Around Jensen’s square principle

Young Researchers in Set Theory

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Introduction
Ladder systems. A discussion

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Remark
The existence of ladder systems follows from the axiom of choice.
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Remark
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Partitioning a stationary set
The standard proof of the fact that any stationary subset of $\omega_1$ can be partitioned into uncountably many mutually disjoint stationary sets builds on an analysis of ladder systems over $\omega_1$.

Strong colorings, $\omega_1 \not\rightarrow [\omega_1]^2_{\omega_1}$
Todorcevic established the existence of a function $f : [\omega_1]^2 \rightarrow \omega_1$ such that $f``[U]^2 = \omega_1$ for every uncountable $U \subseteq \omega_1$. This function $f$ is determined by a ladder system over $\omega_1$. 
A particular ladder system

Definition (Jensen, 1960’s)

\( \square \lambda \) asserts the existence of a ladder system over \( \lambda^+ \),
\( \langle C_\alpha \mid \alpha < \lambda^+ \rangle \), such that for all \( \alpha < \lambda^+ \):

- (Ladders are closed) \( C_\alpha \) is a club in \( \alpha \);
- (Ladders are of bounded type) \( \text{otp}(C_\alpha) \leq \lambda \);
- (Coherence) if \( \text{sup}(C_\alpha \cap \beta) = \beta \), then \( C_\alpha \cap \beta = C_\beta \).
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\[ \langle C_\alpha \mid \alpha < \lambda^+ \rangle, \text{ such that for all } \alpha < \lambda^+: \]
  ▶ (Ladders are closed) \( C_\alpha \) is a club in \( \alpha \);
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  ▶ (Coherence) if \( \sup(C_\alpha \cap \beta) = \beta \), then \( C_\alpha \cap \beta = C_\beta \).

Famous applications
The existence of various sorts of \( \lambda^+ \)-trees; The existence of non-reflecting stationary subsets of \( \lambda^+ \); The existence of other incompact objects.
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$\langle C_\alpha \mid \alpha < \lambda^+ \rangle$, such that for all $\alpha < \lambda^+$:

- (Ladders are closed) $C_\alpha$ is a club in $\alpha$;
- (Ladders are of bounded type) otp($C_\alpha$) $\leq \lambda$;
- (Coherence) if $\sup(C_\alpha \cap \beta) = \beta$, then $C_\alpha \cap \beta = C_\beta$.

Today’s talk would be centered around the above principle, but let us dedicate some time to discuss abstract ladder systems.
Triviality of ladder systems

Means of triviality
A ladder system $\langle A_\alpha \mid \alpha < \kappa \rangle$ is considered to be trivial, if, in some sense, it is determined by a single $\kappa$-sized object.

Example of such sense: "There exists $A \subseteq \kappa$ such that $A_\alpha = A \cap \alpha$ for club many $\alpha < \kappa$."
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Example of such sense:
“There exists $A \subseteq \kappa$ such that $A_\alpha = A \cap \alpha$ for club many $\alpha < \kappa$.”
If $\kappa$ is a large cardinal, then we may necessarily face means of triviality.

Fact (Rowbottom, 1970’s)

*If $\kappa$ is measurable, then every ladder system $\langle A_\alpha \mid \alpha < \kappa \rangle$, admits a set $A \subseteq \kappa$ such that $A_\alpha = A \cap \alpha$ for stationary many $\alpha < \kappa$.***
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Example of such sense:
“There exists \( A \subseteq \kappa \) such that \( A_\alpha = A \cap \alpha \) for club many \( \alpha < \kappa \).”

On the other hand, if \( \kappa \) is non-Mahlo, then for every cofinal \( A \subseteq \kappa \), the following set contains a club:

\[
\{ \alpha < \kappa \mid \text{cf}(\alpha) < \text{otp}(A \cap \alpha) \}.
\]

This suggests that non-triviality may be insured here, by setting a global bound on \( \text{otp}(A_\alpha) \), e.g., letting \( \text{otp}(A_\alpha) = \text{cf}(\alpha) \) for all \( \alpha \).
Triviality of ladder systems

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A ladder system $\langle A_\alpha \mid \alpha < \kappa \rangle$ is considered to be trivial, if, in some sense, it is determined by a single $\kappa$-sized object.

It turns out that requiring that $\text{otp}(A_\alpha) = \text{cf}(\alpha)$ for all $\alpha$ does not eliminate all means of triviality. For instance, it may be the case that any sequence of functions defined on the ladders is necessarily induced from a single $\kappa$-sized object.
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It turns out that requiring that $\text{otp}(A_\alpha) = \text{cf}(\alpha)$ for all $\alpha$ does not eliminate all means of triviality. For instance, it may be the case that any sequence of functions defined on the ladders is necessarily induced from a single $\kappa$-sized object.

Fact (Devlin-Shelah, 1978)

$\text{MA}_{\omega_1}$ implies that any ladder system $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ satisfying $\text{otp}(A_\alpha) = \text{cf}(\alpha)$ for every $\alpha$, is trivial in the following sense. For every sequence of local functions $\langle f_\alpha : A_\alpha \to 2 \mid \alpha < \omega_1 \rangle$ there exists a global function $f : \omega_1 \to 2$ such that for each $\alpha$:

$$f_\alpha = f \upharpoonright A_\alpha \ (\text{mod finite}).$$
Nontrivial ladder systems over $\omega_1$

In contrast, the following concept yields a ladder system which is resistant to Devlin and Shelah’s notion of triviality.

**Definition (Ostaszweski’s ♣)**

♣ asserts the existence of a ladder system $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ such that for every cofinal $A \subseteq \omega_1$, there exists a limit $\alpha < \omega_1$ with $A_\alpha \subseteq A$. 
Nontrivial ladder systems over $\omega_1$

In contrast, the following concept yields a ladder system which is resistant to Devlin and Shelah’s notion of triviality.

