Projective maximal almost disjoint families

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Survey

Definitions and basic facts

Indestructibility

Models of $\mathfrak{b} > \omega_1$

Basic definitions: families of infinite subsets of ω

- $a, b \in [\omega]^\omega$ are *almost disjoint*, if $a \cap b$ is finite.
  An infinite set A is said to be an *almost disjoint family* of infinite subsets of ω (or an almost disjoint subfamily of $[\omega]^\omega$) if $A \subset [\omega]^\omega$ and any two elements of A are almost disjoint.
- $A \subset [\omega]^\omega$ is called a *mad family* of infinite subsets of ω (abbreviated from "maximal almost disjoint"), if it is maximal with respect to inclusion among almost disjoint families of infinite subsets of ω.
- Given $A \subset [\omega]^\omega$, we denote by $\mathcal{L}(A)$ the collection of all positive sets with respect to the ideal generated by A.
  A mad subfamily A of $[\omega]^\omega$ is defined to be ω-*mad*, if for every $B \in [\mathcal{L}(A)]^\omega$ there exists $a \in A$ such that $|a \cap b| = \omega$ for all $b \in B$.

Basic definitions: families of functions from ω to ω

- $a, b \in \omega^\omega$ are *almost disjoint*, if $a \cap b$ is finite.
  An infinite set A is said to be an *almost disjoint family* of functions from ω to ω (or an almost disjoint subfamily of ω^ω) if $A \subset \omega^\omega$ and any two elements of A are almost disjoint.
- $A \subset \omega^\omega$ is called a *mad family* of functions from ω to ω (abbreviated from "maximal almost disjoint"), if it is maximal with respect to inclusion among almost disjoint families of functions from ω to ω.
- Given $A \subset \omega^\omega$, we denote by $\mathcal{L}(A)$ the collection of all $f \in \omega^\omega$ which are positive with respect to the ideal generated by A.
  A mad subfamily A of ω^ω is defined to be ω-*mad*, if for every $B \in [\mathcal{L}(A)]^\omega$ there exists $a \in A$ such that $|a \cap b| = \omega$ for all $b \in B$.

Nonexistence results

Theorem

(Mathias 1977). *There exists no* Σ^1_1 *definable mad family of infinite subsets of* ω.

Theorem

(Kastermans-Steprāns-Zhang 2008). *There exists no* Σ^1_1 *definable* ω*-mad family of functions from* ω *to* ω.

Proof.

Suppose that such a family $A \subset \omega^\omega$ exists. Take $f \in \mathcal{L}(A)$ and consider $B = \{[f = a] : a \in A\}$, where $[f = a] = \{n \in \omega : f(n) = a(n)\}$.

Claim

$C := B \cap [\omega]^\omega$ *is a* Σ^1_1*-definable mad family.*

Proof.

If not, there exists $x \in [\omega]^\omega$ almost disjoint from all elements of C. Fix distinct $a_0, a_1 \in A$ and set $x_i = f \restriction x \cup a_i \restriction (\omega \setminus x)$, $i \in 2$. Observe that $x_i \in \mathcal{L}(A)$. Therefore $|[x_0 = a]| = |[x_1 = a]| = \omega$ for some $a \in A$, which is impossible. □

□

Problem

Is there a Σ^1_1 definable mad family of functions from ω to ω?

Problem

Do ω-mad families exist in ZFC?
(Raghavan: Yes if $\mathfrak{b} = \mathfrak{c}$.)

Existence results

Definition

A subfamily A of ω^ω is called a *Van Douwen mad family* if for any infinite partial function p there is $a \in A$ with $|a \cap p| = \omega$.

Observation

Every ω-mad subfamily of ω^ω is a Van Douwen mad family.

Theorem

(Raghavan 2008). *There exists a Van Douwen mad family.*

Theorem

(A. Miller 1989). *(V=L). There exists a Π^1_1 definable mad family of infinite subsets of ω.*

Theorem

(Kastermans-Steprāns-Zhang 2008). *(V=L). There exists a Π^1_1 definable ω-mad family of functions from ω to ω.*

Corollary

(V=L). There exists a Π^1_1 *definable* ω*-mad family of infinite subsets of* ω*.*

Proof.

