Formalizing Set Theory

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February 2009: Young Set Theory Workshop

Outline

- Definitions and Basics on MCG
- Formalization
- The example
- The formalization

Definitions

S_∞: the group of bijections $\mathbb{N} \to \mathbb{N}$ (permutations) with operation composition.

$f \in S_\infty$ is *cofinitary* iff f is the identity or has only finitely many fixed points.

$G \leq S_\infty$ is a *cofinitary group* iff all $g \in G$ are cofinitary.

$G \leq S_\infty$ is a *maximal cofinitary group* iff G is a cofinitary group and is not properly contained in another cofinitary group.

Some basic properties

Any cofinitary group is contained in a maximal cofinitary group.

(Adeleke, Truss) A maximal cofinitary group cannot be countable.

(Neumann) There is a cofinitary group of size $|\mathbb{R}|$.

(Zhang) If $|\mathbb{N}| < \kappa \leq |\mathbb{R}|$ then it is consistent that there is a maximal cofinitary group G with $|G| = \kappa$.

Some more results (Kastermans)

No maximal cofinitary group has infinitely many orbits.

MA implies there is a maximal cofinitary group with multiple infinite orbits.

MA implies there is a locally finite maximal cofinitary group.

The axiom of constructibility implies there exists a coanalytic maximal cofinitary group.

What is formalization?

Want to construct actual proofs of results like the ones mentioned above.

Were these results then accepted without proof?

What is a proof?

A proof I

$$\dfrac{[A \vee B]^2 \qquad \dfrac{[A]^1}{B \vee A}\,\vee\mathsf{R} \qquad \dfrac{[B]^1}{B \vee A}\,\vee\mathsf{L}}{\dfrac{B \vee A}{A \vee B \to B \vee A}\,2, \to\mathsf{I}}\,1, \vee\mathsf{E}$$

A proof II

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg p]^1}{p \vee \neg p}\,\vee\mathrm{R} \qquad [\neg(p \vee \neg p)]^2}{\bot}\,{\to}\mathrm{E}}{p}\,1,\ \mathrm{RAA}}{p \vee \neg p}\,\vee\mathrm{L} \qquad [\neg(p \vee \neg p)]^2}{\bot}\,{\to}\mathrm{E}}{p \vee \neg p}\,2,\ \mathrm{RAA}$$

System chosen

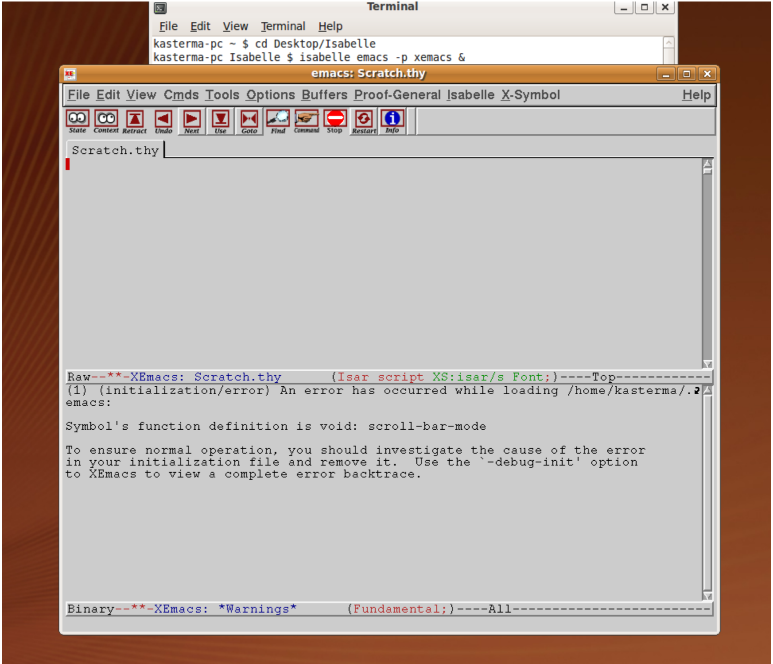
Isabelle/HOL

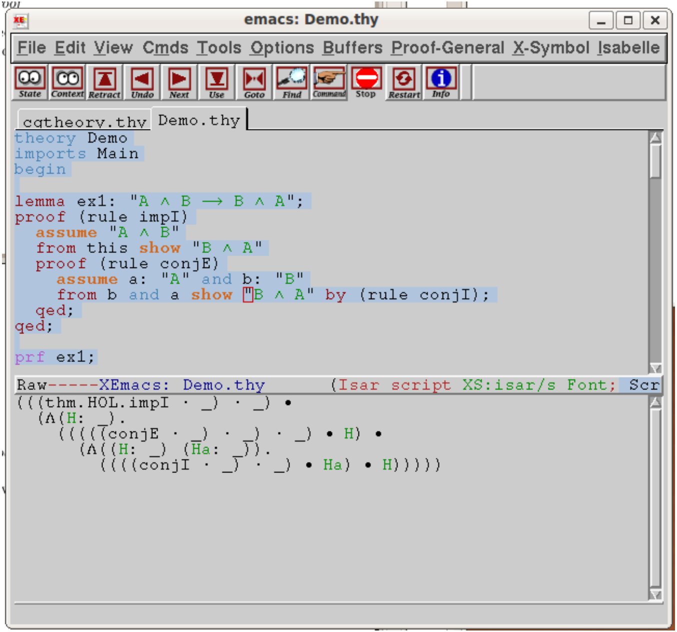
Isabelle is a generic proof assistant.