**Definition (Ostaszewski’s ♣)**

♣ asserts the existence of a ladder system $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ such that for every cofinal $A \subseteq \omega_1$, there exists a limit $\alpha < \omega_1$ with $A_\alpha \subseteq A$.

Indeed, if $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ is a ♣-sequence, then for every global $f : \omega_1 \rightarrow 2$, there exists a limit $\alpha < \omega_1$ for which $f \upharpoonright A_\alpha$ is constant.

Thus, if $f_\alpha : A_\alpha \rightarrow 2$ partitions $A_\alpha$ into two cofinal subsets for all limit $\alpha$, then no global $f$ trivializes the sequence $\langle f_\alpha \mid \alpha < \omega_1 \rangle$. 
Suppose that $\kappa = \lambda^+$ is a successor cardinal. Thus, we are interested in a ladder system $\langle A_\alpha \mid \alpha < \kappa \rangle$ with ALL of the following features:

1. the set $\{\otp(A_\alpha) \mid \alpha < \kappa\}$ is bounded below $\kappa$;
2. the ladders are closed;
3. the ladders cohere;
4. yields a canonical partition of $\kappa$ into mutually disjoint stationary sets;
5. induces strong colorings;
6. a non-triviality condition à la Devlin-Shelah.
The Ostaszewski square
We propose a principle which combines $\square_\lambda$ together with $\clubsuit_\lambda^+$. 

\textit{$\lambda$-sequences}
We propose a principle which combines $\square_\lambda$ together with $\clubsuit_\lambda^+$. For clarity, let us adopt the next ad-hoc terminology:

**Definition**
A sequence $\langle A_i \mid i < \lambda \rangle$ is a $\lambda$-sequence if the following two holds:

1. each $A_i$ is a cofinal subset of $\lambda^+$;
2. if $i < \lambda$ is a limit ordinal, then $A_i$ is moreover closed.

Remark. Clause (2) may be viewed as a continuity condition.
The Ostaszewski square

Definition

\[ \spadesuit \lambda \text{ asserts the existence of a ladder system } \vec{C} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle \]

such that:

- \( \text{otp}(C_\alpha) \leq \lambda \) for all \( \alpha < \lambda^+ \);
- \( C_\alpha \) is a club in \( \alpha \) for all limit \( \alpha < \lambda^+ \);
- if \( \sup(C_\alpha \cap \beta) = \beta \), then \( C_\alpha \cap \beta = C_\beta \);
The Ostaszewski square

**Definition**

\[ \clubsuit_\lambda \] asserts the existence of a ladder system \( \vec{\mathcal{C}} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle \) such that:

- \( \vec{\mathcal{C}} \) is a \( \square_\lambda \)-sequence. Let \( C_\alpha(i) \) denote the \( i_{th} \) element of \( C_\alpha \).
The Ostaszewski square

Definition

♣\(\lambda\) asserts the existence of a ladder system \(\vec{C} = \langle C_\alpha | \alpha < \lambda^+ \rangle\) such that:

- \(\vec{C}\) is a □\(\lambda\)-sequence. Let \(C_\alpha(i)\) denote the \(i\)th element of \(C_\alpha\).
- Suppose that \(\langle A_i | i < \lambda \rangle\) is a \(\lambda\)-sequence. Then for every cofinal \(B \subseteq \lambda^+\), and every limit \(\theta < \lambda\), there exists some \(\alpha < \lambda^+\) such that:
  1. \(\text{otp}(C_\alpha) = \theta\);
  2. for all \(i < \theta\), \(C_\alpha(i) \in A_i\);
  3. for all \(i < \theta\), there exists \(\beta_i \in B\) with \(C_\alpha(i) < \beta_i < C_\alpha(i + 1)\).
The Ostaszewski square (cont.)

The Ostaszewski square \(\clubsuit\lambda\) asserts the existence of a \(\square\lambda\)-sequence \(\langle C_\alpha \mid \alpha < \lambda^+ \rangle\) such that for every \(\lambda\)-sequence \(\langle A_i \mid i < \lambda \rangle\), every cofinal \(B \subseteq \lambda^+\), and every limit \(\theta < \lambda\), there exists some \(\alpha < \lambda^+\) such that:

1. the inverse collapse of \(C_\alpha\) is an element of \(\prod_{i<\theta} A_i\);
\[\clubsuit_{\lambda}\] asserts the existence of a \(\square_{\lambda}\)-sequence \(\langle C_\alpha \mid \alpha < \lambda^+ \rangle\) such that for every \(\lambda\)-sequence \(\langle A_i \mid i < \lambda \rangle\), every cofinal \(B \subseteq \lambda^+\), and every limit \(\theta < \lambda\), there exists some \(\alpha < \lambda^+\) such that:

1. the inverse collapse of \(C_\alpha\) is an element of \(\prod_{i<\theta} A_i\);
2. if \(\gamma < \delta\) belong to \(C_\alpha\), then \(B \cap (\gamma, \delta) \neq \emptyset\).
\(\clubsuit_\lambda\) asserts the existence of a \(\square_\lambda\)-sequence \(<C_\alpha \mid \alpha < \lambda^+>\) such that for every \(\lambda\)-sequence \(<A_i \mid i < \lambda>\), every cofinal \(B \subseteq \lambda^+\), and every limit \(\theta < \lambda\), there exists some \(\alpha < \lambda^+\) such that:

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1. the inverse collapse of \( C_\alpha \) is an element of \( \prod_{i<\theta} A_i \);
2. if \( \gamma < \delta \) belong to \( C_\alpha \), then \( B \cap (\gamma, \delta) \neq \emptyset \).

