Projective measure without projective Baire

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Outline

1. Some context
   - Some classical results on measure and category
   - Seperating category and measure (two ways)
2. Some ideas of the proof
   - Sketch of the iteration
   - Coding
   - Stratified forcing
   - Amalgamation

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Two notions of regularity

This talk is about regularity of sets in the projective hierarchy.

Two ways in which a set of reals can be regular:

- $X \subseteq \mathbb{R}$ is Lebesgue-measurable (LM) $\iff$ $X = B \Delta N$ (B Borel, N null).
- $X \subseteq \mathbb{R}$ has the Baire property (BP) $\iff$ $X = B \Delta M$, where B is Borel (or open), M meager.

We're interested in the projective hierarchy:

projective sets are Σ^1_n or Π^1_n sets, i.e. definable by a formula with quantifiers ranging over reals and real parameters.

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We don't know what's regular...

$V = L$

There is a Δ^1_2 well-ordering of $\mathbb{R}$ and thus irregular Δ^1_2-sets.

Solovay's model

If there is an inaccessible, you can force all projective sets to be measurable and have the Baire property.

Woodin cardinals...

There are models where

- every Σ^1_n set is regular (LM, BP ...)
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Seperating measure and category, one way

Do LM and BP always fail or hold at the *same level* of the projective hierarchy?

Answer: no.

Theorem (Shelah)

From just CON(ZFC) you can force:

- *all projective sets have BP*
- *but there is a projective set without LM (in fact, it's Σ^1_3).*

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Main result and its precursor

What to do next: switch roles of category and measure.

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Assume there is an inaccessible. Then, consistently

- *every set is measurable,*
- *there's a set without the Baire-property.*

Theorem (joint work with S. Friedman)

Assume there is a Mahlo and $V = L$. In a forcing extension,

- *every projective set is measurable,*
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Let κ be the least Mahlo in L.
We will force with an iteration P_κ of length κ.

- κ will be ω_1 in the end but remain Mahlo after $< \kappa$ many steps.
- At limits ξ, we don't know if P_ξ collapses the continuum; so we force to collapse it, as Jensen coding requires GCH.
- We define a set Γ which does not have BP.
- We make Γ projective using Jensen coding.
- The coding makes use of indepent κ^+-Suslin trees, to which we add branches at the very beginning.
- We use amalgamation to ensure P_κ is sufficiently homogeneous.

A sketch of the iteration

1. Force over L with $\prod_{\xi<\kappa}^{<\kappa} T(\xi)$, the κ^+-cc product of constructible κ-closed, κ^+-Suslin trees to add branches $B(\xi)$, $\xi < \kappa$.
2. In $L[\bar{B}]$, iterate for κ steps: $P_{\xi+1} =$
   - $P_\xi * \mathrm{Col}(\omega, c^{L[\bar{B}][G_\xi]})$ (at some stages)
   - $P_\xi \times \mathrm{Add}(\kappa)^L$
   - $P_\xi * J(B(\xi)_{\xi \in I})$ (to make "$r \in \Gamma$" definable for a real r)
   - $(D_\xi)_f^{\mathbb{Z}}$ - f an isomorphism of Random subalgebras of P_ξ, D_ξ dense in P_ξ
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3. Γ (the set w/o BP) $=$ "every other Cohen real" added in the iteration (closed of under automorphisms)

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Getting a projective set without BP

Question: how do we get a set without BP?

Shelah: A set containing every other Cohen real!

Let Γ be s.t. for any $\xi < \kappa$, there's a dense set of reals Cohen over V^{P_ξ} both in Γ and $\neg\Gamma$.

We collapse everthing below a Mahlo, so it's easy to find such Γ.

How do you make Γ projective?

$$r \in \Gamma \iff \exists s \Psi(s, r)$$

(where Ψ is Π^1_2)

We force the above "real by real": for every real added in the iteration, we add s by forcing.

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What's the Σ^1_3 definition of Γ?

