Young Set Theory Workshop 2010, Raach, Austria.

The Proper Forcing Axiom: a tutorial

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In these notes we will present an exposition of the Proper Forcing Axiom (PFA). We will first discuss examples of the consequences of PFA. We will then present two proper partial orders which are used to force two combinatorial principles which follow from PFA: The P-Ideal Dichotomy (PID) and Todorcevic's formulation of the Open Coloring Axiom. On one hand, these posets and the proofs of their properness are quite typical of direct applications of PFA. On the other hand, these principles already capture a large number of the consequences of PFA and do not use any terminology or technical tools from the theory of forcing.

1 Consequences of PFA

We start with the definition of PFA.

Definition 1.1. PFA holds if, given a proper poset $\mathbb{Q}$ and a collection $\mathcal{D}$ of dense subsets of $\mathbb{Q}$, with $|\mathcal{D}| \leq \aleph_1$, then there is a filter $G \subseteq \mathbb{Q}$ such that $G \cap D \neq \varnothing$ for all $D \in \mathcal{D}$.

While we will give a definition of *proper* later on, it is sufficient for the moment to say that properness is a weakening of the *countable chain condition (c.c.c.)*. In particular, PFA is a strengthening of MA($\aleph_1$).

The following two theorems predated PFA and were important in its formulation.

Theorem 1.2. *(Solovay, Tennenbaum [15]) Souslin Hypothesis is consistent with ZFC.*

Theorem 1.3. *(Baumgartner [4]) The following statement is consistent with ZFC: every pair of $\aleph_1$-dense*[1] *subsets of $\mathbb{R}$ are isomorphic.*

[1] A set $X \subseteq \mathbb{R}$ is said to be κ-dense, if every interval meets X in κ points.

Martin observed that Solovay and Tennenbaum's proof of the consistency of Souslin's Hypothesis could be adapted to prove the consistency of a stronger statement, now known as MA($\aleph_1$). The model which Baumgartner constructed to establish the above theorem is, like Solovay and Tennenbaum's model, obtained by a finite support iteration of forcings which satisfy the c.c.c.. Unlike Souslin's Hypothesis, however, the conclusion of Baumgartner's theorem did not apparently follow from MA($\aleph_1$) (this was later confirmed by work of Abraham and Shelah [2]).

Let us now consider two principles which we will later demonstrate are consequences of PFA. Recall that an ideal $\mathcal{I} \subseteq [S]^\omega$ is a *P-ideal* if

- $\mathcal{I}$ contains every finite subsets of S,
- for every family $\{X_n : n \in \omega\} \subseteq \mathcal{I}$ there is an $X \in \mathcal{I}$ such that X_n is contained in X modulo a finite set for every $n \in \omega$.

Definition 1.4. (P-Ideal Dichotomy (PID)) If S is a set and $\mathcal{I} \subseteq [S]^\omega$ is a *P-ideal*, then either

- there is an uncountable $Z \subseteq S$ such that $[Z]^\omega \subseteq \mathcal{I}$, or
- $S = \bigcup_{n\in\omega} S_n$ such that no infinite subset of any S_n is in $\mathcal{I}$.

Definition 1.5. (Open Coloring Axiom (OCA)) If G is an open graph on a separable metric space X, then either

- there is an uncountable $Z \subseteq X$ such that $[Z]^2 \subseteq G$, or
- $X = \bigcup_{n\in\omega} X_n$ such that $[X_n]^2 \cap G = \varnothing$, for every $n \in \omega$.

Remark 1.6. The formulation of this principle is due to Todorcevic and is based on the different formulations of OCA presented in [1].

We will now turn to some of the consequences of these principles.

Theorem 1.7. *(Todorcevic [21]) PFA implies that if X is a Banach space of density $\aleph_1$, then X has a quotient with a basis of length ω_1.*

In fact for Todorcevic's result, the conjunction of PID and MA($\aleph_1$) is sufficient to derive the desired conclusion.