If $A \subset \omega^\omega$ is ω-mad, then $A \cup \{\text{vertical lines}\}$ is an ω-mad family of infinite subsets of ω. □

Indestructibility of mad families

Definition

Let $\mathcal{A}$ be a mad family and $\mathbb{P}$ be a poset. $\mathcal{A}$ is $\mathbb{P}$ *indestructible*, if $\mathcal{A}$ stays mad in $V^{\mathbb{P}}$.

Theorem

(Kurilić 2001). *A mad family $\mathcal{A} \subset [\omega]^\omega$ is Cohen indestructible iff for every $B \in \mathcal{L}(\mathcal{A})$ there exists $\mathcal{L}(\mathcal{A}) \ni C \subset B$ such that $\mathcal{A}|C = \{A \cap C : A \in \mathcal{A}, |A \cap C| = \omega\}$ is an ω-mad subfamily of $[C]^\omega$.*

Proof

We prove the "only if" part. Suppose that for every $B \in \mathcal{L}(\mathcal{A})$ there exists a countable $\mathcal{B}_B \subset [B]^\omega \cap \mathcal{L}(\mathcal{A})$ witnessing for $\mathcal{A}|B$ being not ω-mad. Fix $B_\emptyset \in \mathcal{L}(\mathcal{A})$ and consider a map $\omega^{<\omega} \ni \langle s_0, \dots, s_n \rangle \mapsto B_{\langle s_0, \dots, s_n \rangle} \in \mathcal{L}(\mathcal{A})$ such that $\{B_{s\frown n} : n \in \omega = \mathcal{B}_{B_s}\}$ for all $s \in \omega^{<\omega}$.

Now let $c \in \omega^\omega$ be a Cohen real (i.e., a generic subset of $\omega^{<\omega}$). In $V[c]$, find a set $X \in [\omega]^\omega$ such that $X \subset^* B_{c \restriction n}$ for all n.

Claim

X is almost disjoint from all elements of $\mathcal{A}$.

Proof.

Given $A \in \mathcal{A}$, the set $D_A := \{s \in \omega^{<\omega} : |A \cap B_s| < \omega\}$ is dense in $\omega^{<\omega}$. □

Fix $A \in \mathcal{A}$ and find $n \in \omega$ such that $c \restriction n \in D_A$. The latter menas that $B_{c \restriction n} \cap A$ is finite. Since $X \subset^* B_{c \restriction n}$, $X \cap A$ is finite either.

□

More indestructibility

Definition

(Raghavan 2009). Let $\mathbb{P}$ be a poset. $\mathbb{P}$ has *diagonal fusion* if there exist a sequence $\langle \leq_n : n \in \omega \rangle$ of partial orderings on $\mathbb{P}$, a strictly increasing sequence of natural numbers $\langle i_n : n \in \omega \rangle$ with $i_0 = 0$, and for each $p \in \mathbb{P}$ a sequence $\langle p_i : i \in \omega \rangle \in \mathbb{P}^\omega$ such that the following hold:

- P has fusion with respect to $\langle \leq_n : n \in \omega \rangle$;
- For all $i \in \omega$, $p_i \leq p$;
- If $q \leq p$, then $q \not\leq p_i$ for infinitely many i;
- If $q \leq_n p$, then $q_i \leq p_i$ for all $i \leq i_n$;
- If $\langle r_i : i_n \leq i < i_{n+1} \rangle$ is a sequence such that $r_i \leq p_i$ for all $i \in [i_n, i_{n+1})$, then exists $q \leq_n p$ such that $q_i \leq r_i$ for all $i \in [i_n, i_{n+1})$.

More indestructibility, continued

Theorem

(Raghavan 2009.) *Suppose that* $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \gamma, \eta < \gamma \rangle$ *is a countable support iteration forcing construction such that* $\Vdash_\xi$ *"*$\dot{\mathbb{Q}}_\xi$ *has a diagonal fusion" for all* ξ. *Then all ground model* ω*-mad subfamilies of* ω^ω *are* $\mathbb{P}_\gamma$*-indestructible.*

Example.

