ARITHMETIC, SET THEORY, AND THEIR MODELS

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This tutorial will survey old and new results that illustrate the remarkable connections between set theory and models of arithmetic. Our emphasis, naturally, will be on the set-theoretical side of the story. Some longstanding open questions regarding models of arithmetic that are intimately connected with higher set theory will be included in our discussion.

Part I of the tutorial is centered on the notion of *end embeddings*. We will begin with the *McDowell-Specker-Gaifman Theorem* (dealing with models of arithmetic), which we will establish via an iterated ultrapower construction. As we shall see, this theorem has an impressive number of analogues and variants, some pertaining to models of arithmetic, and even more dealing with models of set theory (including classical results of Scott, Kunen, and Keisler-Morley, as well as fairly recent ones). Part I will conclude with a close look at the theory $\mathsf{ZFC} + \Lambda$, where Λ is the *Lévy scheme* that asserts, for each standard natural number n, the existence of an n-Mahlo cardinal κ that is n-reflective (i.e., V_κ is a Σ_n-elementary submodel of the universe).

Part II of the tutorial concerns *endomorphims* (self-embeddings). Our discussion will start with a proof of a striking result of Harvey Friedman, which states that every countable nonstandard model of PA or ZF is isomorphic to a proper (rank) initial segment of itself. We then turn to automorphisms and explain how they can be used as a 'lens' to detect *canonical* set theories, including the aforementioned $\mathsf{ZFC} + \Lambda$. Part II includes a fairly detailed discussion of the proof of the following result:

Theorem [E, 2004].
(a) *If j is a nontrivial automorphism of a model $\mathcal{M}$ of a fragment of* ZFC (*that is even weaker than Kripke-Platek set theory*) *such that the fixed-point set of j forms a* ***rank initial segment*** *of $\mathcal{M}$, then the fixed-point set of j is a model of* $\mathsf{ZFC} + \Lambda$.
(b) *Given any consistent extension T of* $\mathsf{ZFC} + \Lambda$*, there is a model $\mathcal{M}$ of set theory* (*indeed of T*)*, and a nontrivial automorphism j of $\mathcal{M}$, such that the fixed-point set of j forms a* ***rank initial segment*** *of $\mathcal{M}$ and is a model of T.*