**Feature 1. Club guessing**

For every club \( D \subseteq \lambda^+ \), there exists \( \alpha < \lambda^+ \) such that \( C_\alpha \subseteq D \).
The Ostaszewski square (cont.)

\[\clubsuit \lambda\] asserts the existence of a \(\square \lambda\)-sequence \(\langle C_\alpha \mid \alpha < \lambda^+ \rangle\) such that for every \(\lambda\)-sequence \(\langle A_i \mid i < \lambda \rangle\), every cofinal \(B \subseteq \lambda^+\), and every limit \(\theta < \lambda\), there exists some \(\alpha < \lambda^+\) such that:

1. the inverse collapse of \(C_\alpha\) is an element of \(\prod_{i<\theta} A_i\);
2. if \(\gamma < \delta\) belong to \(C_\alpha\), then \(B \cap (\gamma, \delta) \neq \emptyset\).

Feature 2. \[\spadesuit \lambda^+\]

For every cofinal \(A \subseteq \lambda^+\), there exists \(\alpha < \lambda^+\) s.t. \(nacc(C_\alpha) \subseteq A\).\(^a\)

\[^a\]nacc\((C_\alpha) = C_\alpha \setminus \text{acc}(C_\alpha)\), where acc\((C_\alpha) := \{\beta \in C_\alpha \mid \sup(C_\alpha \cap \beta) = \beta\}\).
The Ostaszewski square (cont.)

\( \clubsuit_{\lambda} \) asserts the existence of a \( \square_{\lambda} \)-sequence \( \langle C_{\alpha} \mid \alpha < \lambda^+ \rangle \) such that for every \( \lambda \)-sequence \( \langle A_i \mid i < \lambda \rangle \), every cofinal \( B \subseteq \lambda^+ \), and every limit \( \theta < \lambda \), there exists some \( \alpha < \lambda^+ \) such that:

1. the inverse collapse of \( C_{\alpha} \) is an element of \( \prod_{i<\theta} A_i \);  
2. if \( \gamma < \delta \) belong to \( C_{\alpha} \), then \( B \cap (\gamma, \delta) \neq \emptyset \).

Feature 3. Canonical partition to stationary sets

Denote \( S_\theta := \{ \alpha < \lambda^+ \mid \text{otp}(C_{\alpha}) = \theta \} \).

Then \( \langle S_\theta \mid 0 \in \theta \in \text{acc}(\lambda) \rangle \) is a canonical partition of the set of limit ordinals \( < \lambda^+ \) into \( \lambda \) many mutually disjoint stationary sets.
\( \spadesuit_\lambda \) asserts the existence of a \( \square_\lambda \)-sequence \( \langle C_\alpha \mid \alpha < \lambda^+ \rangle \) such that for every \( \lambda \)-sequence \( \langle A_i \mid i < \lambda \rangle \), every cofinal \( B \subseteq \lambda^+ \), and every limit \( \theta < \lambda \), there exists some \( \alpha < \lambda^+ \) such that:

1. the inverse collapse of \( C_\alpha \) is an element of \( \prod_{i<\theta} A_i \);
2. if \( \gamma < \delta \) belong to \( C_\alpha \), then \( B \cap (\gamma, \delta) \neq \emptyset \).

**Feature 4. Simultaneous \( \clubsuit_\lambda^{\lambda^+} \) & Club guessing**

For every cofinal \( A \subseteq \lambda^+ \), every club \( D \subseteq \lambda^+ \), and every \( \theta < \lambda \), there exists \( \alpha \in S_\theta \) such that \( \text{nacc}(C_\alpha) \subseteq A \), and \( \text{acc}(C_\alpha) \subseteq D \).
\[\clubsuit_\lambda\] asserts the existence of a \(\square_\lambda\)-sequence \(\langle C_\alpha \mid \alpha < \lambda^+ \rangle\) such that for every \(\lambda\)-sequence \(\langle A_i \mid i < \lambda \rangle\), every cofinal \(B \subseteq \lambda^+\), and every limit \(\theta < \lambda\), there exists some \(\alpha < \lambda^+\) such that:

1. the inverse collapse of \(C_\alpha\) is an element of \(\prod_{i<\theta} A_i\);
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Further features
We shall now turn to discuss further features.
Simple constructions of higher Souslin trees
\[\lambda^+-\text{Souslin trees}\]

Jensen proved that “GCH + \square_{\lambda} + \diamond_S\] for all stationary \(S \subseteq \lambda^+\)” yields the existence of a \(\lambda^+\)-Souslin tree, for every singular \(\lambda\). We now suggest a simple construction from a related hypothesis.
λ⁺-Souslin trees

Jensen proved that “GCH + ♠_λ + ♦_S for all stationary S ⊆ λ⁺” yields the existence of a λ⁺-Souslin tree, for every singular λ. We now suggest a simple construction from a related hypothesis.

Proposition

Suppose that λ is an uncountable cardinal. If ♣_λ + ♦_λ⁺ holds, then there exists a λ⁺-Souslin tree.
Jensen proved that “GCH $+ \Box_\lambda + \diamond S$ for all stationary $S \subseteq \lambda^+$” yields the existence of a $\lambda^+$-Souslin tree, for every singular $\lambda$. We now suggest a simple construction from a related hypothesis.

Proposition
Suppose that $\lambda$ is an uncountable cardinal.
If $\clubsuit_\lambda + \diamond_{\lambda^+}$ holds, then there exists a $\lambda^+$-Souslin tree.

