Projective measure without projective Baire

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YST 2011
Outline

1. Some context
   - Some classical results on measure and category
   - Separating category and measure (two ways)

2. Some ideas of the proof
   - Sketch of the iteration
   - Coding
   - Stratified forcing
   - Amalgamation
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Two notions of regularity

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We’re interested in the projective hierarchy:

- projective sets are $\Sigma^1_n$ or $\Pi^1_n$ sets, i.e. definable by a formula with quantifiers ranging over reals and real parameters.
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Two ways in which a set of reals can be regular:

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D. Schrittesser  Projective (LM without BP)
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There is a \( \Delta^1_2 \) well-ordering of \( \mathbb{R} \) and thus irregular \( \Delta^1_2 \)-sets.

Solovay’s model

If there is an inaccessible, you can force all projective sets to be measurable and have the Baire property.

Woodin cardinals...

There are models where

- every \( \Sigma^1_n \) set is regular (LM, BP ...)
- irregular \( \Delta^1_{n+1} \) sets (from a well-ordering).
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- every \( \Sigma_n^1 \) set is regular (LM, BP ...)
- irregular \( \Delta_{n+1}^1 \) sets (from a well-ordering).
Do LM and BP always fail or hold at the same level of the projective hierarchy?
Some context

Some classical results on measure and category
Seperating category and measure (two ways)

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Some ideas of the proof

Sketch of the iteration
Coding
Stratified forcing
Amalgamation
Do LM and BP always fail or hold at the same level of the projective hierarchy?
Answer: no.

Theorem (Shelah)

*From just CON(ZFC) you can force:*

- all projective sets have BP
- but there is a projective set without LM (in fact, it’s $\Sigma^1_3$).
Do LM and BP always fail or hold at the same level of the projective hierarchy?
Answer: no.

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Main result and its precursor

What to do next: switch roles of category and measure.

**Theorem (Shelah)**

Assume there is an inaccessible. Then, consistently
- every set is measurable,
- there’s a set without the Baire-property.

**Theorem (joint work with S. Friedman)**

Assume there is a Mahlo and $V = L$. In a forcing extension,
- every projective set is measurable,
- there’s a $\Delta^1_3$ set without the Baire-property.

By a theorem of Shelah, we need to assume at least an inaccessible.
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Let $\kappa$ be the least Mahlo in $L$.
We will force with an iteration $P_\kappa$ of length $\kappa$.

- $\kappa$ will be $\omega_1$ in the end but remain Mahlo after $< \kappa$ many steps.
- At limits $\xi$, we don’t know if $P_\xi$ collapses the continuum; so we force to collapse it, as Jensen coding requires GCH.
- We define a set $\Gamma$ which does not have BP.
- We make $\Gamma$ projective using Jensen coding.
- The coding makes use of indepent $\kappa^+$-Suslin trees, to which we add branches at the very beginning.
- We use amalgamation to ensure $P_\kappa$ is sufficiently homogeneous.
A sketch of the iteration

1. Force over $L$ with $\prod_{\xi < \kappa}^{<\kappa} T(\xi)$, the $\kappa^+$-cc product of constructible $\kappa$-closed, $\kappa^+$-Suslin trees to add branches $B(\xi)$, $\xi < \kappa$.

2. In $L[\bar{B}]$, iterate for $\kappa$ steps: $P_{\xi+1} =$
   - $P_\xi \times \text{Col}(\omega, c^{L[\bar{B}][G_\xi]})$ (at some stages)
   - $P_\xi \times \text{Add}(\kappa)^L$
   - $P_\xi \times J(B(\xi)_{\xi \in I})$ (to make “$r \in \Gamma$” definable for a real $r$)
   - $(D_\xi)^f - f$ an isomorphism of Random subalgebras of $P_\xi$, $D_\xi$
   - Dense in $P_\xi$
   - $(P_\xi)^\mathbb{Z} - \Phi$ an automorphism added by a previous amalgamation

3. $\Gamma$ (the set w/o BP) = “every other Cohen real” added in the iteration (closed of under automorphisms)
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Getting a projective set without BP

Question: how do we get a set without BP?
Shelah: A set containing every other Cohen real!
Let $\Gamma$ be s.t. for any $\xi < \kappa$, there’s a dense set of reals Cohen over $V^{P_\xi}$ both in $\Gamma$ and $\neg \Gamma$.
We collapse everything below a Mahlo, so it’s easy to find such $\Gamma$.

How do you make $\Gamma$ projective?

$$r \in \Gamma \iff \exists s \Psi(s, r)$$

(where $\Psi$ is $\Pi^1_2$)
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What’s the $\Sigma^1_3$ definition of $\Gamma$?