At some stage ξ we are given r by book-keeping, and we pick $\dot{Q}_\xi$ so that the following holds in $L[\bar{B}][G_{\xi+1}]$:

$$r \in \Gamma \iff \exists s \text{ s.t. all } T(\xi) \text{ with } \xi \in I(r) \text{ have a branch in } L[s],$$

where $I(r) \subset \kappa$ and r can be obtained from $I(r)$.

I.e. let Q_ξ be Jensen coding to add s coding the right branches. In fact, we use a variant (David's trick), which makes a stronger statement true:

$$r \in \Gamma \iff \exists s \forall^* \alpha < \kappa L_\alpha[s] \models \text{just the right } T(\xi) \text{ have branches}$$

This second, stronger statement is Σ^1_3.
That $\Leftarrow$ holds (in $L[\bar{B}][G_\kappa]$) requires a careful choice of $I(r)$.

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What's $I(r)$? The Problem

The most obvious choice

$$I(r) = \{\xi \cdot \omega + n \mid n \in r\}$$

must fail: this would force a well-ordering of reals of length ω_1 in $L[\bar{B}][G_\kappa]$. Observe: if

$$1 \Vdash_{\bar{T}*P_\kappa} \exists s L_\alpha[s] \models \xi \in I(\dot{r}) \Rightarrow T(\xi) \text{ has a branch.}$$

and Φ is an automorphism of $\bar{T} * P_\kappa$, then also

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What's $I(r)$? The Solution

Let C be an $\mathrm{Add}(\kappa)^L$ generic added at stage $\xi - 1$. Set

$$I(r) = \{(\sigma, n, i) \mid \sigma \lhd C, r(n) = i\}$$

where $\lhd$ denotes "initial segment".

One can show $\Phi(\dot{C}) \neq \dot{C}$ whenever $\dot{r} \neq \Phi(\dot{r})$, for any automorphism coming from amalgamation. This uses that C is κ-closed. Thus $I(r)$ and $\Phi(I(r))$ are almost disjoint.

Finally, Ψ

$$\forall^* \alpha < \kappa \quad L_\alpha[s] \models \exists \text{ a large set } C \text{ s.t.}$$
$$(r(n) = i \text{ and } \sigma \lhd C) \Rightarrow T^\alpha(\sigma, n, i, 0) \text{ has a branch.}$$

Excuse the change of notation in the indexing of the trees.

Outline

To show we preserve cardinals:

We need a property that is

- iterable with the right support
- Jensen coding has it
- it is preserved by amalgamation.

- Jensen coding is nice because for every regular λ, you can write it as $P^\lambda * \dot{P}_\lambda$, where P^λ is (almost) λ^+-closed and $P^\lambda \Vdash P_\lambda$ is λ-centered.
- Does this iterate? We formulate an abstraction, called "stratified", satisfying above requirements.

Careful!

We do collapse everything below κ. Stratification does not help much at the final stage κ. The Mahlo-ness of κ is used to show:

- κ remains a cardinal in $L[\bar{B}]^{P_\kappa}$
- No reals are added at stage κ, every real is contained in some $L[\bar{B}]^{P_\xi}$, $\xi < \kappa$.

We need to use Easton-like Jensen coding!

P is stratified above λ_0 means we have relations for each regular $\lambda \geq \lambda_0$ such that:

1. $\preccurlyeq^\lambda$ is a pre-order on P stronger than $\leq$: a notion of direct extension
2. $\langle P, \preccurlyeq^\lambda \rangle$ is closed under definable, strategic sequences
3. $\mathbf{C}^\lambda \subseteq P \times \lambda$ is similar to a centering function
4. $\precsim^\lambda$ is a binary relation on P weaker than $\leq$
5. If $\mathbf{C}^\lambda(r) \cap \mathbf{C}^\lambda(q) \neq \emptyset$ and $r \precsim^\lambda q$ then $r \cdot q \neq 0$
6. If $r \leq q$ there is $p \preccurlyeq^\lambda q$ such that $p \precsim^\lambda r$
7. $\mathrm{dom}(\mathbf{C}^\lambda)$ is dense (in the sense of $\preccurlyeq^{\lambda'}$ for any $\lambda' < \lambda$)
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A closer look at “quasi-closure”