Conventions
A $\kappa$-tree $T$ is a tree of height $\kappa$, whose underlying set is $\kappa$, and levels are of size $< \kappa$.
The $\alpha_{th}$-level is denoted $T_\alpha$, and we write $T \upharpoonright \beta := \bigcup_{\alpha < \beta} T_\alpha$.
$T$ is $\kappa$-Souslin if it is ever-branching and has no $\kappa$-sized antichains.
\( \lambda^+ \)-Souslin trees

**Proposition**

Suppose that \( \lambda \) is an uncountable cardinal.
If \( \clubsuit_\lambda + \diamondsuit_{\lambda^+} \) holds, then there exists a \( \lambda^+ \)-Souslin tree.

**Proof.**

Let \( \langle C_\alpha \mid \alpha < \lambda^+ \rangle \) witness \( \clubsuit_\lambda \), and \( \langle S_\gamma \mid \gamma < \lambda^+ \rangle \) witness \( \diamondsuit_{\lambda^+} \).
We build the \( \lambda^+ \)-Souslin tree, \( T \), by recursion on the levels.
\( \lambda^+ \)-Souslin trees

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Suppose that \( \lambda \) is an uncountable cardinal.
If \( \clubsuit \lambda + \diamondsuit_{\lambda^+} \) holds, then there exists a \( \lambda^+ \)-Souslin tree.

Proof.
Let \( \langle C_{\alpha} \mid \alpha < \lambda^+ \rangle \) witness \( \clubsuit \lambda \), and \( \langle S_{\gamma} \mid \gamma < \lambda^+ \rangle \) witness \( \diamondsuit_{\lambda^+} \).
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Set \( T_0 := \{0\} \).
 Proposition
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If $\clubsuit_\lambda + \diamondsuit_{\lambda^+}$ holds, then there exists a $\lambda^+$-Souslin tree.

 Proof.
Let $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ witness $\clubsuit_\lambda$, and $\langle S_\gamma \mid \gamma < \lambda^+ \rangle$ witness $\diamondsuit_{\lambda^+}$. We build the $\lambda^+$-Souslin tree, $T$, by recursion on the levels. Set $T_0 := \{0\}$. If $T \upharpoonright \alpha + 1$ is defined, $T_{\alpha+1}$ is obtained by providing each element of $T_\alpha$ with two successors in $T_{\alpha+1}$. 
Proposition

Suppose that $\lambda$ is an uncountable cardinal. If $\clubsuit_\lambda + \diamondsuit_{\lambda^+}$ holds, then there exists a $\lambda^+$-Souslin tree.

Proof.

Let $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ witness $\clubsuit_\lambda$, and $\langle S_\gamma \mid \gamma < \lambda^+ \rangle$ witness $\diamondsuit_{\lambda^+}$. We build the $\lambda^+$-Souslin tree, $T$, by recursion on the levels. Set $T_0 := \{0\}$. If $T \upharpoonright \alpha + 1$ is defined, $T_{\alpha+1}$ is obtained by providing each element of $T_\alpha$ with two successors in $T_{\alpha+1}$. Assume now that $\alpha$ is a limit ordinal; for every $x \in T \upharpoonright \alpha$, we attach a sequence $x_\alpha$ which is increasing and cofinal in $T \upharpoonright \alpha$, and then $T_\alpha$ is defined as the limit of all these sequences. Consequently, the outcome $T_\alpha$ is of size $\leq |T \upharpoonright \alpha| \leq \lambda$.  

$\lambda^+$-Souslin trees
\( \lambda^+ \)-Souslin trees (cont.)

For every \( x \in T \upharpoonright \alpha \), pick \( x_\alpha = \langle x_\alpha(\gamma) \mid \gamma \in C_\alpha \setminus \text{ht}(x) + 1 \rangle \) s.t.:

1. \( \text{ht}(x_\alpha(\gamma)) = \gamma \) for all \( \gamma \in \text{dom}(x_\alpha) \);
2. \( x < x_\alpha(\gamma_1) < x_\alpha(\gamma_2) \) whenever \( \gamma_1 < \gamma_2 \);
3. If \( \gamma \in \text{nacc}(\text{dom}(x_\alpha)) \), and \( S_\gamma \) is a maximal antichain in \( T \upharpoonright \gamma \), then \( x_\alpha(\gamma) \) happens to be above some element from \( S_\gamma \).
\(\lambda^+\)-Souslin trees (cont.)

For every \(x \in T \upharpoonright \alpha\), pick \(x_\alpha = \langle x_\alpha(\gamma) \mid \gamma \in C_\alpha \setminus \text{ht}(x) + 1 \rangle\) s.t.:

1. \(\text{ht}(x_\alpha(\gamma)) = \gamma\) for all \(\gamma \in \text{dom}(x_\alpha)\);
2. \(x < x_\alpha(\gamma_1) < x_\alpha(\gamma_2)\) whenever \(\gamma_1 < \gamma_2\);
3. If \(\gamma \in \text{nacc}(\text{dom}(x_\alpha))\), and \(S_\gamma\) is a maximal antichain in \(T \upharpoonright \gamma\), then \(x_\alpha(\gamma)\) happens to be above some element from \(S_\gamma\).

If we make sure to choose \(x_\alpha(\gamma)\) in a canonical way (e.g., using a well-ordering), then the coherence of the square sequence implies that the branches cohere: \(\sup(C_\alpha \cap \delta) = \delta\) implies \(x_\delta = x_\alpha \upharpoonright \delta\).

In turn, we get that the whole construction may be carried, ending up with a \(\lambda^+\)-tree.
$\lambda^+$-Souslin trees (cont.)