Miller and Sacks forcings have diagonal fusion, while Laver does not.

Theorem

(Brendle-Yatabe 2005) *Suppose* $\mathbb{P}$ *is a forcing notion that adds a new real, and suppose* $\mathcal{A}$ *is a mad subfamily (either of* $[\omega]^\omega$ *or of* ω^ω*). If* $\mathcal{A}$ *is* $\mathbb{P}$*-indestructible, then* $\mathcal{A}$ *is also Sacks indestructible.*

Problem

(Brendle-Yatabe 2005) *Do Sacks indestructible mad families exist in ZFC?*

Definability with higher continuum

If $\mathcal{A} \in V$ is a Π^1_1 definable almost disjoint family whose Π^1_1 definition is provided by formula $\varphi(x)$, then $\varphi(x)$ defines an almost disjoint family in any extension V' of V. This is a straightforward consequence of the Shoenfield's Absoluteness Theorem:

$\forall x \in \omega^\omega \forall y \in \omega^\omega \left(\varphi(x) \wedge \varphi(y) \rightarrow (|x \cap y| < \omega)\right)$ is a Π^1_2 statement.

Thus if a ground model Π^1_1 definable mad family *remains mad* in a forcing extension, it remains Π^1_1 definable by means of the same formula.

It follows that the Π^1_1 definable ω-mad family in L of functions constructed by Kastermans, Steprāns, and Zhang remains Π^1_1 definable and ω-mad in $L[G]$, where G is a generic over L for the countable support iteration of Miller forcing of length ω_2.

Corollary

Let κ be a regular cardinal. The existence of a Π^1_1 definable ω-mad family is consistent with $2^\omega = \kappa$.

Models of $\mathfrak{b} > \omega_1$

Theorem

(Friedman-Z. 2009). *It is consistent that* $2^\omega = \mathfrak{b} = \omega_2$ *and there exists a* Π^1_2 *definable* ω*-mad family of infinite subsets of* ω *(of functions from* ω *to* ω*).*

Proof in case of subfamilies of $[\omega]^\omega$

Some auxiliary facts:

Proposition

- *There exists an almost disjoint family* $R = \{r_{\langle\zeta,\xi\rangle} : \zeta \in \omega \cdot 2, \xi \in \omega_1^L\} \in L$ *of infinite subsets of* ω *such that* $R \cap M = \{r_{\langle\zeta,\xi\rangle} : \zeta \in \omega \cdot 2, \xi \in (\omega_1^L)^M\}$ *for every transitive model* M *of* ZF^-*.*
- *There exists a* Σ_1 *definable over* L_{ω_2} *sequence* $\bar{S} = \langle S_\alpha : \alpha < \omega_2\rangle$ *of pairwise almost disjoint* L*-stationary subsets of* ω_1 *such that whenever* M, N *are suitable models of* ZF^- *such that* $\omega_1^M = \omega_1^N$, $\bar{S}^M$ *agrees with* $\bar{S}^N$ *on* $\omega_2^M \cap \omega_2^N$*. Moreover, we can additionally assume that* $\omega_1 \setminus \bigcup_{\xi<\omega_2} S_\xi$ *is stationary in* L*.*

We say that transitive ZF^- model M is *suitable* if $M \models$"ω_2 exists and $\omega_2 = \omega_2^L$"

The poset

We start with the ground model $V = L$. Recursively, we shall define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$. The desired family A is constructed along the iteration: for cofinally many α's the poset $\mathbb{Q}_\alpha$ takes care of some countable family B of infinite subsets of ω which might appear in $\mathcal{L}(A)$ in the final model, and adds to A some $a_\alpha \in [\omega]^\omega$ almost disjoint from all elements of A_α such that $|a \cap b| = \omega$ for all $b \in B$ (here A_α stands for the set of all elements of A constructed up to stage α). Our forcing construction may be slightly modified to allow for further applications.