We work in a model of the form $L[A]$. There is a function $F\colon \lambda \times V \times P \to P$ definable by a ´$^A_1$ formula such that for any $\lambda \leq \bar\lambda$, both regular

- $F(\lambda, x, p) \preccurlyeq^{\lambda} p$
- if $p \preccurlyeq^{\bar\lambda} 1$ then $F(\lambda, x, p) \preccurlyeq^{\bar\lambda} 1$
- every λ-adequate sequence $\bar p = (p_\xi)_{\xi<\rho}$ has a greatest lower bound

where $\bar p$ is adequate iff $\rho \leq \lambda$, $\bar p$ is $\preccurlyeq^{\lambda}$-descending and there is x such that

- $p_{\xi+1} \preccurlyeq^{\lambda'} F(\lambda, x, p_\xi)$ for some regular λ'
- $\bar p$ is $\Delta^A_1(\lambda, x)$
- for limits $\bar\xi < \rho$, $p_{\bar\xi}$ is a greatest lower bound of $(p_\xi)_{\xi<\bar\xi}$.

We also need that $p \preccurlyeq^{\lambda} p_\xi$ for each $\xi < \rho$ and if all $p_\xi \preccurlyeq^{\bar\lambda} 1$, then $p \preccurlyeq^{\bar\lambda} 1$.

Diagonal support

The right support to iterate stratified forcing is diagonal support: Let λ be regular. Let $\bar{P} = (P_\xi, \dot{Q}_\xi)_{\xi<\theta}$ be an iteration of stratified forcings, and let π_ξ be the projection to P_ξ.

Definition

$$\operatorname{supp}^\lambda(p) = \{\xi \mid \pi_{\xi+1}(p) \not\preccurlyeq^\lambda \pi_\xi(p)\}$$

For diagonal support on P_θ we demand that $\operatorname{supp}(p)$ be of size $< \lambda$.
We also need to demand of $\bar{P}$ that for each regular λ there is $\iota < \lambda^+$ such that

$$\forall p \in P_\theta \quad p \preccurlyeq^\lambda \pi_\iota(p).$$

Stratified extension

When $P_{\xi+1}$ results from an amalgamation of P_ξ, $P_{\xi+1} : P_\xi$ is not forced to be stratified by P_ξ.
Therefore we introduce the notion of (Q, P) being a stratified extension above λ_0.

- $(P, P * \dot{Q})$ is a stratified extension, if $\Vdash_P Q$ is stratified
- So is $(P, P \times Q)$ if P and Q are stratified
- Same for $(P, A(P))$, where $A(P)$ denotes an amalgamation of P
- P is stratified $\iff$ $(\{1_P\}, P)$ is a stratified extension
- If (Q, P) is a stratified extension, P is stratified

Stratified extension and iteration

Most importantly:

Theorem

If $(P_\xi)_{\xi\leq\theta}$ has diagonal supports and for all $\xi < \theta$, $(P_\xi, P_{\xi+1})$ is a stratified extension, then P_θ is stratified.

Outline

How to get all sets LM.

Why do all projective sets have a measure in Solovays model? If we force with an iteration $(P_\xi, \dot{Q}_\xi)_{\xi<\kappa}$ of length κ and the following holds in V^{P_κ}:

- $\mathbb{R} \cap V^{P_\xi}$ is null (meager) for any $\xi < \kappa$
- every real is small generic, i.e. every $r \in \mathbb{R}$ is in some V^{P_ξ}, for $\xi < \kappa$.
- P_κ has many automorphisms.