At some stage $\xi$ we are given $r$ by book-keeping, and we pick $\check{Q}_\xi$ so that the following holds in $L[\check{B}][G_{\xi+1}]$:

$$r \in \Gamma \iff \exists s \text{ s.t. all } T(\xi) \text{ with } \xi \in I(r) \text{ have a branch in } L[s],$$

where $I(r) \subset \kappa$ and $r$ can be obtained from $I(r)$.

I.e. let $Q_\xi$ be Jensen coding to add $s$ coding the right branches.

In fact, we use a variant (David’s trick), which makes a stronger statement true:

$$r \in \Gamma \iff \exists s \forall^* \alpha < \kappa L_\alpha[s] \models \text{just the right } T(\xi) \text{ have branches}$$

This second, stronger statement is $\Sigma^1_3$.

That $\iff$ holds (in $L[\check{B}][G_{\kappa}]$) requires a careful choice of $I(r)$.
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where $I(r) \subset \kappa$ and $r$ can be obtained from $l(r)$.

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What’s \( I(r) \)? The Problem

The most obvious choice

\[
I(r) = \{ \xi \cdot \omega + n \mid n \in r \}
\]

must fail: this would force a well-ordering of reals of length \( \omega_1 \) in \( L[\bar{B}][G_\kappa] \). Observe: if

\[
1 \models \bar{T} \ast P_\kappa \exists s L_\alpha[s] \models \xi \in I(\dot{r}) \Rightarrow T(\xi) \text{ has a branch.}
\]

and \( \Phi \) is an automorphism of \( \bar{T} \ast P_\kappa \), then also

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1 \models \bar{T} \ast P_\kappa \exists s L_\alpha[s] \models \xi \in \Phi(I(\dot{r})) \Rightarrow T(\xi) \text{ has a branch.}
\]

I.e. we should expect \( \Gamma \) to be closed under such \( \Phi \). This makes it harder to show \( r \in \Gamma \Leftarrow \exists s \Psi(s, r) \).
What’s $I(r)$? The Problem

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What's \( l(r) \)? The Problem

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I.e. we should expect \( \Gamma \) to be closed under such \( \Phi \). This makes it harder to show \( r \in \Gamma \iff \exists s \Psi(s, r) \).
Let $C$ be an $\text{Add}(\kappa)^L$ generic added at stage $\xi - 1$. Set

$$I(r) = \{(\sigma, n, i) \mid \sigma \triangleleft C, r(n) = i\}$$

where $\triangleleft$ denotes “initial segment”.

One can show $\Phi(\dot{C}) \neq \dot{C}$ whenever $\dot{r} \neq \Phi(\dot{r})$, for any automorphism coming from amalgamation. This uses that $C$ is $\kappa$-closed. Thus $I(r)$ and $\Phi(I(r))$ are almost disjoint.
Finally, $\psi$

$$\forall^* \alpha < \kappa \quad L_\alpha[s] \models \exists \text{ a large set } C \text{ s.t.}$$

$$(r(n) = i \text{ and } \sigma \triangleleft C) \Rightarrow T^\alpha(\sigma, n, i, 0) \text{ has a branch.}$$

Excuse the change of notation in the indexing of the trees.
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To show we preserve cardinals:

We need a property that is
- iterable with the right support
- Jensen coding has it
- it is preserved by amalgamation.

Jensen coding is nice because for every regular $\lambda$, you can write it as $P^\lambda \ast \dot{P}_\lambda$, where $P^\lambda$ is (almost) $\lambda^+$-closed and $P^\lambda \Vdash P_\lambda$ is $\lambda$-centered.

Does this iterate? We formulate an abstraction, called “stratified”, satisfying above requirements.
Careful!

We do collapse everything below $\kappa$. Stratification does not help much at the final stage $\kappa$. The Mahlo-ness of $\kappa$ is used to show:

- $\kappa$ remains a cardinal in $L[\mathcal{B}]^{P_\kappa}$
- No reals are added at stage $\kappa$, every real is contained in some $L[\mathcal{B}]^{P_\xi}$, $\xi < \kappa$.

We need to use Easton-like Jensen coding!
$P$ is stratified above $\lambda_0$ means we have relations for each regular $\lambda \geq \lambda_0$ such that:

1. $\preccurlyeq^\lambda$ is a pre-order on $P$ stronger than $\leq$: a notion of direct extension
2. $\langle P, \preccurlyeq^\lambda \rangle$ is closed under definable, strategic sequences
3. $C^\lambda \subseteq P \times \lambda$ is similar to a centering function
4. $\preccurlyeq^\lambda$ is a binary relation on $P$ weaker than $\leq$
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8. $C^\lambda$ is “continuous”.
Some context
Some ideas of the proof
Questions

Sketch of the iteration
Coding
Stratified forcing
Amalgamation

$P$ is stratified above $\lambda_0$ means we have relations for each regular $\lambda \geq \lambda_0$ such that:

1. $\triangleleft^\lambda$ is a pre-order on $P$ stronger than $\leq$: a notion of direct extension
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D. Schrittesser
Projective (LM without BP)
$P$ is stratified above $\lambda_0$ means we have relations for each regular $\lambda \geq \lambda_0$ such that:

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Projective (LM without BP)
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D. Schrittesser
Projective (LM without BP)
A closer look at “quasi-closure”

We work in a model of the form $L[A]$. There is a function $F: \lambda \times V \times P \to P$ definable by a $\mathcal{A}^1$ formula such that for any $\lambda \leq \bar{\lambda}$, both regular

- $F(\lambda, x, p) \leq^\lambda p$
- if $p \leq^\lambda 1$ then $F(\lambda, x, p) \leq^\bar{\lambda} 1$
- every $\lambda$-adequate sequence $\bar{p} = (p_\xi)_{\xi<\rho}$ has a greatest lower bound

where $\bar{p}$ is adequate iff $\rho \leq \lambda$, $\bar{p}$ is $\leq^\lambda$-descending and there is $x$ such that

- $p_{\xi+1} \leq^{\lambda'} F(\lambda, x, p_\xi)$ for some regular $\lambda'$
- $\bar{p}$ is $\Delta^A_1(\lambda, x)$
- for limits $\xi < \rho$, $p_\xi$ is a greatest lower bound of $(p_\xi)_{\bar{\xi}<\bar{\xi}}$

We also need that $p \leq^\lambda p_\xi$ for each $\xi < \rho$ and if all $p_\xi \leq^\bar{\lambda} 1$, then $p \leq^\bar{\lambda} 1$. 
The right support to iterate stratified forcing is diagonal support: Let \( \lambda \) be regular. Let \( \bar{P} = (P_\xi, Q_\xi)_{\xi < \theta} \) be an iteration of stratified forcings, and let \( \pi_\xi \) be the projection to \( P_\xi \).

**Definition**

\[
\text{supp}^\lambda(p) = \{ \xi \mid \pi_{\xi+1}(p) \not\leq^\lambda \pi_\xi(p) \}
\]

For diagonal support on \( P_\theta \) we demand that \( \text{supp}(p) \) be of size \(< \lambda\).

We also need to demand of \( \bar{P} \) that for each regular \( \lambda \) there is \( \iota < \lambda^+ \) such that

\[
\forall p \in P_\theta \quad p \leq^\lambda \pi_\iota(p).
\]
When $P_{\xi+1}$ results from an amalgamation of $P_{\xi}$, $P_{\xi+1} : P_{\xi}$ is not forced to be stratified by $P_{\xi}$. Therefore we introduce the notion of $(Q, P)$ being a stratified extension above $\lambda_0$.

- $(P, P \ast \dot{Q})$ is a stratified extension, if $\Vdash_P Q$ is stratified
- So is $(P, P \times Q)$ if $P$ and $Q$ are stratified
- Same for $(P, A(P))$, where $A(P)$ denotes an amalgamation of $P$
- $P$ is stratified $\iff (\{1_P\}, P)$ is a stratified extension
- If $(Q, P)$ is a stratified extension, $P$ is stratified
Most importantly:

**Theorem**

If \( (P_\xi)_{\xi \leq \theta} \) has diagonal supports and for all \( \xi < \theta \), \( (P_\xi, P_{\xi+1}) \) is a stratified extension, then \( P_\theta \) is stratified.
Outline

1. Some context
   - Some classical results on measure and category
   - Separating category and measure (two ways)

2. Some ideas of the proof
   - Sketch of the iteration
   - Coding
   - Stratified forcing
   - Amalgamation
How to get all sets LM.

Why do all projective sets have a measure in Solovay's model? If we force with an iteration \((P_\xi, \dot{Q}_\xi)_{\xi<\kappa}\) of length \(\kappa\) and the following holds in \(V^{P_\kappa}\):

- \(\mathbb{R} \cap V^{P_\xi}\) is null (meager) for any \(\xi < \kappa\)
- every real is small generic, i.e. every \(r \in \mathbb{R}\) is in some \(V^{P_\xi}\), for \(\xi < \kappa\).
- \(P_\kappa\) has many automorphisms.

