Descriptive set theory at uncountable cardinals

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Topics:

regularity properties for subsets of ${}^\kappa\kappa$ and ${}^\kappa 2$: Baire property and perfect set property

definable equivalence relations on ${}^\kappa\kappa$ and ${}^\kappa 2$

Motivation:

framework for classification problems for uncountable structures

1. Setting

Let κ always be a regular uncountable cardinal with $\kappa^{<\kappa} = \kappa$, e.g. ω_1 under CH.

- ${}^{\kappa}\kappa$ is the space of functions $\kappa \to \kappa$ with basic open sets $O(s) = \{f \in {}^{\kappa}\kappa : s \subseteq f\}$ for $s \in {}^{<\kappa}\kappa$

- closed sets $[T]$ for trees $T \subseteq {}^{<\kappa}\kappa$

The intersection of κ many open dense sets is nonempty. Borel sets are generated from the open sets by unions of length κ and complements. Meager sets are unions of κ many nowhere dense sets.

Σ^1_1 formulas are of the form $\exists x \in {}^\kappa\kappa\, \forall\alpha_0 < \kappa\, \exists\alpha_1 < \kappa ... \forall\alpha_n < \kappa\, \phi(x, \vec{y}, \vec{\alpha})$ where ϕ is quantifier-free.

Lemma 1. *The following are equivalent for $A \subseteq {}^\kappa\kappa$:*

1. *A is Σ^1_1 over ${}^\kappa\kappa$ in some parameter $h \in {}^{\kappa\times\kappa}\kappa$*
2. *$A = p[T]$ for a tree $T \subseteq {}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$*
3. *A is $\Sigma_1(H_{\kappa^+})$ in some parameter $h \in {}^\kappa\kappa$*

The set of wellfounded binary relations on κ is a closed subset of ${}^\kappa 2$.

There are non-Borel Δ^1_1 sets.

Suppose two players try to form a decreasing sequence $(p_\alpha : \alpha < \kappa)$ in a forcing $\mathbb{P}$ with player 2 playing at limit stages, and player 2 wins if she can always extend.

Definition 1. *$\mathbb{P}$ is $< \kappa$-strategically closed if player 2 has a winning strategy.*

Lemma 2. *If $\mathbb{P}$ is $< \kappa$-strategically closed, then $V \prec_{\Sigma^1_1({}^\kappa\kappa)} V^{\mathbb{P}}$.*

Proof. Deny. Then $\mathbb{P}$ adds a κ-branch to a tree T which doesn't have a branch in V. Let σ be a name for this branch. Player 2 can choose a condition p_α in move α which decides $\sigma \restriction \alpha$, so T has a κ-branch in V. □

2. Regularity properties

For $\kappa = \omega$, it is consistent that all subset of ${}^{\omega}\omega$ in $L(\mathbb{R})$ are Lebesgue measurable and have the Baire property and the perfect set property.

Lemma 3. *(Halko-Shelah, Kovachev) The club filter on κ does not have the property of Baire in ${}^{\kappa}2$.*

The club filter is a Σ_1^1 set. It's not known whether it is consistent that the Banach-Mazur game is determined for all Σ_1^1 sets.

Definition 2. *A tree $T \subseteq {}^{<\kappa}\kappa$ is perfect if it is $< \kappa$-closed and its set of splitting nodes is cofinal.*

Definition 3. *A set $A \subseteq {}^{\kappa}\kappa$ is perfect if $A = [T]$ for some perfect tree T.*

Definition 4. *A set $A \subseteq {}^{\kappa}\kappa$ has the perfect set property if $|A| \leq \kappa$ or A contains a perfect subset.*

If there is a ω_1-Kurepa tree T, i.e. its levels are countable and there are ω_2 many branches, then $[T]$ is a closed set of size ω_2 without a perfect subset.

Proposition 1. *Suppose $\lambda > \kappa$ is inaccessible. Then in $V^{Col(\kappa,<\lambda)}$ every Σ^1_1 set has the perfect set property.*

Proof. Let G be $Col(\kappa, < \lambda)$-generic over V. Let $T \subseteq {}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ be a tree in $V[G]$ with $|p[T]| > \kappa$. Let's assume $T \in V$.