For every $x \in T \upharpoonright \alpha$, pick $x_\alpha = \langle x_\alpha(\gamma) \mid \gamma \in C_\alpha \setminus \text{ht}(x) + 1 \rangle$ s.t.:

1. $\text{ht}(x_\alpha(\gamma)) = \gamma$ for all $\gamma \in \text{dom}(x_\alpha)$;
2. $x < x_\alpha(\gamma_1) < x_\alpha(\gamma_2)$ whenever $\gamma_1 < \gamma_2$;
3. If $\gamma \in \text{nacc}(\text{dom}(x_\alpha))$, and $S_\gamma$ is a maximal antichain in $T \upharpoonright \gamma$, then $x_\alpha(\gamma)$ happens to be above some element from $S_\gamma$.

Sousliness: towards a contradiction, suppose that $A \subseteq \lambda^+$ is an antichain in $T$ of size $\lambda^+$. By $\Diamond_{\lambda^+}$, the following set is stationary

$$A' := \{\gamma < \lambda^+ \mid A \cap \gamma = S_\gamma \text{ is a maximal antichain in } T \upharpoonright \gamma\}.$$
For every $x \in T \upharpoonright \alpha$, pick $x_\alpha = \langle x_\alpha(\gamma) \mid \gamma \in C_\alpha \setminus \text{ht}(x) + 1 \rangle$ s.t.:

1. $\text{ht}(x_\alpha(\gamma)) = \gamma$ for all $\gamma \in \text{dom}(x_\alpha)$;
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$$A' := \{ \gamma < \lambda^+ \mid A \cap \gamma = S_\gamma \text{ is a maximal antichain in } T \upharpoonright \gamma \}$$

Let $\langle A_i \mid i < \lambda \rangle$ be a $\lambda$-sequence with $A_{i+1} = A'$ for all $i < \lambda$. Pick $\alpha < \lambda^+$ such that $C_\alpha(i) \in A_i$ for all $i < \text{otp}(C_\alpha)$.

Then $\text{nacc}(C_\alpha) \subseteq A'$, and hence clause (3) above applies to the construction of $x_\alpha$ for each and every $x \in T \upharpoonright \alpha$. 

λ⁺-Souslin trees (cont.)
For every $x \in T \upharpoonright \alpha$, pick $x_\alpha = \langle x_\alpha(\gamma) \mid \gamma \in C_\alpha \setminus \text{ht}(x) + 1 \rangle$ s.t.:

1. $\text{ht}(x_\alpha(\gamma)) = \gamma$ for all $\gamma \in \text{dom}(x_\alpha)$;
2. $x < x_\alpha(\gamma_1) < x_\alpha(\gamma_2)$ whenever $\gamma_1 < \gamma_2$;
3. If $\gamma \in \text{nacc}(\text{dom}(x_\alpha))$, and $S_\gamma$ is a maximal antichain in $T \upharpoonright \gamma$, then $x_\alpha(\gamma)$ happens to be above some element from $S_\gamma$.

Sousliness: towards a contradiction suppose that $A \subseteq \lambda^+$ is an antichain in $T$ of size $\lambda^+$. By $\diamondsuit_{\lambda^+}$, the following set is stationary

$$A' := \{ \gamma < \lambda^+ \mid A \cap \gamma = S_\gamma \text{ is a maximal antichain in } T \upharpoonright \gamma \}$$

Let $\langle A_i \mid i < \lambda \rangle$ be a $\lambda$-sequence with $A_{i+1} = A'$ for all $i < \lambda$. Pick $\alpha < \lambda^+$ such that $C_\alpha(i) \in A_i$ for all $i < \text{otp}(C_\alpha)$.

Then $\text{nacc}(C_\alpha) \subseteq A'$, and hence clause (3) above applies to all $x_\alpha$. As every element of $T_\alpha$ is the limit of some $x_\alpha$, every element of $T_\alpha$ happens to be above some element from $A \cap \alpha$. So, $A \cap \alpha$ is a maximal antichain in $T$. This is a contradiction. ■
So, what do we gain from the fact that we may guess a $\lambda$-sequence if at the end of the day we are only concerned with guessing a single set?

Suppose we wanted the resulted tree to be, in addition, rigid. Then fix a $\diamondsuit_{\lambda^+}$ sequence that guesses functions $\langle f_\gamma \mid \gamma < \lambda^+ \rangle$.

Given an hypothetical maximal antichain $A$, and a non-trivial automorphism $f$, the following sets would be cofinal (in fact, stat.):

$A_0 := \{ \gamma < \lambda^+ \mid A \cap \gamma = S_{\gamma} \text{ is a maximal antichain in } T \upharpoonright \gamma \}$;

$A_1 := \{ \gamma < \lambda^+ \mid f \upharpoonright \gamma = f_{\gamma} \text{ is a n.t. automorphism of } T \upharpoonright \gamma \}$.

So, we could find $C_\alpha$ whose odd nacc points are in $A_0$, and even nacc points are in $A_1$. Meaning that we could overcome $A$ and $f$ along the way.

Similarly, we may overcome $\lambda^+$ many obstructions in a very elegant way.
\( \lambda^+\)-Souslin trees. The aftermath

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So, we could find $C_\alpha$ whose odd nacc points are in $A_0$, and even nacc points are in $A_1$. Meaning that we could overcome $A$ and $f$ along the way. Similarly, we may overcome $\lambda$ many obstructions in a very elegant way.
Question
What do we gain from the fact that we may guess a $\lambda$-sequence if we are only concerned with guessing a single cofinal set?

Answer
We can smoothly construct complicated objects, taking into account $\lambda$ many independent considerations.
\(\lambda^+\)-Souslin trees. The aftermath

We can smoothly construct complicated objects, having in mind \(\lambda\) many independent considerations.

**Question**

“smoothly”?
We can smoothly construct complicated objects, having in mind \( \lambda \) many independent considerations.

**Question**

“smoothly”? 