Theorem 1.8. *(Farah [6]) OCA implies that if H is a separable infinite dimensional Hilbert space, then all automorphisms of $\mathcal{B}(H)/\mathcal{K}(H)$ are inner.*

Theorem 1.9. *(Moore [12]) PFA implies that every uncountable linear order contains an isomorphic copy of one of the following: $X, \omega_1, -\omega_1, C, -C$. Here X is a set of reals of cardinality $\aleph_1$ and C is a Countryman line.*

Theorem 1.10. *(Moore [13]) PFA implies that η_C is universal for the Aronszajn lines. Here η_C is the direct limit of the finite lexicographic products of the form $C \times (-C) \times \ldots \times (\pm C)$.*

Theorem 1.11. *(Martinez [8]) PFA implies that the Aronszajn lines are well quasi ordered by embeddability.*

Theorem 1.12. *(Todorcevic, Veličković [5] [23]) PFA implies that $2^{\aleph_0} = \aleph_2$.*

Theorem 1.13. *(Todorcevic [16]) PFA implies that for every regular $\kappa > \aleph_1$, $\square(\kappa)$ fails.*

Theorem 1.14. *(Viale [26]) PFA implies $2^\mu = \mu^+$ whenever μ is a singular strong limit cardinal.*

The two last theorems are consequences of PID [20] [27].

Theorem 1.15. *(Todorcevic [17]) OCA implies that every (κ, λ^*)-gap in $\omega^\omega/\mathrm{fin}$ is of the form $\kappa = \lambda = \omega_1$ or $\min(\kappa, \lambda) = \omega$ and $\max(\kappa, \lambda) \geq \omega_2$.*

2 Some technical background

Both when defining properness and when building proper forcings, it will be convenient to work with set models of a sufficiently strong fragment of ZFC. The most appropriate fragment in the present setting is ZFC with the power set axiom omitted. Models of this theory are provided by the structures $(H(\theta), \in)$, where θ is an uncountable regular cardinal and $H(\theta)$ is the collection of all sets of hereditary cardinality less than θ. Note in particular that if A is in $H(\theta)$, then $\mathcal{P}(A) \subseteq H(\theta)$ and if A and B are in $H(\theta)$, then so is $A \times B$. In particular, $H(\theta) \models |A| \leq |B|$ if and only if $|A| \leq |B|$. Since we will frequently be working with elementary substructures of some fixed $H(\theta)$, it will be useful to also fix a well ordering $\triangleleft$ of $H(\theta)$. This provides a nice set of Skolem functions for $H(\theta)$.

In what follows, we will only be interested in countable elementary submodels of $H(\theta)$ and will write $M \prec H(\theta)$ to mean that M is a countable elementary submodel of $H(\theta)$. We will say that θ is *sufficiently large* with respect to X if $\mathcal{P}(X)$ is in $H(\theta)$. A *suitable model* for X is an $M \prec H(\theta)$ where X is in M and θ is sufficiently large for X.

Definition 2.1. If $\mathbb{Q}$ is a forcing notion and M is a suitable model for $\mathbb{Q}$, then a condition $q \in \mathbb{Q}$ is $(M, \mathbb{Q})$-generic if whenever $r \leq q$ and $D \subseteq \mathbb{Q}$ is dense and in M, there is a $s \in D \cap M$, such that s and r are compatible (i.e. $D \cap M$ is predense below q).

The above definition is equivalent to saying that

$$q \Vdash \dot{G} \cap \check{M} \text{ is } \check{M}\text{-generic}$$

where $\dot{G}$ is the $\mathbb{Q}$-name for the generic filter.

Definition 2.2. A forcing notion $\mathbb{Q}$ is *proper* if whenever M is a suitable model for $\mathbb{Q}$ and q is in $\mathbb{Q} \cap M$, q has an extension which is $(M, \mathbb{Q})$-generic.

The following are useful facts about countable elementary submodels.