Suppose $p \Vdash |p[T]| > \kappa$ and σ, τ are names with $p \Vdash (\sigma, \tau) \in [T]$ and $p \Vdash \sigma \notin V$.

We build $(p_u, s_u, t_u : u \in {}^{<\kappa}2)$ with

- $u \subsetneq v$ implies $s_u \subsetneq s_v$ and $t_u \subsetneq t_v$
- $p_u \Vdash s_u \subseteq \sigma, t_u \subseteq \tau$
- $s_u \perp s_v$ for $u \neq v \in {}^{\alpha}2$, $\alpha < \kappa$

$\square$

Proposition 2. *It is consistent that 2^κ is arbitrarily large and every Σ^1_2 subset of ${}^\kappa\kappa$ has size $\leq \kappa^+$ or a perfect subset.*

For trees S and T we write $S \leq T$ if there is a strict order-preserving map from S to T.

A universal family for for the class of trees of height and size κ is a family of such trees so that for every such tree S there is a tree T in the universal family with $S \leq T$.

Theorem 1. *(Mekler-Väänänen) Such a family of arbitrary regular size μ with $\kappa^+ \leq \mu \leq 2^\kappa$ can be added by $< \kappa$-closed κ^+-c.c. forcing.*

The forcing is an iteration which in every step adds a tree T such that $S \leq T$ for all previous trees S.

The forcing can be modified to get a universal family $(T_\alpha : \alpha <\kappa^+)$ and add many Cohen subsets of κ.

Suppose $A = p[B]$ is a Σ^1_2 subset of ${}^\kappa\kappa$. Suppose T is a tree on $\kappa \times \kappa \times \kappa$ with $B = ({}^\kappa\kappa \times {}^\kappa\kappa) - p[T]$. Then $x \in A$ iff $\exists y(x, y) \in B$ iff there is y such that $T_{x,y}$ does not have κ-branches.

We build a tree U on $\kappa \times \kappa^+$ which searches for $y \in {}^\kappa\kappa$ and a strict order preserving map from $T_{x,y}$ into some T_α, $\alpha <\kappa$, for any given $x \in {}^\kappa\kappa$.

Let $V[g]$ be an intermediate extension with $U \in V[g]$ and σ a $V[g]$-name for an element of $A - V[g]$. We can build a perfect subset of A.

3. A counterexample to Silver's theorem

Theorem 2. *(Silver) Every coanalytic equivalence relation on* ${}^{\omega}\omega$ *either has countably many equivalence classes, or there's a perfect set of inequivalent reals.*

A natural question is whether there is any generalization to equivalence relations on ${}^{\kappa}\kappa$.

The prewellorder which compares the ranks of wellorders with domain κ is Δ^1_1. There are κ^+ many ranks.

Let $\mathbb{P}$ be the forcing which adds a Cohen subset of κ.

Lemma 4. *Suppose $\leq$ is a Σ^1_1 prewellorder on ${}^\kappa\kappa$ so that forcing with $\mathbb{P}$ preserves ranks, and suppose this is true in every $\mathbb{P}$-generic extension. Then there is no perfect set of elements of ${}^\kappa\kappa$ of pairwise different ranks.*

Proof. Suppose $[T]$ is a perfect set of elements of ${}^\kappa\kappa$ of pairwise different ranks. The elements of $[T]$ are inequivalent in every $\mathbb{P}$-generic extension. Let σ be a name for a new element of $[T]$ of rank α. $\square$

Corollary 1. *There is no perfect set of wellorders with domain κ of pairwise different ranks.*

Lemma 5. *Suppose* $\leq$ *is a* $\mathbb{P}^i$*-absolutely* Δ^1_1 *prewellorder for* $i \leq 3$*. Then forcing with* $\mathbb{P}$ *preserves ranks and no element of* ${}^\kappa\kappa \cap V^{\mathbb{P}}$ *bounds* ${}^\kappa\kappa \cap V$*.*