**Answer**

Jensen’s original construction consists of two distinct components; one which is responsible for insuring that the construction may be carried up to height \( \lambda^+ \), and the other responsible for sealing potential large antichains. This distinction affects the completeness degree of the tree. In contrast, here, the potential antichains are sealed along the way.
\(\lambda^+\)-Souslin trees. The aftermath

We can smoothly construct complicated objects, having in mind \(\lambda\) many independent considerations.

A complaint

“smoothly”… okay! But Jensen’s construction is from

\[\text{GCH} + \square_\lambda + \diamond S \text{ for all stationary } S \subseteq \lambda^+,\]

while the other construction requires \(\clubsuit_\lambda!!\)
We can smoothly construct complicated objects, having in mind $\lambda$ many independent considerations.

A complaint

"smoothly"... okay! But Jensen’s construction is from

$$\text{GCH} + \Box_\lambda + \Diamond_S \text{ for all stationary } S \subseteq \lambda^+,$$

while the other construction requires $\spadesuit_\lambda!!$

Answer

If you are serious about purchasing my $\spadesuit_\lambda$, let me make a price quote.
Ostaszewski square - the price

It should be clear that the usual fine-structural-type of arguments yield that ♣_λ holds in L for all λ. But that’s an high price to pay.
Ostaszewski square - the price

It should be clear that the usual fine-structural-type of arguments yield that $\clubsuit_\lambda$ holds in $L$ for all $\lambda$. But that’s an high price to pay.

Main Theorem
Suppose that $\square_\lambda$ holds for a given cardinal $\lambda$.

1. If $\lambda$ is a limit cardinal, then $\lambda^\lambda = \lambda^+$ entails $\clubsuit_\lambda$.
2. If $\lambda$ is a successor, then $\lambda^{<\lambda} < \lambda^\lambda = \lambda^+$ entails $\clubsuit_\lambda$.

Corollary
Assume GCH. Then for every uncountable cardinal $\lambda$, TFAE:

- $\square_\lambda$;
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So, for the Jensen setup, you pay no extra!
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Corollary
Assume GCH. Then for every uncountable cardinal $\lambda$, TFAE:

- $\Box_\lambda$;
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So, for the Jensen setup, you pay no extra! In fact, you pay less, since $\Box_\lambda + \text{GCH}$ implies $\clubsuit_\lambda + \Diamond_{\lambda^+}$. 
Reflection
Reflection of stationary sets

**Definition**

We say that a stationary subset $S \subseteq \kappa$ reflects at an ordinal $\alpha < \kappa$, if $S \cap \alpha$ is stationary (as a subset of $\alpha$).

**Fact (Hanf-Scott, 1960’s)**

*If $\kappa$ is a weakly compact cardinal, then every stationary subset of $\kappa$ reflects at some $\alpha < \kappa$.***
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Fact (Hanf-Scott, 1960’s)
If $\kappa$ is a weakly compact cardinal, then every stationary subset of $\kappa$ reflects at some $\alpha < \kappa$.

Proof.
By Todorcevic, $\kappa$ is weakly compact iff every ladder system $\langle A_\alpha \mid \alpha < \kappa \rangle$ whose ladders are closed, is trivial in the following sense. There exists a club $C \subseteq \kappa$ such that for all $\beta < \kappa$, there exists $\alpha \geq \beta$ for which $A_\alpha \cap \beta = C \cap \beta$. 
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Suppose now that $S \subseteq \kappa$ is stationary and non-reflecting. Then there exists a ladder system as above with $A_\alpha \cap S = \emptyset$ for all limit $\alpha$. This contradicts the fact that there exists a limit $\beta \in S \cap C$. \qed
A $\square_\lambda$-sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ is non-trivial in the above sense.
Weak square

A $\square_\lambda$-sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ is non-trivial in the above sense. Indeed, since the ladders cohere, $S_\theta = \{ \alpha < \lambda^+ \mid \text{otp}(C_\alpha) = \theta \}$ does not reflect for any $\theta < \lambda$, whereas by Fodor’s lemma, there must exist some $\theta < \lambda$ for which $S_\theta$ is stationary.
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**Definition (Jensen, 1960’s)**

$\square_\lambda^*$ asserts the existence of a ladder system, $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$, s.t.:

- $\otp(C_\alpha) \leq \lambda$;
- $C_\alpha$ is closed;
- for all $\beta < \lambda^+$, $\{ C_\alpha \cap \beta \mid \alpha < \lambda^+ \}$ is of size at most $\lambda$. 

$\square_\lambda^*$ follows from $\square_\lambda$, but also from $\lambda < \lambda = \lambda$, hence the main interest in $\square_\lambda^*$ is whenever $\lambda$ is singular.
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$\square^*_\lambda$ follows from $\square_\lambda$, but also from $\lambda^{<\lambda} = \lambda$, hence the main interest in $\square^*_\lambda$ is whenever $\lambda$ is singular.
Squares and reflection of stationary sets

Theorem (Cummings-Foreman-Magidor, 2001)

It is relatively consistent with the existence of infinitely many supercompact cardinals, that all of the following holds simultaneously:

- GCH;
- $\square^*_{\aleph_\omega}$;
- every stationary subset of $\aleph_{\omega + 1}$ reflects.

So, unlike square, weak square does not imply non-reflection.
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It is relatively consistent with the existence of infinitely many supercompact cardinals, that all of the following holds simultaneously:

- GCH;
- \( \Box^*_{\aleph_\omega} \);
- every stationary subset of \( \aleph_{\omega+1} \) reflects.