Fact 2.3. *If X is definable in $H(\theta)$ from parameters in $M \prec H(\theta)$, then $X \in M$.*

Fact 2.4. *If $X \in M \prec H(\theta)$, then X is countable if and only if $X \subseteq M$*

Proof. Only the reverse implication requires argument. Observe that since elements of ω are definable, $\omega \subseteq M$. If X is countable, then, by elementarity, M will contain a function from X into ω. Since ω is contained in X, its preimage under this function is as well and hence $X \subseteq M$. □

Fact 2.5. *If $M \prec H(\theta)$, then $M \cap \omega_1$ is a countable ordinal.*

Proof. If $\alpha \in M \cap \omega_1$, then $\alpha \subseteq M$ by the previous fact. Hence $M \cap \omega_1$ is a transitive set of ordinals and therefore an ordinal. □

3 Forcings associated to OCA and PID

We will now define posets associated to OCA and to PID and prove that they are proper.

3.1 The poset for OCA

Suppose that G is an open graph on X. Let $\mathbb{Q}_{X,G}$ be the poset consisting of all the pairs $q = (H_q, \mathcal{N}_q)$ such that:

1. $H_q \subseteq X$ is a finite clique in G;
2. $\mathcal{N}_q$ is an increasing finite $\in$-chain of elementary submodels of $H({2^{\aleph_0}}^+)$ which each contain (X, G);

3. if $x \neq y$ are in H_q, then there is an N in $\mathcal{N}_q$ such that $|N \cap \{x, y\}| = 1$;

4. if N is in $\mathcal{N}_q$ and $E \subseteq X$ is in N with $[E]^2 \cap G = \varnothing$, then $E \cap H_q \subseteq N$.

Define the order on $\mathbb{Q}_{X,G}$ by $p \leq q$ if $H_q \subseteq H_p$ and $\mathcal{N}_q \subseteq \mathcal{N}_p$. Notice that, by *condition 3*, $\mathcal{N}_q$ induces an order on H_q corresponding to the number of elements of $\mathcal{N}_q$ which do not contain a given element of H_q. It will be convenient to let $x_q(i)$ denote the i^{th} element of H_q in this enumeration.

Remark 3.1. If CH holds, then can modify the forcing $\mathbb{Q}_{X,G}$ as follows. Let $\mathcal{N}$ be a continuous $\in$-chain of length ω_1 of countable elementary submodels of $H({2^{\aleph_0}}^+)$, each containing the relevant objects. Define, e.g., $\mathbb{Q}_{X,G}$ to be all H_q such that $(H_q, \mathcal{N}_q)$ is in $\mathbb{Q}_{X,G}$ whenever $\mathcal{N}_q$ is a finite subset of $\mathcal{N}$. One can verify that this modified forcing in fact satisfies the countable chain condition. The role of CH here can be explained as follows: if X is a separable metric space and if M and N are two countable elementary submodels of $(H({2^{\aleph_0}}^+), \in, X)$ such that $M \cap \omega_1 = N \cap \omega_1$, then CH implies that $M \cap H(\aleph_1) = N \cap H(\aleph_1)$ and hence M and N have the same intersection with the closed subsets of X. Notice that in *condition 4*, the set E can be assumed to be closed since if $[E]^2 \cap G$ is empty, the same is true for the closure of E.

3.2 The poset for PID

Let $\mathcal{I}$ be a P-ideal on a set S and suppose that θ is a regular cardinal such that $\mathcal{I}$ is in $H(\theta)$. For each countable subset X of $\mathcal{I}$, let I_M be an element of $\mathcal{I}$ such that $I \subseteq_* I_X$ for every I in X. Here $I \subseteq_* J$ means that the set $I \setminus J$ is finite. If M is a countable elementary submodel of $H(\theta)$, will let I_M denote $I_{M \cap \mathcal{I}}$. Define $\mathcal{I}^\perp$ to be the collection of all subsets of S which have finite intersection with every element of $\mathcal{I}$.

