta^1_2(y)$ in a countable ordinal iff $x \in L[y]$.

A recent result.

Theorem (S.)

MSC holds in the minimal model of $AD_{\mathbb{R}}$ + *"*Θ *is regular".*

The use of *MSC*.

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Theorem (S.)

The HOD *of the minimal model of* $AD_{\mathbb{R}}$ + "Θ *is regular" is essentially a mouse. Actually, it is a hod mouse.*

Where is this going?

Theorem (S.)

PFA, in fact the failure of square at a singular strong limit cardinal, implies that there is a model containing the reals and ordinals and satisfying $AD_{\mathbb{R}} +$ *"Θ is regular".*

How about getting large cardinals?

Theorem (S. and Steel)

Assume $AD_{\mathbb{R}}$ + "Θ is regular". Then there is an inner model with a proper class of Woodin cardinals and strong cardinals.

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Theorem (S.)

$AD_{\mathbb{R}}$ + "Θ is regular" is consistent relative to a Woodin limit of Woodins.

The main conjecture.

Conjecture

The following is true.

1. *Supercompact cardinal is equiconsistent with the theory $AD^+ + V_\Theta^{\mathrm{HOD}} \vDash$ "there is a supercompact cardinal".*

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The following is true.

1. *Supercompact cardinal is equiconsistent with the theory* $AD^+ + V_\Theta^{\mathrm{HOD}} \vDash$ *"there is a supercompact cardinal".*
2. *Superstrong cardinal is equiconsistent with* $AD^+ + V_\Theta^{\mathrm{HOD}} \vDash$ *"there is a proper class of Woodins and strongs".*

The end.