Cummings-Foreman-Magidor and Aspero-Krueger-Yoshinobu found that (for a singular \( \lambda \),) \( \Box^*_\lambda \) implies sorts of non-reflection, but of generalized stationary sets (in the sense of \( P_\kappa(\lambda) \), \( P_\kappa(\lambda^+) \)).
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- GCH;
- □^*_\omega;
- every stationary subset of \(\aleph_{\omega+1}\) reflects.

We found out that □^*_\lambda does entail ordinary non-reflection; it is just that the non-reflection takes place in an outer universe...
Weak squares and reflection of stationary sets

**Theorem**

Suppose that $2^\lambda = \lambda^+$ for a strong limit singular cardinal $\lambda$. If $\square_\lambda^*$ holds, then in $V^{\text{Add}(\lambda^+ , 1)}$, there exists a non-reflecting stationary subset of $\lambda^+$. So, this aspect of non-triviality of the weak square system is witnessed in a generic extension.
Weak squares and reflection of stationary sets

Theorem

Suppose that $2^\lambda = \lambda^+$ for a strong limit singular cardinal $\lambda$. If $\square^*_\lambda$ holds, then in $V^{\text{Add}(\lambda^+, 1)}$, there exists a non-reflecting stationary subset of $\{\alpha < \lambda^+ \mid \text{cf}(\alpha) = \text{cf}(\lambda)\}$.

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Suppose that $2^\lambda = \lambda^+$ for a strong limit singular cardinal $\lambda$. If $\Box^*_\lambda$ holds, then in $V^{Add(\lambda^+, 1)}$, there exists a non-reflecting stationary subset of $\{\alpha < \lambda^+ | \text{cf}(\alpha) = \text{cf}(\lambda)\}$. So, this aspect of non-triviality of the weak square system is witnessed in a generic extension.

Compare with the following.

Example
Suppose that $\lambda > \kappa > \text{cf}(\lambda)$, where $\lambda$ is a strong limit, and $\kappa$ is a Laver-indestructible supercompact cardinal. Then $2^\lambda = \lambda^+$ holds for the strong limit singular cardinal $\lambda$, while in $V^{Add(\lambda^+, 1)}$, every stationary subset of $\{\alpha < \lambda^+ | \text{cf}(\alpha) = \text{cf}(\lambda)\}$ do reflect.
Strong Colorings
Suppose that $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ is a ladder system whose ladders are closed. For every $\alpha < \beta < \kappa$, let $\beta = \beta_0 > \cdots > \beta_{k+1} = \alpha$ denote the minimal walk from $\beta$ down to $\alpha$ along $\vec{C}$. Let $[\alpha, \beta]_n$ denote the $n_{th}$ element in the walk from $\beta$ to $\alpha$. 

Fact (Todorcevic, Shelah, 1980's) Suppose that $S$ is a stationary subset of $\kappa$ such that $S \cap C_\alpha = \emptyset$ for every limit $\alpha < \kappa$. (So, $S$ is non-reflecting). Then there exists an oscillating function $o : [\kappa]^2 \to \omega$ such that $S \setminus \bigcup \{[\alpha, \beta]_o(\alpha, \beta) \mid \alpha < \beta \text{ in } A \}$ is non-stationary for every cofinal $A \subseteq \kappa$. 
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Then there exists an oscillating function $o : [\kappa]^2 \to \omega$ such that

$$S \setminus \bigcup \left\{ [\alpha, \beta]_{o(\alpha, \beta)} \mid \alpha < \beta \text{ in } A \right\}$$

is non-stationary for every cofinal $A \subseteq \kappa$. 
Simply definable strong colorings

Suppose that $\vec{C} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle$ witnesses ♣_λ, and let $[\alpha, \beta]_n$ denote the $n_{th}$ element in the $\vec{C}$-walk from $\beta$ to $\alpha$. 
Simply definable strong colorings

Suppose that \( \vec{C} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle \) witnesses \( \clubsuit \lambda \), and let \( [\alpha, \beta]_n \) denote the \( n_{th} \) element in the \( \vec{C} \)-walk from \( \beta \) to \( \alpha \).

**Proposition**

For every cofinal \( B \subseteq \lambda^+ \), there exists an \( n < \omega \) such that for every cofinal \( A \subseteq \lambda^+ \), the set

\[
\{ [\alpha, \beta]_n \mid \alpha \in A, \beta \in B, \alpha < \beta \}
\]

is co-bounded in \( \lambda^+ \).
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Suppose that $\vec{C} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle$ witnesses ♣\_\lambda, and let $[\alpha, \beta]_n$ denote the $n_{th}$ element in the $\vec{C}$-walk from $\beta$ to $\alpha$.

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is co-bounded in $\lambda^+$.

**Corollary**

For every cofinal $B \subseteq \lambda^+$, there exists an $n < \omega$ such that for every cofinal $A \subseteq \lambda^+$, the set

$$\{ \otp(C[\alpha, \beta]_n) \mid \alpha \in A, \beta \in B, \alpha < \beta \}$$

contains each and every limit ordinal $< \lambda$. 
Simply definable strong colorings

Suppose that $\vec{C} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle$ witnesses ♠_\lambda, and let $[\alpha, \beta]_n$ denote the $n_{th}$ element in the $\vec{C}$-walk from $\beta$ to $\alpha$.

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$$\{[\alpha, \beta]_n \mid \alpha \in A, \beta \in B, \alpha < \beta\}$$

is co-bounded in $\lambda^+$.

**Remark**

The above is optimal in the sense that for every $n < \omega$, there exists a cofinal $B \subseteq \lambda^+$, such that

$$\{[\alpha, \beta]_n \mid \alpha, \beta \in B, \alpha < \beta\}$$

omits any limit ordinal $< \lambda^+$.
Simply definable strong colorings

Suppose that $\vec{C} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle$ witnesses $\clubsuit\lambda$, and let $[\alpha, \beta]_n$ denote the $n_{th}$ element in the $\vec{C}$-walk from $\beta$ to $\alpha$.