Let $\mathbb{Q}_\mathcal{I}$ be the poset consisting of all the pairs $q = (Z_q, \mathcal{N}_q)$ such that

1. $Z_q \subseteq S$ is finite;

2. $\mathcal{N}_q$ is an increasing finite $\in$-chain of elementary submodels of $H(\theta)$ which each contain $\mathcal{I}$;

3. if $x, y \in S$, then there is an N in $\mathcal{N}_q$ such that $|N \cap \{x, y\}| = 1$;

4. if N is in $\mathcal{N}_q$ and X is in $N \cap \mathcal{I}^\perp$, then $X \cap Z_q \subseteq N$.

Define the order on $\mathbb{Q}_\mathcal{I}$ by:

1. $p \leq q$ if $Z_q \subseteq Z_p$;

2. $\mathcal{N}_q \subseteq \mathcal{N}_p$;

3. $(Z_p \setminus Z_q) \cap N \subseteq I_N$, whenever N is in $N \in \mathcal{N}_q$.

4 Verification of properness

The above examples of forcings all have a common from. For instance, their elements consist of pairs $q = (X_q, \mathcal{N}_q)$ where X_q is a finite approximation of some desired object and $\mathcal{N}_q$ is a finite $\in$-chain of countable elementary submodels of some suitably chosen $H(\theta)$. In order to verify properness of such forcings, the general strategy is to argue that if M is a suitable model for $\mathbb{Q}$ and $M \cap H(\theta)$ is in $\mathcal{N}_q$, then q is $(M, \mathbb{Q})$-generic. The forcing $\mathbb{Q}$ is defined in such a way that it is trivial to verify that if q_0 is in $M \cap \mathbb{Q}$, then

$$q = (X_{q_0}, \mathcal{N}_{q_0} \cup \{M \cap H(\theta)\})$$

a condition in $\mathbb{Q}$ which extends q_0.

In order to verify that $M \cap H(\theta) \in \mathcal{N}_q$ implies the $(M, \mathbb{Q})$-genericity of q, one usually argues that:

(*): if $D \subseteq \mathbb{Q}$ is in M and q is in D, then q is compatible with an element of $D \cap M$.

Notice that this implies genericity: if $D \subseteq \mathbb{Q}$ is dense and in M, then we can first extend q to an element r of D and then appeal to (*) to find a condition s in $D \cap M$ which is compatible with r and hence with q.

We now show that $\mathbb{Q}_{X,G}$ and $\mathbb{Q}_{\mathcal{I}}$ are proper and argue that $\mathbb{Q}_{A,B}$ can be viewed as a special case of a modified version of $\mathbb{Q}_{X,G}$.

Theorem 4.1. *$\mathbb{Q}_{X,G}$ is proper.*

Proof. Suppose that M is suitable for $\mathbb{Q}_{X,G}$, r is in $\mathbb{Q}_{X,G}$, and $N = M \cap H(2^{\aleph_0 +})$ is in $\mathcal{N}_r$. It is sufficient to verify that r is $(M, \mathbb{Q}_{X,G})$-generic and this will be done by verifying (*). To this end, let $D \subseteq \mathbb{Q}_{X,G}$ be in M and contain r. Fix disjoint open sets U_i $(i < n)$ which are in N such that $x_r(i)$ is in U_i and $U_i \times U_j \subseteq G$ whenever $i \neq j < n$.

By replacing D with a subset if necessary, we may assume that if s is in D, then

- $s \leq r_0$,
- for some $N_s \in \mathcal{N}_s$, $r_0 = (H_s \cap N_s, \mathcal{N}_s \cap N_s)$,

- $|H_s| = |H_r| = n$, and
- $x_s(i) \in U_i$ for all $i < n$.

Let $r_0 = r \cap M$ and note that $r_0 \in N$.

To find a condition s in $D \cap M$ compatible with r we have to check that $[H_r \cup H_s]^2 \subseteq G$. By construction of the set D, we just need to show that

$$\{x_r(i), x_s(j)\} \in G \text{ for } i = j,$$

because for $i \neq j$, U_i and U_j already witness the fact that $\{x_r(i), x_s(j)\} \in G$.

The following is now the key lemma.