We proceed with the definition of $\mathbb{P}_{\omega_2}$. For successor α let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for some proper forcing of size ω_1 adding a dominating real. For a subset s of ω and $l \in |s|$ $(= \mathrm{card}(s) \leq \omega)$ we denote by $s(l)$ the l'th element of s. In what follows we shall denote by $E(s)$ and $O(s)$ the sets $\{s(2i) : 2i \in |s|\}$ and $\{s(2i+1) : 2i+1 \in |s|\}$, respectively. Let us consider some limit α and a $\mathbb{P}_\alpha$-generic filter G_α.

The poset

Suppose also that

$$(*) \quad \forall B \in [A_\alpha]^{<\omega}\, \forall r \in R\, (|E(r) \setminus \cup B| = |O(r) \setminus \cup B| = \omega)$$

Observe that equation $(*)$ yields $|E(r) \setminus \cup B| = |O(r) \setminus \cup B| = \omega$ for every $B \in [R \cup A_\alpha]^{<\omega}$ and $r \in R \setminus B$. Let us fix some function $F : Lim \cap \omega_2 \to L_{\omega_2}$ such that $F^{-1}(x)$ is unbounded in ω_2 for every $x \in L_{\omega_2}$. Unless the following holds, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the trivial poset. Suppose that $F(\alpha)$ is a sequence $\langle \dot{b}_i : i \in \omega \rangle$ of $\mathbb{P}_\alpha$-names such that $b_i = \dot{b}_i^{G_\alpha} \in [\omega]^\omega$ and none of the b_i's is covered by a finite subfamily of A_α. In this case $\mathbb{Q}_\alpha$ defined as follows. Find a limit ordinal $\eta_\alpha \in \omega_1$ such that there are no finite subsets J, E of $(\omega \cdot 2) \times (\omega_1 \setminus \eta_\alpha)$, A_α, respectively, and $i \in \omega$, such that $b_i \subset \bigcup_{\langle \zeta, \xi \rangle \in J} r_{\langle \zeta, \xi \rangle} \cup \bigcup E$. (The almost disjointness of the $r_{\langle \zeta, \xi \rangle}$'s imply that if $b_i \subset \bigcup R' \cup \bigcup A'$ for some $R' \in [R]^{<\omega}$ and $A' \in [A_\alpha]^{<\omega}$, then $b_i \setminus \bigcup A'$ has finite intersection with all elements of $R \setminus R'$. Together with equation $(*)$ this easily yields the existence of such an η_α.)

The poset, continued

Let z_α be an infinite subset of ω coding a surjection from ω onto η_α. For a subset s of ω we denote by $\bar{s}$ the set $\{2k+1 : k \in s\} \cup \{2k : k \in (\sup s \setminus s)\}$.
In $V[G_\alpha]$, $\mathbb{Q}_\alpha$ consists of sequences $\langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ satisfying the following conditions:

- (i) c_k is a closed, bounded subset of $\omega_1 \setminus \eta_\alpha$ such that $S_{\alpha+k} \cap c_k = \emptyset$ for all $k \in \omega$;
- (ii) $y_k : |y_k| \to 2$, $|y_k| > \eta_\alpha$, $y_k \restriction \eta_\alpha = 0$, and $\mathrm{Even}(y_k) = (\{\eta_\alpha\} \cup (\eta_\alpha + X_\alpha)) \cap |y_k|$;
- (iii) $s \in [\omega]^{<\omega}$, $s^* \in \left[\{r_{\langle m,\xi\rangle} : m \in \bar{s}, \xi \in c_m\} \cup \{r_{\langle \omega+m,\xi\rangle} : m \in \bar{s}, y_m(\xi) = 1\} \cup A_\alpha\right]^{<\omega}$. In addition, for every $2n \in |s \cap r_{\langle 0,0\rangle}|$, $n \in z_\alpha$ if and only if there exists $m \in \omega$ such that $(s \cap r_{\langle 0,0\rangle})(2n) = r_{\langle 0,0\rangle}(2m)$; and