HOL is for higher order logic.

Chosen because of Isar language.

Proof General




emacs: Demo.thy
File Edit View Cmds Tools Options Buffers Proof-General X-Symbol Isabelle
State Context Retract Undo Next Use Goto Find Command Stop Restart Info
cqtheory.thy Demo.thy
theory Demo
imports Main
begin
lemma ex1: "A ∧ B ⟶ B ∧ A";
proof (rule impI)
assume "A ∧ B"
from this show "B ∧ A"
proof (rule conjE)
assume a: "A" and b: "B"
from b and a show "B ∧ A" by (rule conjI);
qed;
qed;
prf ex1;
Raw-----XEmacs: Demo.thy (Isar script XS:isar/s Font; Scr
(((thm.HOL.impI · _) · _) •
(Λ(H: _).
(((((conjE · _) · _) · _) • H) •
(Λ((H: _) (Ha: _)).
((((conjI · _) · _) • Ha) • H)))))

```
lemma ex2: "A ∧ B ⟶ B ∧ A"
proof
  assume "A ∧ B"
  from this show "B ∧ A"
  proof
    assume "A" and "B"
    thus ?thesis by auto;
  qed;
qed;
```

```
lemma ex3: "A ∧ B ⟶ B ∧ A" by auto;
```

```
lemma ex4: assumes "A ∧ B"
  shows "B ∧ A"
proof
  from `A ∧ B` show "B" ..;
  from `A ∧ B` show "A" ..;
qed;
```

```
lemma assumes Pf: "∃x. P(f x)" shows "∃ y. P y"
proof -
  from Pf obtain x where "P(f x)" ..
  thus " ∃  y. P y" ..
qed;
```

```
lemma intcom2: "A ∩ B = B ∩ A" by fast;
```