Lemma 4.2. *Suppose that:*

- $N \prec H(2^{\aleph_0 +})$ *with* $(X, G) \in N$*;*
- $A \subseteq X^n$ *is in* N*;*
- $\bar{x} \in \mathrm{cl}(A)$ *and* $\bar{x} \restriction n-1 \in M$*;*
- *for every* $E \subseteq X$ *in* M *with* $[E]^2 \cap G = \varnothing$*,* $x(n-1)$ *is not in* $\mathrm{cl}(E)$*.*

Then there is a basic open $U \subseteq X$ *not containing* $x(n-1)$ *such that*

$$\{x(n-1), y\} \in G \textit{ for every } y \in U,$$

$$\bar{x} \restriction n-1 \in \mathrm{cl}(\{\bar{y} \restriction (n-1) : \bar{y} \in A \wedge y(n-1) \in U\}).$$

Proof. Let $\{W_i : i \in \omega\}$ be a neighborhood base for $\bar{x} \restriction (n-1)$ which is in M and define

$$E_0 = \bigcap_i \mathrm{cl}(\{\bar{y}(n-1) : \bar{y} \in A \text{ and } \bar{y} \restriction (n-1) \in W_i\}).$$

Notice that $x(n-1) \in E_0$. Observe that E_0 is in M since it is definable from parameters in M. Similarly,

$$E = \{z \in E_0 : \forall y \in E_0 \ \{z, y\} \notin G\}$$

is in M. By the fourth condition in the definition of $\mathbb{Q}_{X,G}$, $x(n-1) \notin E$. Pick then $y \in E_0$ such that $\{x(n-1), y\} \in G$. Since G is open, we can find a basic open neighborhood U of y which does not contain x such that $\{x(n-1)\} \times U \subseteq G$.

We now must show that

$$\bar{x} \restriction n-1 \in \mathrm{cl}(\{\bar{y} \restriction (n-1) : \bar{y} \in A \wedge y(n-1) \in U\}).$$

Let $j \in \omega$ be given. Since U is open and y is in E_0, there is a $\bar{z}$ in A such that $\bar{z} \restriction (n-1)$ is in W_j and $z(n-1)$ is in U. Since $\{W_i : i \in \omega\}$ is a neighborhood base at $\bar{x} \restriction (n-1)$ and j was arbitrary, we have the desired conclusion. $\square$

We are now ready to complete the proof that $\mathbb{Q}_{X,G}$ is proper. Set $A = \{\bar{x}_s : s \in D\}$ and, using the above lemma, inductively construct a sequence of sets A_i $(i \leq n)$ in N such that:

- $A_0 = A$ and $A_i \subseteq A_{i-1}$ if $i > 0$;
- $x_r \upharpoonright i$ is an accumulation point of $\{y \upharpoonright i : y \in A_i\}$;
- if $i < j < n$ and y is in A_i, then $\{x(j), y(j)\} \in G$.

Now if s is in N and $\bar{x}_s$ is in A_n, it follows that s is compatible with r. □

Finally we turn to the proof that the PID forcing is proper.

Lemma 4.3. *$\mathbb{Q}_{\mathcal{I}}$ is proper.*

Proof. Let M be a suitable model for $\mathbb{Q}_{\mathcal{I}}$ and suppose that r is in $\mathbb{Q}_{\mathcal{I}}$ with $M \cap H(\theta)$ in $\mathcal{N}_r$. Define $r_0 = (Z_r \cap M, \mathcal{N}_r \cap M)$ and let $n = |Z_r \setminus M|$. We will verify that (*) holds. To this end, suppose that $D \subseteq \mathbb{Q}_{\mathcal{I}}$ with D in M and r in D. By replacing D with a subset if necessary, we may assume that if s is in D then:

- there is an N_s in $\mathcal{N}_s$ such that $r_0 = (Z_s \cap N_s, \mathcal{N}_s \cap N_s)$;
- $|Z_s \setminus N_s| = n$.