The poset, continued

(iv) For all $k \in \bar{s} \cup (\omega \setminus (\max\ \bar{s}))$, limit ordinals $\xi \in \omega_1$ such that $\eta_\alpha < \xi \leq |y_k|$, and suitable ZF^- models M containing $y_k \restriction \xi$ and $c_k \cap \xi$ with $\omega_1^M = \xi$, ξ is a limit point of c_k, and the following holds in M: $(\mathrm{Even}(y_k) - \min \mathrm{Even}(y_k)) \cap \xi$ codes a limit ordinal $\bar{\alpha}$ such that $S^M_{\bar{\alpha}+k}$ is non-stationary.

The tuples $\langle s, s^* \rangle$ and $\langle c_k, y_k : k \in \omega \rangle$ will be referred to as the *finite part* and the *infinite part* of the condition $\langle \langle s, s^* \rangle, \langle c_k, y_k : k \in \omega \rangle \rangle$, respectively.

The poset, continued

For conditions $\vec{p} = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ and $\vec{q} = \langle\langle t, t^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ in $\mathbb{Q}_\alpha$, we let $\vec{q} \leq \vec{p}$ (by this we mean that $\vec{q}$ is stronger than $\vec{p}$) if and only if

(v) (t, t^*) extends (s, s^*) in the almost disjoint coding, i.e. t is an end-extension of s and $t \setminus s$ has empty intersection with all elements of s^*;

(vi) If $m \in \bar{t} \cup (\omega \setminus (\max \bar{t}))$, then d_m is an end-extension of c_m and $y_m \subset z_m$.

This finishes our definition of $\mathbb{P}_{\omega_2}$.

Proposition

$\dot{\mathbb{Q}}_\alpha$ is $\omega_1 \setminus \bigcup_{\xi<\omega_2} S_\xi$-proper. Consequently, $\mathbb{P}_{\omega_2}$ is $\omega_1 \setminus \bigcup_{\xi<\omega_2} S_\xi$-proper and hence preserves cardinals.

More precisely, for every condition $\vec{p} = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}^1_\alpha$ the poset $\{\vec{r} \in \mathbb{K}^1_\alpha : \vec{r} \leq \vec{p}\}$ is $\omega_1 \setminus \bigcup_{n\in\bar{s}\cup(\omega\setminus(\max\bar{s}))} S_{\alpha+n}$-proper.

Consequently, $S_{\alpha+n}$ remains stationary in $V^{\mathbb{P}_{\omega_2}}$ for all $n \in \omega \setminus \overline{a_\alpha}$.

Why is the constructed family Π^1_2 definable?

Lemma

In $L[G]$ the following conditions are equivalent:

(1) $a \in A$;

(2) *For every countable suitable model M of ZF^- containing a as an element there exists $\bar{\alpha} < \omega_2^M$ such that $S^M_{\bar{\alpha}+k}$ is nonstationary in M for all $k \in \overline{a}$.*

The condition in (2) provides a Π^1_2 definition of A.

Combining two methods

Fischer and Friedman have recently proved that some inequalities between cardinal invariants are consistent with the existence of a Δ^1_3 definable wellorder of the reals.

Theorem

(Friedman-Z. 2009). *It is consistent with Martin's Axiom that there exists a* Δ^1_3 *definable wellorder of the reals and a* Π^1_2 *definable* ω*-mad family of infinite subsets of* ω*.*

Some questions

Question

Is it consistent to have $\mathfrak{b} > \omega_1$ *with a* Σ^1_2 *definable (*ω*-)mad family?*

Question

Is it consistent to have $\omega_1 < \mathfrak{b} < 2^\omega$ *with a* Π^1_2 *definable (*ω*-)mad family?*

Question

Is it consistent to have $\mathfrak{b} < \mathfrak{a}$ *and a* Π^1_2 *definable (*ω*-)mad family?*

Question

Is a projective (ω*-)mad family consistent with* $\mathfrak{b} \geq \omega_3$*?*

The last slide

Thank you for your attention.