The Simple Example

Let $g(k) = k + 1$ on the integers. Let $Ex_1 = \langle g \rangle$. Pick a bijection from the natural numbers to the integers (say evens map to the positive integers, and the odds to the negative integers). Then conjugating $Ex_1 1$ by that bijection gives a cofinitary group Ex_2.

```
definition S_inf :: "(nat ⇒ nat) set"
where
"S_inf = {f::(nat ⇒ nat). bij f}";   (* next;; *)
```

```
locale CofinitaryGroup =
  fixes
    dom :: "(nat ⇒ nat) set"
  assumes
    type_dom : "dom ⊆ S_inf" and
    id_com : "id ∈ dom" and
    mult_closed : "f ∈ dom ∧ g ∈ dom ⟹ f ∘ g ∈ dom" and
    inv_closed : "f ∈ dom ⟹ inv f ∈ dom" and
    cofinitary : "f ∈ dom ∧ f ≠ id ⟹ finite (Fix f)"; (* ne
```

```
definition upOne :: "int ⇒ int"
where
"upOne n = n + 1";   (* next;; *)
```

```
theorem bij_upOne: "bij upOne"
by (unfold bij_def, rule conjI [OF inj_upOne surj_upOne]);
```

```
theorem "Fix upOne = {}"
proof -
  from Fix_def[of upOne]
  have "Fix upOne = {n . upOne n = n}" by auto;
  with no_fix_upOne have "Fix upOne = {n . False}" by auto;
  with Set.empty_def show "Fix upOne = {}" by auto;
qed;
```

```
inductive_set Ex1 :: "(int ⇒ int) set" where
base_func: "upOne ∈ Ex1" |
comp_func: "f ∈ Ex1 ⟹ (upOne ∘ f) ∈ Ex1" |
comp_inv : "f ∈ Ex1 ⟹ ((inv upOne) ∘ f) ∈ Ex1";
```

```
theorem Ex1_Normal_form: "(f ∈ Ex1) = (∃k. ∀n. f(n) = n + k)";
proof
  assume "f ∈ Ex1"
  with Ex1_Normal_form_part1 [of f]
    show "(∃k. ∀n. f(n) = n + k)" by auto;
next;
  assume "∃k. ∀n. f(n) = n + k"
  with Ex1_Normal_form_part2
    show "f ∈ Ex1" by auto;
qed;    (* next;; *)
```

```
theorem no_fixed_pt:
  assumes f_Ex1: "f ∈ Ex1"
  and f_not_id: "f ≠ id"
  shows "Fix f = {}";     (*;
```

```
theorem closed_comp: "f ∈ Ex1 ∧ g ∈ Ex1 ⟹ f ∘ g ∈ Ex1"    (* ne
proof (rule Ex1.induct [of f], blast);
  assume "f ∈ Ex1 ∧ g ∈ Ex1";
  with Ex1.comp_func[of g] show "upOne ∘ g ∈ Ex1" by auto;
next
  fix fa
  assume "fa ∘ g ∈ Ex1"
  with Ex1.comp_func [of "fa ∘ g"]
    and Fun.o_assoc [of "upOne" "fa" "g"]
    show "upOne ∘ fa ∘ g ∈ Ex1" by auto;
next
  fix fa
  assume "fa ∘ g ∈ Ex1"
  with Ex1.comp_inv [of "fa ∘ g"]
    and Fun.o_assoc [of "inv upOne" "fa" "g"]
    show "(inv upOne) ∘ fa ∘ g ∈ Ex1" by auto;
qed;
```

```
definition ni_bij:: "nat ⇒ int"
where
"ni_bij n = (if ((n mod (2)) = 0)
              then int (n div 2)
              else -int (n div 2) - 1)"
```

```
theorem ni_bij_bij: "bij ni_bij";    (* nex
proof (unfold bij_def, rule conjI);

  show INJ: "inj ni_bij"
  proof (rule injI)
    fix x::nat and y::nat
    assume eq_ass: "ni_bij x = ni_bij y";
    show "x = y"
```

```
theorem conj_fix_pt: "⋀f::('a ⇒ 'b). ⋀g::('b ⇒ 'b). (bij f)
  ⟹ ((inv f)`(Fix g)) = Fix ((inv f) ∘ g ∘ f)"; █(* next; *)
proof -
  fix f::"'a ⇒ 'b"
  assume bij_f: "bij f"
  with bij_def have inj_f: "inj f" by auto;
  fix g::"'b⇒'b"
  show "((inv f)`(Fix g)) = Fix ((inv f) ∘ g ∘ f)";
  thm set_eq_subset[of "(inv f)`(Fix g)" "Fix((inv f) ∘ g ∘ f)"
  proof
    show "(inv f)`(Fix g) ⊆ Fix ((inv f) ∘ g ∘ f)"
    proof
      fix x
      assume "x ∈ (inv f)`(Fix g)"
      with image_def have "∃y ∈ Fix g. x = (inv f) y" by auto;
```

```
definition CONJ :: "(int ⇒ int) ⇒ (nat ⇒ nat)"
where
"CONJ f = (inv ni_bij) ∘ f ∘ ni_bij";  (* next;; *)

declare CONJ_def [simp] -- "automated tools can use the definition";
```

```
lemma type_CONJ: "f ∈ Ex1 ⟹ (inv ni_bij) ∘ f ∘ ni_bij ∈ S_inf"
(* next;; *)
proof -
  assume f_Ex1: "f ∈ Ex1"
  with all_bij have "bij f" by auto;
  with ni_bij_bij and comp_bij
    have bij_f_nibij: "bij (f ∘ ni_bij)" by auto;
  with ni_bij_bij and bij_imp_bij_inv have "bij (inv ni_bij)" by auto;
  with bij_f_nibij and comp_bij[of  "f ∘ ni_bij" "inv ni_bij"]
    and o_assoc[of "inv ni_bij" "f" "ni_bij"]
    have "bij ((inv ni_bij) ∘ f ∘ ni_bij)" by auto;
  with S_inf_def show "((inv ni_bij) ∘ f ∘ ni_bij) ∈ S_inf"; by auto;
qed;
```

```
lemma inv_CONJ:
  assumes bij_f: "bij f"
  shows "inv (CONJ f) = CONJ (inv f)" (is "?left = ?right")
(* next; *)
proof -
  have st1: "?left = inv ((inv ni_bij) ∘ f ∘ ni_bij)"
    using CONJ_def by auto;
  from ni_bij_bij and bij_imp_bij_inv
    have inv_ni_bij_bij: "bij (inv ni_bij)" by auto;
  with bij_f and comp_bij have "bij (inv ni_bij ∘ f)" by auto;
  with o_inv_distrib[of "inv ni_bij ∘ f" ni_bij] and ni_bij_bij
  have "inv ((inv ni_bij) ∘ f ∘  ni_bij) =
    (inv ni_bij) ∘ (inv ((inv ni_bij) ∘ f))" by auto;
  with st1 have st2: "?left =
    (inv ni_bij) ∘ (inv ((inv ni_bij) ∘ f))" by auto;
  from inv_ni_bij_bij and `bij f` and o_inv_distrib
    have h1: "inv (inv ni_bij ∘ f) = inv f ∘ inv (inv (ni_bij))" by auto;
  from ni_bij_bij and inv_inv_eq[of ni_bij]
    have "inv (inv ni_bij) = ni_bij" by auto;
  with st2 and h1 have "?left = (inv ni_bij ∘ (inv f ∘ ( ni_bij)))" by auto;
  with o_assoc have "?left = inv ni_bij ∘ inv f ∘ ni_bij" by auto;
  with CONJ_def[of "inv f"] show ?thesis by auto;
qed;
```

```
definition Ex2 :: "(nat ⇒ nat) set"
where
"Ex2 = CONJ`Ex1";
(* next: *)
```

```
theorem Ex2_cofinitary:
  assumes f_Ex2: "f ∈ Ex2"
  and f_nid: "f ≠ id"
  shows "Fix f = {}";
(* next; *)
proof -
  from f_Ex2 and mem_Ex2_rule
  obtain g where g_Ex1: "g ∈ Ex1" a
  with id_CONJ and f_nid have "g ≠
  with g_Ex1 and no_fixed_pt[of g]
  from conj_fix_pt[of ni_bij g] and
  have "(inv ni_bij)`(Fix g) = Fix(
```

```
lemma comp_Ex2:
  assumes f_Ex2: "f ∈ Ex2" and
  g_Ex2: "g ∈ Ex2"
  shows "f ∘ g ∈ Ex2"
proof -
  from f_Ex2 obtain f_1
    where f_1_Ex1: "f_1 ∈ Ex1" and "f = CONJ f_1"
    using mem_Ex2_rule by auto;
  moreover
  from g_Ex2 obtain g_1
    where g_1_Ex1: "g_1 ∈ Ex1" and "g = CONJ g_1"
    using mem_Ex2_rule by auto;
  ultimately
  have "f ∘ g = (CONJ f_1) ∘ (CONJ g_1)" by auto;
  hence "f ∘ g = CONJ (f_1 ∘ g_1)" using comp_CONJ by auto;
  moreover
  have "f_1 ∘ g_1 ∈ Ex1" using closed_comp and f_1_Ex1 and g_1_Ex1 by auto;
  ultimately
  show "f ∘ g ∈ Ex2" using mem_Ex2_rule by auto;
qed;
```

```
interpretation CofinitaryGroup Ex2;
proof;
  show "Ex2 ⊆ S_inf"
  proof;
    fix f
    assume "f ∈ Ex2"
    with mem_Ex2_rule obtain g where "g ∈ Ex1" and "f = CONJ g" by auto;
    with type_CONJ show "f ∈ S_inf" by auto;
  qed;
next
  from id_Ex2 show "id ∈ Ex2" .;
next
  fix f g
  assume "f ∈ Ex2 ∧ g ∈ Ex2"
  with comp_Ex2 show "f ∘ g ∈ Ex2"; by auto;
next
  fix f
  assume "f ∈ Ex2"
  with inv_Ex2 show "inv f ∈ Ex2" by auto;
next;
  fix f
  assume "f ∈ Ex2 ∧ f ≠ id"
  with Ex2_cofinitary have "Fix f = {}" by auto;
  thus "finite (Fix f)" using finite_def by auto;
qed;
```

