ARITHMETIC, SET THEORY, AND THEIR MODELS

PART ONE: END EXTENSIONS

Ali Enayat

YOUNG SET THEORY WORKSHOP

KÖNIGSWINTER, MARCH 21-25, 2011

SET THEORETICAL TWINS OF ARITHMETICAL THEORIES: THE GLOBAL VIEW

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- ZFC$\backslash\{\text{Power}\} + \mathbf{V} = \mathrm{H}(\aleph_1) \approx$

- KMC$^-$.

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- **(5)** [Szmielew-Tarski]

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- **Example C:** von Neumann's interpretation of PA in ZF.

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- An interpretation of a theory S in a theory T, written, $\mathcal{I}: S \to T$ consists of a translation of each formula φ of S into a formula $\varphi^{\mathcal{I}}$ in the language of T such that

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- The translation $\varphi \longmapsto \varphi^{\mathcal{I}}$ is induced by the following:
  (a) A universe of discourse designated by a first order formula U of T;
  (b) A distinguished definable equivalence relation E on to interpret equality on U;
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- We write $S \leq_{\mathrm{I}} T$ if S can be interpreted in T.

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- **(2)** S can verify that $\mathcal{B} \cong \mathcal{B}_{\mathcal{A}_{\mathcal{B}}}$.

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- Therefore "interpretability strength" is a *refinement* of "consistency strength".
- **Theorem.** [Mostowski-Robinson-Tarski] If T is axiomatizable and $\mathrm{Q} \leq_{\mathrm{I}} T$, then T is *essentially undecidable*.

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- **Theorem.** EFA $\vdash$ Con(ZF) $\rightarrow$ Con(ZF + $\neg$CH).
- **Proof:** Move within **L** and build $\mathbf{L}^{\mathbb{B}}$, where $\mathbb{B} = \mathrm{c.b.a}$ for adding $\aleph_2$ Cohen reals; then mod out $\mathbf{L}^{\mathbb{B}}$ by the **L**-least ultrafilter on $\mathbb{B}$.

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- Second order arithmetic (Z_2) is a **two-sorted theory**; one sort for *numbers*, and the other sort for *reals*. Models of Z_2 are of the form $(\mathcal{M}, \mathcal{A})$, where $\mathcal{A} \subseteq \mathcal{P}(M)$, such that

  (1) For each $X \in \mathcal{A}$, $(\mathcal{M}, X) \models \mathrm{PA(X)}$, and

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- The Choice Scheme Π^1_∞-AC consists of the universal closure of formulae of the form

$$\forall n \, \exists X \; \varphi(n, X) \rightarrow \exists Y \; \forall n \; \varphi(n, (Y)_n)$$

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- To interpret an model $\mathfrak{A} \models \mathsf{T}_{\mathsf{set}}$ within a model $(\mathbb{N}^*, \mathcal{A}) \models \mathsf{T}_{\mathsf{analysis}}$, one defines the notion of "suitable trees", and an equivalence relation $=^*$ among suitable trees, and a binary relation $\in^*$ among the equivalence classes of $=^*$ This yields a model $\mathfrak{A} = (A, E)$ of $\mathsf{T}_{\mathsf{set}}$; where A is the set of equivalence classes of $=^*$ and $E = \in^*$.

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- Conversely, if $\mathfrak{A}$ is a model of $\mathsf{T}_{\mathsf{set}}$, then the "standard model of second order arithmetic" in the sense of $\mathfrak{A}$ is a model of $\mathsf{T}_{\mathsf{analysis}}$.

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- **Theorem.** [Ackermann, 1940] *Let* E *be defined by* $a\mathsf{E}b$ *iff the* a*-th component of the base-2 expansion of* b *is 1. Then*

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- **Theorem.** $\mathrm{Z_2} + \Pi^1_\infty$-AC is bi-interpretable with $\mathrm{KMC}^{-\infty}$.