Then every projective set is is measurable (has BP). In Solovays model, projective sets are both BP and LM because $\mathrm{Col}(\omega, <\kappa)$ is *very* homogeneous.
Shelah: only *just enough* automorphism to get *one* kind of regularity.

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To get all projective sets LM, P_κ has enough automorphisms means:

Extend isomorphisms of Random subalgebras

Say r_0, r_1 are Random reals over V^{P_ι}.
Let $\dot{B}_i$ be the complete sub-abgebra of $\mathrm{ro}(P_\xi : P_\iota)$ generated by r_i in V^{P_ι}, let $B_i = P_\iota * \dot{B}_i$ and let f be the isomorphism:

$$f\colon B_0 \to B_1$$

Then there is an automorphism

$$\Phi\colon P_\kappa \to P_\kappa$$

which extends f.

Here's an adaptation of Shelah's amalgamation more apt to preserve closure:
Let $f\colon B_0 \to B_1$ be an isomorphism of two sub-algebras of $\mathrm{ro}(P)$. Let $\pi_i\colon P_\xi \to B_i$ denote the canonical projection.

Amalgamation

$P_f^{\mathbb{Z}}$ consists of all $\bar{p}\colon \mathbb{Z} \to P \cdot B_0 \cdot B_1$ such that

$$\forall i \in \mathbb{Z} \quad f(\pi_0(\bar{p}(i)) = \pi_1(\bar{p}(i+1))$$

- The map $p \mapsto (\dots, f^{-1}(\pi_1(p)), p, f(\pi_0(p)), \dots)$ is a complete embedding
- The left shift is an automorphism extending f.

How amalgamation is used

- For any $\iota < \kappa$ and any two reals r_0, r_1 random over $L[\bar{B}]^{P_\iota}$ there should be $\xi < \kappa$ such that

$$P_{\xi+1} = (P_\xi)^{\mathbb{Z}}_f$$

  where $B_i = P_\iota * \dot{B}(r_i)$ and f is the isomorphism of B_0 and B_1.
- Then $P_{\xi+1}$ has an automorphism Φ
- Of course you have to extend this Φ to $\Phi' \colon P_{\xi'} \to P_{\xi'}$, for cofinally many $\xi' < \kappa$.
- Amalgamation may collapse the current ω_1.

Amalgamation and stratification

Problem: preserve some closure

- P carries an auxillary ordering $\preccurlyeq$
- Certain "adequate" $\preccurlyeq$-descending sequences have lower bounds in P
- π_i not continuous, why should

$$f(\pi_0(\bar{p}(i)) = \pi_1(\bar{p}(i+1))$$

hold for the coordinatewise limit of a sequence $\bar{p}_\xi \in P_f^{\mathbb{Z}}$?

Amalgamation and stratification

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Why should $f(\pi_0(\bar{p}(i)) = \pi_1(\bar{p}(i+1))$ hold for the coordinatewise limit of a sequence $\bar{p}_\xi \in P_f^{\mathbb{Z}}$?

Solution:

Replace P by a dense subset D, where $p \in D \iff$

$$\forall q \preccurlyeq p \quad \forall b \in B_0 \quad \pi_1(q \cdot b) = \pi_1(p \cdot b)$$

Fine point:

To show D completely embedds into $D_f^{\mathbb{Z}}$, we need

- $Q \subseteq D$
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A few questions

So projective measure does not imply projective Baire.

Questions:

- Can we make Γ Δ^1_{k+1}, keeping the Baire-property for all Σ^1_k sets, $k \geq 3$?
- For which σ-ideals can we substitute "Borel modulo I" for either of them?
- Force $\neg$CH at the same time?
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Another question

Again, the question:

How do you separate regularity properties in the projective hierarchy?

Theorem (A blueprint for a theorem)

The following is consistent, assuming small large cardinals (for any k,n):

1. *Every Σ^1_n set is regular, but there is a non-regular Δ^1_{n+1} set.*
2. *Every Σ^1_k set is regular, but there is a non-regular Δ^1_{k+1} set.*