9 The Conclusion

With all that we have shown we have already clearly shown *Ex2* to be a cofinitary group. The formalization also shows this, we just have to refer to the correct theorems proved above.

interpretation *CofinitaryGroup Ex2*
proof
  show $Ex2 \subseteq S\text{-}inf$
  proof
    fix f
    assume $f \in Ex2$
    with *mem-Ex2-rule* **obtain** g **where** $g \in Ex1$ **and** $f = CONJ\ g$ **by** *auto*
    with *type-CONJ* **show** $f \in S\text{-}inf$ **by** *auto*
  qed
next
  from *id-Ex2* **show** $id \in Ex2$ **.**
next
  fix $f\ g$
  assume $f \in Ex2 \wedge g \in Ex2$
  with *comp-Ex2* **show** $f \circ g \in Ex2$ **by** *auto*
next
  fix f
  assume $f \in Ex2$
  with *inv-Ex2* **show** $inv\ f \in Ex2$ **by** *auto*
next
  fix f
  assume $f \in Ex2 \wedge f \neq id$
  with *Ex2-cofinitary* **have** $Fix\ f = \{\}$ **by** *auto*
  thus $finite\ (Fix\ f)$ **using** *finite-def* **by** *auto*
qed

end

References

Isabelle: http://isabelle.in.tum.de
Has the software for download, installation instructions, documentation, links to more information.

Proof General: http://proofgeneral.inf.ed.ac.uk/

Bart Kastermans, An Example of a Cofinitary Group in Isabelle/HOL, www.bartk.nl/files.php