Then every projective set is measurable (has BP). In Solovay's model, projective sets are both BP and LM because \(\text{Col}(\omega, < \kappa)\) is very homogeneous. Shelah: only just enough automorphism to get one kind of regularity.
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To get all projective sets LM, $P_\kappa$ has enough automorphisms means:

**Extend isomorphisms of Random subalgebras**

Say $r_0, r_1$ are Random reals over $V^{P_\iota}$. Let $\hat{B}_i$ be the complete sub-algebra of $ro(P_\xi : P_\iota)$ generated by $r_i$ in $V^{P_\iota}$, let $B_i = P_\iota \ast \hat{B}_i$ and let $f$ be the isomorphism:

$$f : B_0 \to B_1$$

Then there is an automorphism

$$\Phi : P_\kappa \to P_\kappa$$

which extends $f$. 
Here’s an adaptation of Shelah’s amalgamation more apt to preserve closure:
Let \( f : B_0 \to B_1 \) be an isomorphism of two sub-algebras of \( \text{ro}(P) \). Let \( \pi_i : P_\xi \to B_i \) denote the canonical projection.

### Amalgamation

\( P_f^Z \) consists of all \( \bar{p} : \mathbb{Z} \to P \cdot B_0 \cdot B_1 \) such that

\[
\forall i \in \mathbb{Z} \quad f(\pi_0(\bar{p}(i))) = \pi_1(\bar{p}(i + 1))
\]

- The map \( p \mapsto (\ldots, f^{-1}(\pi_1(p)), p, f(\pi_0(p)), \ldots) \) is a complete embedding
- The left shift is an automorphism extending \( f \).
How amalgamation is used

- For any \( \iota < \kappa \) and any two reals \( r_0, r_1 \) random over \( L[\bar{B}]^{P_{\iota}} \) there should be \( \xi < \kappa \) such that
  \[
  P_{\xi+1} = (P_{\xi})^{\mathbb{Z}}
  \]
  where \( B_i = P_{\iota} \ast \dot{B}(r_i) \) and \( f \) is the isomorphism of \( B_0 \) and \( B_1 \).
- Then \( P_{\xi+1} \) has an automorphism \( \Phi \)
- Of course you have to extend this \( \Phi \) to \( \Phi' : P_{\xi'} \rightarrow P_{\xi'} \), for cofinally many \( \xi' < \kappa \).
- Amalgamation may collapse the current \( \omega_1 \).
Amalgamation and stratification

Problem: preserve some closure

- $P$ carries an auxiliary ordering $\preccurlyeq$
- Certain “adequate” $\preccurlyeq$-descending sequences have lower bounds in $P$
- $\pi_i$ not continuous, why should

$$f(\pi_0(\bar{p}(i))) = \pi_1(\bar{p}(i + 1))$$

hold for the coordinatewise limit of a sequence $\bar{p}_\xi \in P_f^\mathbb{Z}$?
Problem: preserve some closure

Why should $f(\pi_0(\bar{p}(i))) = \pi_1(\bar{p}(i + 1))$ hold for the coordinatewise limit of a sequence $\bar{p}_\xi \in P^\mathbb{Z}_f$?

Solution:

Replace $P$ by a dense subset $D$, where $p \in D$ if:

$$\forall q \preceq p \quad \forall b \in B_0 \quad \pi_1(q \cdot b) = \pi_1(p \cdot b)$$

Fine point:

To show $D$ completely embeds into $D^\mathbb{Z}_f$, we need:

- $Q \subseteq D$
- $Q \cdot D \subseteq D$. 
Amalgamation and stratification

Problem: preserve some closure

Why should \( f(\pi_0(\bar{p}(i))) = \pi_1(\bar{p}(i + 1)) \) hold for the coordinatewise limit of a sequence \( \bar{p}_\xi \in P_f^\mathbb{Z} \)?

Solution:

Replace \( P \) by a dense subset \( D \), where \( p \in D \iff \forall q \preceq p \forall b \in B_0 \pi_1(q \cdot b) = \pi_1(p \cdot b) \)

Fine point:
To show \( D \) completely embedds into \( D_f^\mathbb{Z} \), we need

- \( Q \subseteq D \)
- \( Q \cdot D \subseteq D \).
A few questions

So projective measure does not imply projective Baire.

Questions:

- Can we make $\Gamma \Delta^1_{k+1}$, keeping the Baire-property for all $\Sigma^1_k$ sets, $k \geq 3$?
- For which $\sigma$-ideals can we substitute “Borel modulo $I$” for either of them?
- Force $\neg$CH at the same time?
- Prove the Mahlo is necessary or get rid of it?
So projective measure does not imply projective Baire.

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- For which $\sigma$-ideals can we substitute “Borel modulo $I$” for either of them?
- Force $\neg$CH at the same time?
- Prove the Mahlo is necessary or get rid of it?
Again, the question:
How do you separate regularity properties in the projective hierarchy?

Theorem (A blueprint for a theorem)

The following is consistent, assuming small large cardinals (for any $k,n$):

1. Every $\Sigma^1_n$ set is regular, but there is a non-regular $\Delta^1_{n+1}$ set.
2. Every $\Sigma^1_k$ set is regular, but there is a non-regular $\Delta^1_{k+1}$ set.