ARITHMETIC, SET THEORY, AND THEIR MODELS

PART TWO: ENDOMORPHISMS

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YOUNG SET THEORY WORKSHOP

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- **Theorem**. [Scott]
  (a) $\mathrm{SSy}(\mathcal{M})$ *is a Scott set for every* $\mathcal{M} \models \mathsf{ZF}^{\pm\infty}$.
  (b) *If* $\mathcal{A}$ *is a countable Scott set, then* $\mathcal{A}$ *can be realized as* $\mathrm{SSy}(\mathcal{M})$ *for some model of* $\mathsf{ZF}^{\pm\infty}$.

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- **Theorem** [E, Shelah] *There exists $\mathcal{A} \subseteq \mathcal{P}(\omega)$ that is is arithmetically closed and $\mathcal{A}/\mathrm{fin}$ is proper; indeed $\mathcal{A}$ can be arranged to be Borel.*

- **Proposition.** *$\mathcal{M}$ is recursively saturated iff* **(1)** *$\mathcal{M}$ is not ω-standard, and* **(2)** *$V_\alpha^{\mathcal{M}} \prec \mathcal{M}$ for cofinally many $\alpha \in \mathbf{Ord}^{\mathcal{M}}$.*

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- **(1)** Recursively saturated models are *homogeneous*, i.e., if $(\mathcal{M}, a_1, \cdots, a_n) \equiv (\mathcal{M}, b_1, \cdots, b_n)$, then for every $c \in M$ there is $d \in M$ such that $(\mathcal{M}, a_1, \cdots, a_n, c) \equiv (\mathcal{M}, b_1, \cdots, b_n, d)$.

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- **(2)** The set of *n*-types that are coded in a recursively saturated model of arithmetic are precisely those finitely satisfiable types whose Gödel numbers are coded in SSy($\mathcal{M}$).
- **(3)** Any two countable homogeneous models that satisfy the same set of types are isomorphic. This is established by a back-and-forth argument.

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- Let $\mathcal{N} \succ_{\mathrm{end}} \mathcal{M}$. There is some $\beta \in \mathbf{Ord}^{\mathcal{N}} \backslash \mathbf{Ord}^{\mathcal{M}}$ such that $\mathrm{Th}^{\leq c}(V_\beta^{\mathcal{M}}) = \mathrm{Th}^{\leq c}(V_{\alpha_0}^{\mathcal{M}})$.

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- $V_\beta^{\mathcal{M}} \cong V_{\alpha_0}^{\mathcal{M}}$. By restricting any isomorphism between them to $\mathcal{M}$ we obtain an embedding of $\mathcal{M}$ into a proper rank initial segment of itself.

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- There is a group embedding

$$j \mapsto \hat{\jmath}$$

of $\mathrm{Aut}(\mathbb{L})$ into $\mathrm{Aut}(\mathcal{M}_{\mathcal{U},\mathbb{L}})$ such that

$$\mathrm{fix}(\hat{\jmath}) = \mathcal{M},$$

for every fixed-point free j.

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6. Does T have a *rigid* model?

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- Λ is also axiomatized by formulas of the form $\psi_{C,n} :=$ $C(x)$ is CUB $\to \exists\kappa \ C(\kappa)$ and κ is n-Mahlo.

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- $\frac{\mathrm{I}\Delta_0}{\mathrm{PA}} \quad \sim \quad \frac{\mathrm{EST}(\in,\lhd)+\mathrm{GW}}{\mathrm{ZFC}+\Lambda}$