**Proposition**

For every cofinal $B \subseteq \lambda^+$, there exists an $n < \omega$ such that for every cofinal $A \subseteq \lambda^+$, the set

$$\{[\alpha, \beta]_n \mid \alpha \in A, \beta \in B, \alpha < \beta\}$$

is co-bounded in $\lambda^+$.

**Conjecture**

There exists a one-place function $o : \lambda^+ \rightarrow \omega$ such that for every cofinal $A, B \subseteq \lambda^+$, the set

$$\{[\alpha, \beta]_{o(\beta)} \mid \alpha \in A, \beta \in B, \alpha < \beta\}$$

is co-bounded in $\lambda^+$. 
Squares and small forcings
Some people (including the speaker) speculated at some point in time that $\square_\lambda$ cannot be introduced by a forcing notion of size $\ll \lambda$. This indeed sounds plausible, However:
Squares and small forcing notions

Some people (including the speaker) speculated at some point in time that □_λ cannot be introduced by a forcing notion of size \( \ll \lambda \). This indeed sounds plausible, However:

Theorem (Cummings-Foreman-Magidor, 2001)

It is relatively consistent with the existence of a supercompact cardinal that □_{\aleph_\omega} is introduced by a forcing of size \( \aleph_1 \).
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It is relatively consistent with the existence of a supercompact cardinal that $\square_{\aleph_\omega}$ is introduced by a forcing of size $\aleph_1$.

The idea of the proof is to cook up a model in which $\square_{\aleph_\omega}$ fails, while $\{\alpha < \aleph_{\omega+1} \mid \text{cf}(\alpha) > \omega_1\}$ does carry a so-called partial square. Then, to overcome the lack of coherence over $\{\alpha < \aleph_{\omega+1} \mid \text{cf}(\alpha) = \omega_1\}$, they Levy collapse $\aleph_1$ into countable cardinality.
Squares and small forcing notions

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The idea of the proof is to cook up a model in which $\Box_{\aleph_\omega}$ fails, while $\{\alpha < \aleph_{\omega+1} \mid \text{cf}(\alpha) > \omega_1\}$ does carry a so-called partial square. Then, to overcome the lack of coherence over $\{\alpha < \aleph_{\omega+1} \mid \text{cf}(\alpha) = \omega_1\}$, they Levy collapse $\aleph_1$ into countable cardinality. The latter trivially overcomes the failure of $\Box_{\aleph_\omega}$, and is a forcing notion of size $\aleph_1$. 
Squares and small forcing notions

Theorem (Cummings-Foreman-Magidor, 2001)

It is relatively consistent with the existence of a supercompact cardinal that $\square_{\aleph_\omega}$ is introduced by $\text{coll}(\omega, \omega_1)$.

A rant

Insuring coherence by collapsing cardinals? this is cheating!!
Let me correct my conjecture.
Theorem (Cummings-Foreman-Magidor, 2001)

It is relatively consistent with the existence of a supercompact cardinal that □_{\aleph_\omega} is introduced by coll(\omega, \omega_1).

Speculation, revised

Square/weak square cannot be introduced by a small forcing that does not collapse cardinals.
Squares and small forcing notions

**Theorem (Cummings-Foreman-Magidor, 2001)**

It is relatively consistent with the existence of a supercompact cardinal that \( \Box_{\aleph_\omega} \) is introduced by \( \text{coll}(\omega, \omega_1) \).

**False speculation**

Square/weak square cannot be introduced by a small forcing that does not collapse cardinals.

**Theorem**

*It is relatively consistent with the existence of two supercompact cardinals that \( \Box^*_{\aleph_\omega_1} \) is introduced by a cofinality preserving forcing of size \( \aleph_3 \).*
Squares and small forcing notions

Theorem (Cummings-Foreman-Magidor, 2001)
It is relatively consistent with the existence of a supercompact cardinal that $\square_{\aleph_\omega}$ is introduced by $\text{coll}(\omega, \omega_1)$.

Theorem
It is relatively consistent with the existence of two supercompact cardinals that $\square^*_{\aleph_{\omega_1}}$ is introduced by a cofinality preserving forcing of size $\aleph_3$.

Conjecture
As $\aleph_1$-sized notion of forcing suffices to introduce $\square_{\aleph_\omega}$, then $\aleph_2$-sized notion of forcing should suffice to introduce (in a cofinality-preserving manner!) $\square^*_{\aleph_{\omega_1}}$. 
Open Problems
Two problems

Question
Suppose that $\clubsuit_\lambda + \diamondsuit_{\lambda^+}$ holds for a given singular cardinal $\lambda$. Does there exists an homogenous $\lambda^+$-Souslin tree?
Two problems

Question
Suppose that ♣_λ + ◊_λ⁺ holds for a given singular cardinal λ. Does there exists an homogenous λ⁺-Souslin tree?

Theorem (Dolinar-Džamonja, 2010)
□_ω₁ may be introduced by a forcing notion whose working parts are finite. (that is, the part in the forcing conditions which approximates the generic square sequence is finite.)

Conjecture
□^*_N_ω₁ may be introduced by a small, cofinality preserving forcing notion whose working parts are finite.
Epilogue

Summary

- ♣_\lambda is a particular form of □_\lambda whose intrinsic complexity allows to derive complex objects (such as trees, partitions of stationary sets, and strong colorings) in a canonical way;
- ♣_\lambda and □_\lambda are equivalent, assuming GCH;
- weak square may be introduced by a small forcing that preserves the cardinal structure;
- weak square implies the existence of a non-reflecting stationary set in a generic extension by Cohen forcing.