By definition of extension in $\mathbb{Q}_{\mathcal{I}}$, our goal will be to find an $s \in D \cap M$ such that $Z_s \setminus N_s \subseteq I_P$, for each $P \in \mathcal{N}_r \setminus M$. This is sufficient, since then $(Z_s \cup Z_r, \mathcal{N}_s \cup \mathcal{N}_r)$ is in $\mathbb{Q}_{\mathcal{I}}$ and is a common extension of r and s.

Define $T = \{t_s : t_s = Z_s \setminus N_s \text{ and } s \in D\}$.

Claim 4.4. *There is a $T' \subseteq T$ which is $\mathcal{J}^+$-splitting, where $\mathcal{J}$ is the σ-ideal generated by $\mathcal{I}^\perp$.*

Proof. If $U \subseteq S^n$, define ∂U to be the set of all u in U such that there is no $k < n$ with

$$\{v(k) : (v \in U) \wedge (u \upharpoonright k = v \upharpoonright k)\} \in \mathcal{J}.$$

By definition, U is $\mathcal{J}^+$-splitting if $\partial U = U$ and U is non empty. Observe that for any $U \subseteq S^n$, if u is in $\partial^{i+1} U \setminus \partial^i U$, then the k which witnesses this must be less than $n - i$. In particular, $\partial^{n+1} T = \partial^n T$. Observe that $T' = \partial^n T$ is in N and contains $t_r = Z_r \setminus N$. □

Now we inductively build $\sigma_0, \sigma_1, \ldots \sigma_n \in M$ such that $\sigma_0 = \varnothing$ and each σ_i is the initial part of some $s \in T'$ with σ_{i+1} having σ_i as an initial part whenever $i < n$. Given σ_i consider

$$\{x \in S : \sigma_i{}^\frown x \text{ has a extension in } T'\}.$$

This set is in M, since it is definable from parameters which are in M. By elementarity of M and the definition of $\mathcal{J}$, the above set contains a countably infinite subset H in M that is in $\mathcal{I}$. In particular, $H \subseteq_* I_P$ for all $P \in \mathcal{N}_r$. Hence there is an element $x \in H \cap \bigcap_{P \in \mathcal{N}_r} I_P$ such that $\sigma_{i+1} = \sigma_i{}^\frown x$ has an extension in T'. Finally, at stage n, $\sigma_n = \sigma = Z_s \setminus N_s \subseteq \bigcap_{P \in \mathcal{N}_r} I_P$, for some $s \in D$. Such an s is now compatible with r. We have therefore established that $\mathbb{Q}_\mathcal{I}$ is proper. □

5 Density arguments

While the main difficulty in proving implications such as *PFA implies PID* lies in the verification that the relevant forcing is proper, some additional argument is usually required to show that certain sets are dense. In the case of PID and OCA, this verification is straightforward. In order to illustrate the argument, we will go through the remainder of the proof that PFA implies PID.

Theorem 5.1. *(*PFA*)* PFA *implies* PID*.*

Proof. Suppose that $\mathcal{I} \subseteq [S]^\omega$ is a P-ideal. If S can be covered by countably many sets in $\mathcal{I}^\perp$, then there is nothing to show. Suppose that this is not the case and let M be a suitable model for $\mathbb{Q}_\mathcal{I}$ and $x \in X$ be outside of every element of $\mathcal{I}^\perp \cap M$. Define $q = (\{x\}, \{M \cap H(\theta)\})$. We have established that q is $(M, \mathbb{Q}_\mathcal{I})$-generic and therefore q forces that

$$\dot{Z} = \bigcup_{p \in \dot{G}} Z_p$$

is uncountable. Also observe that q forces that every countable subset of S is contained in N for some N in $\dot{\mathcal{N}} = \bigcup_{p \in \dot{G}}$. This is easily seen for countable sets in V and holds for sets in $V[G]$ as well since $\mathbb{Q}_\mathcal{I}$ is proper. Furthermore, q forces that if N is in $\dot{\mathcal{N}}$, then $N \cap \dot{Z} \subseteq^* I_N$ and hence is in $\mathcal{I}$.