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- **Theorem** [Keisler-Morely]. *Every ω-standard countable $\mathcal{M} \models \mathrm{ZF}$ has a nonstandard elementary extension that is ω-standard.*

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- $\mathcal{M} \subseteq_{rank} \mathcal{N}$ for every $x \in N \backslash M$, and every $y \in M$, $\mathcal{N} \models \rho(x) > \rho(y)$.

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- $\mathcal{M} \subseteq_{cons} \mathcal{N}$ if the intersection of any parametrically definable subset of $\mathcal{N}$ with $\mathcal{M}$ is also parametrically definable in $\mathcal{M}$.

- **Proposition.**
  (a) *Rank extensions are end extensions, but not vice versa.*
  (b) *If* $\mathcal{M} \preceq_{end} \mathcal{N} \models \mathrm{ZFC}$, then $\mathcal{M} \subseteq_{rank} \mathcal{N}$.
  (c) *If* $\mathcal{M} \models \mathrm{ZF}_{\mathrm{fin}}$ and $\mathcal{M} \preceq_{cons} \mathcal{N}$, *then* $\mathcal{M} \subseteq_{end} \mathcal{N}$.

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- **Theorem** [Splitting Theorem]. *Suppose* $\mathcal{M} \prec \mathcal{N}$ *where* $\mathcal{M}$ *is a model of* $\mathrm{ZF}^{\pm\infty}$. *There exist a model* $\mathcal{N}^*$ *such that*

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MODEL THEORETIC PRELIMINARIES -PART III

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- **Theorem** [E]. *If* $\mathcal{M} \preceq_{cons} \mathcal{N} \models \text{ZFC}$ *and* $\mathcal{N}$ *fixes* $\omega^{\mathcal{M}}$, *then* $\mathcal{M}$ *is cofinal in* $\mathcal{N}$.

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- **Theorem** [Keisler-Tarski; Scott] κ *is a measurable cardinal iff* $\mathbf{V} \prec_{\kappa\text{-}end} \mathcal{N}$ *for some* $\mathcal{N}$ (note: $\kappa = \omega$ allowed!).

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KAPPA ELEMENTARY END EXTENSIONS: KEISLER-MORLEY, KUNEN

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KAPPA ELEMENTARY END EXTENSIONS, CONT'D

- **Theorem.** *Suppose* $\mathcal{M}$ *is a model of* ZFC, κ *is a regular cardinal of* $\mathcal{M}$, *and* $\left|\mathcal{P}^{\mathcal{M}}(\kappa)\right| = \aleph_0$. *Then* $\mathcal{M}$ *has a proper* κ*-e.e.e.*

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- Moreover, if κ *is weakly compact in* $\mathcal{M}$, *then the extension can be arranged to be* κ*-conservative.*
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- **Theorem** [E] *There is no set Φ of first order sentences in the language $\{\in, \kappa\}$ such that for all countable models $\mathcal{M}$ of* ZFC, *$\mathcal{M} \models \Phi$ iff there is an ultrafilter $\mathcal{U}$ on $\mathcal{P}^{\mathcal{M}}(\kappa)$ such that $\mathcal{U}$ is both $\mathcal{M}$-iterable and $\mathcal{M}$-normal.*

ELEMENTARY END EXTENSIONS: "GOOD" NEWS

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  (a) *Every model* $\mathcal{M}$ *of* $\mathrm{ZF}^{-\infty} + \mathrm{TC}$ *has an e.e.e* $\mathcal{N}$.

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ELEMENTARY END EXTENSIONS: "BAD" NEWS

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- **Theorem** [Kaufmann-E] *Every consistent extension of* ZFC *has a model* $\mathcal{M}$ *of power* $\aleph_1$ *such that* $\mathcal{M}$ *has no e.e.e.*

ELEMENTARY END EXTENSIONS: "GOOD" NEWS REGAINED

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- **Theorem** [Kaufmann-E] *The following are equivalent for a consistent complete extension T of ZFC:*
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  (b) *For each natural number n,*
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- **SLOGAN:** ZFC $+ \Lambda$ is the weakest extension of ZFC that allows infinite set theory to model-theoretically catch-up with finite set theory!