Lemma 6. *Suppose* $\leq$ *is a* $\mathbb{P}^i$*-absolutely* Δ^1_1 *prewellorder for* $i \leq 3$*. Then there is no perfect set of elements of* ${}^\kappa\kappa$ *of different ranks.*

Corollary 2. *There is no absolutely* Δ^1_1 *wellorder of* ${}^\kappa\kappa$*.*

5. A weak variant of Silver's theorem

Suppose $2^\kappa = \kappa^+$ and $\mathbb{Q}$ is the forcing for adding $\mu > \kappa$ many Cohen subsets of κ.

Proposition 3. *If E is a coanalytic equivalence relation on ${}^\kappa\kappa$ in $V^{\mathbb{Q}}$, then E has $\leq \kappa^+$ many equivalence classes or there is a perfect set of inequivalent elements of ${}^\kappa\kappa$.*

Let G be $\mathbb{Q}$-generic over V. Let σ, τ denote nice $\mathbb{Q}$-names for subsets of κ.

Let $\mathbb{Q}_\kappa$ be the subforcing of $\mathbb{Q}$ of the first κ many factors. There are $(\kappa^+)^{V[G]}$ many nice $\mathbb{Q}_\kappa$-names for subsets of κ.

Case 1: $(1,1) \Vdash_{\mathbb{Q}\times\mathbb{Q}} (\sigma,\sigma) \in E$ for all σ.

Then for every $x \in {}^\kappa\kappa \cap V[G]$, there is a $\mathbb{Q}_\kappa$-name τ with $xE\tau^G$.

Case 2: There is σ and a condition $p \in \mathbb{Q}$ with $\forall q \leq p \exists r, s \leq q : (r, s) \Vdash (\sigma, \sigma) \notin E$.

Suppose $E = ({}^{\kappa}\kappa \times {}^{\kappa}\kappa) - p[T]$. Let's assume $T \in V$.

We build $(p_u, s_u : u \in {}^{<\kappa}2)$ with

- $u \subsetneq v$ implies $s_u \subsetneq s_v$
- $p_u \Vdash s_u \subseteq \sigma$
- $s_u \perp s_v$ for $u \neq v \in {}^{\alpha}2$, $\alpha < \kappa$

together with witnesses that pairs of branches are in $p[T]$.

Let $\{t_n : n < \omega\} \subseteq {}^{<\omega}2$ be dense with $lh(s_n) = n$.

Let G_0 be the graph on ${}^{\omega}\omega$ whose edges are the pairs $(t_n \frown i \frown x, t_n \frown j \frown x)$ for $i \neq j$, $i, j = 0, 1$, and $x \in {}^{\omega}2$.

Theorem 3. *(Kechris-Solecki-Todorcevic) Suppose G is an analytic graph on ${}^{\omega}\omega$. Then either there is a Borel ω-coloring of G, or there is a continuous homomorphism from G_0 to G.*

Let $\{t_\alpha : \alpha < \kappa\} \subseteq {}^{<\kappa}2$ be dense with $lh(s_\alpha) = \alpha$.

Let G_0 be the graph on ${}^{\kappa}\kappa$ whose edges are the pairs $(t_\alpha \frown i \frown x, t_\alpha \frown j \frown x)$ for $i \neq j$, $i, j = 0, 1$, and $x \in {}^{\kappa}2$.

Lemma 7. *G_0 is acyclic.*

Lemma 8. *There is no Baire measurable coloring of G_0 with $\leq \kappa$ many colors.*

Suppose $2^\kappa = \kappa^+$ and $\mathbb{Q}$ is the forcing for adding $\mu > \kappa$ many Cohen subsets of κ.

Proposition 4. *If G is an analytic graph on ${}^\kappa\kappa$ in $V^{\mathbb{Q}}$, then there is a κ^+-coloring of G or a continuous homomorphism $G_0 \to G$.*

Thank you for listening!