Let $\dot{f}$ be a $\mathbb{Q}_\mathcal{I}$-name for an injection from ω_1 into $\dot{Z}$ and let $\dot{g}$ be a $\mathbb{Q}_\mathcal{I}$-name for a function from ω_1 into $\mathcal{I}$ such that q forces $\forall \delta < \omega_1 (\dot{f}''\delta \subseteq \dot{g}(\delta))$.

If we let D_ξ be the set of all p such that p decides $\dot{f}(\xi)$ and $\dot{g}(\xi)$, then any filter G meeting D_ξ for all $\xi < \omega_1$ must satisfy that $Z = \dot{Z}[G]$ is an uncountable set such that every countable subset of Z is in $\mathcal{I}$. □

6 Some application of PID

We now present some consequences of PID.

Definition 6.1. An ideal $\mathcal{K}$ on a set S is *countably generated in* $\mathcal{X}$ if there is a countable family $\{K_n : n \in \omega\} \subseteq \mathcal{X}$ such that for all $K \in \mathcal{K}$, there is an n such that $K \subseteq K_n$. We will say that $\mathcal{K}$ is *countably generated* if it is countably generated in $\mathcal{K}$.

In what follows we will use the notational convention that $\mathcal{K} \restriction X = \mathcal{K} \cap \mathcal{P}(X)$.

Theorem 6.2. *PID implies that if $\mathcal{K}$ is an ideal on S that is not countably generated in $\mathcal{K}^{\perp\perp}$, then there is an $X \subseteq S$ of size less or equal to $\aleph_1$ such that $\mathcal{K} \restriction X$ is not countably generated in $\mathcal{K}^{\perp\perp}$.*

Proof. Define $\mathcal{I} = \mathcal{K}^{\perp}$ and assume that, for every countable $X \subseteq S$, $\mathcal{K} \restriction X$ is countably generated in $\mathcal{K}^{\perp\perp}$.

Claim 6.3. *$\mathcal{I}$ is a P-ideal*

Proof. Let $I_n \in \mathcal{I}$, for $n \in \omega$ and set $X = \bigcup_n I_n$. Since $X \subseteq S$ is countable, $\mathcal{K} \restriction X$ is countably generated in $\mathcal{K}^{\perp\perp}$, and so fix a family $\{K_i : i \in \omega\} \subseteq \mathcal{K}^{\perp\perp}$ which witnesses this. Without loss of generality we can also suppose that $K_i \subseteq K_{i+1}$.

Now define

$$I_* = \bigcup_i (I_i \setminus K_i).$$

To see that this set is in $\mathcal{I}$, it is sufficient to show that it has finite intersection with K_n for each $n < \omega$. For a given n, the intersection of I_* with K_n is contained in $\bigcup_{i<n} I_i$, a set which is in $\mathcal{I}$ and has finite intersection with K_n. Hence I_* is in $\mathcal{I}$ as desired. Moreover $I_n \subseteq_* I_*$, since $I_n \setminus I_* \subseteq K_n \cap I_n$, which is finite. $\square$

We can now apply PID. If $S = \bigcup_n S_n$ such that, for all n, $S_n \in \mathcal{I}^{\perp}$, then $\{S_n : n \in \omega\}$ witnesses that $\mathcal{K}$ is countably generated in $\mathcal{K}^{\perp\perp}$.

On the other hand, if $Z \subseteq S$ is uncountable and $[Z]^{\omega} \subseteq \mathcal{I}$, then $Z \cap K$ is finite, for every $K \in \mathcal{K}$, because if $Z \cap K$, were infinite, then there would be a countable $Y \subseteq Z \cap K$, but then $Y \in [Z]^{\omega} \subseteq \mathcal{I}$ and hence $Y \cap K$ would be finite. Z witnesses that $\mathcal{K} \restriction Z$ is not countably generated in $\mathcal{K}^{\perp\perp}$, because there is no family $\{K_n\}_{n\in\omega}$ in $\mathcal{K}^{\perp\perp}$, such that, for all $K \in \mathcal{K}$, there is an n such that $K \subseteq K_n$. $\square$

Corollary 6.4. *PID implies SCH.*

Proof. We can state SCH in the following way: for every strong limit cardinal μ, with $\text{cof}(\mu) < \mu$, $2^\mu = \mu^+$. Suppose that SCH fails and let μ be the least witness to this. By Silver's theorem, $\text{cof}(\mu) = \omega$. Fix an increasing sequence $\{\mu_n\}_{n\in\omega}$ that converges to μ. For each $\beta < \mu^+$, find $K_{\beta,n} \subseteq \beta$ such that

- $|K_{\beta,n}| \leq \mu_n$,
- $\bigcup_n K_{\beta,n} = \beta$,
- $\forall \beta < \beta'$, for every m there exists an n such that $K_{\beta,m} \subseteq K_{\beta',n}$

The key point here is that if $X \subseteq \mu^+$ is countable, then there is a β such that for every $\beta' > \beta$, the ideal on X generated by $\{K_{\beta,n} \cap X : n \in \omega\}$ equals the ideal on X generated by $\{K_{\beta',n} : n \in \omega\}$. This follows from the fact that $2^{2^{\aleph_1}} < \mu^+$.

The above fact implies that, if $\mathcal{K}$ is the ideal on μ^+ generated by $\{K_\beta, n : \beta < \mu^+, n \in \omega\}$, then $\mathcal{K} \restriction X$ is countably generated $\forall X \in [\mu^+]^{\aleph_1}$.

So, by Theorem 6.2 and by PID, $\mathcal{K}$ is countably generated in $\mathcal{K}^{\perp\perp}$. In particular there is a $Z \subseteq \mu^+$ cofinal in μ^+ such that all countable subsets of Z are contained in $K_{\beta,n}$, for some β and n.

Thus $|Z^\omega| = \mu^+ = |\mu^\omega|$ and so $2^\mu = \mu^+$, since any subset $X \subseteq \mu$ can be seen as $X = \bigcup_{n\in\omega} X \cap \mu_n$. $\square$

Corollary 6.5. *PID the failure of $\square(\kappa)$, for every regular $\kappa > \aleph_1$.*

Proof. We just sketch the proof. Recall that $\square(\kappa)$ is the following principle: there is a sequence $\langle C_\alpha : \alpha < \kappa\rangle$ such that:

- $C_\alpha \subseteq \alpha$ is club in α;
- if α is a limit point of C_β, then $C_\alpha = C_\beta \cap \alpha$,
- there is no club $C \subseteq \kappa$ such that, for all limit points α of C, $C \cap \alpha = C_\alpha$.

Now, for $\alpha < \beta < \kappa$ inductively define

$$\varrho_2(\alpha, \beta) = 1 + \varrho_2(\alpha, min(C_\beta \setminus \alpha)),$$

$$\varrho_2(\alpha, \alpha) = 0.$$

Thus $\varrho_2(\alpha, \beta)$ is the length of the walk from β to α.

We define, for $\beta < \kappa$,

$$K_{\beta,n} = \{\alpha < \beta : \varrho_2(\alpha, \beta) \leq n\}.$$

It can be verified that ϱ_2 has the following properties:

- if $\beta < \beta'$, then there is an n such that $|\varrho_2(\alpha,\beta) - \varrho_2(\alpha,\beta')| \leq n$ for all $\alpha < \beta$;
- if $X \subseteq \kappa$ is unbounded, then there is a β such that $\{\varrho_2(\alpha,\beta) : \alpha \in X \cap \beta\}$ is infinite.

It follows that the ideal $\mathcal{K}$ generated by $\{K_{\beta,n} : (\beta < \kappa) \wedge (n < \omega)\}$ is not countably generated in $\mathcal{K}^{\perp\perp}$ and yet $\mathcal{K} \restriction \beta$ is countably generated in $\mathcal{K}^{\perp\perp}$ for all $\beta < \kappa$. $\square$

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