Vienna notes on effective descriptive set theory and admissible sets

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1 Preface

These notes are based around the exposition of a sequence of connected ideas: Δ^1_1 and Π^1_1 and Σ^1_1 sets; admissible structures; ω_1^{ck}; recursive well founded relations; ordinal analysis of trees; Barwise compactness; Gandy-Harrington forcing. I am not going to discuss applications of these techniques – though there are striking applications such as Harrington's proof of Silver's theorem, or the Harrington-Kechris-Louveau dichotomy theorem, or Louveau's sharp analysis of $\Delta^1_1 \cap \underset{\sim}{\Pi}^\alpha_0$. In these notes I am simply going to develop the techniques from the ground up, in the hope that this will provide a useful calculative tool.

In the background is a philosophy: That in certain situations the *right* way to understand the structure of some pointclasses is to understand them in the context of an appropriate inner model which still reflects their behaviour, and has some degree of absoluteness. For instance one could try to understand Σ^1_2 sets using Gödel's constructible universe L, and there is some truth to the idea that Σ^1_2 shares some structural properties with L. We can understand Borel sets by simply going to any model of set theory whose ω is isomorphic to the real ω. Perhaps at the level of Π^1_3 we should consider sufficiently iterable inner models of a Woodin cardinal. At the level of Π^1_1, the right inner models are well founded models of KP – that is to say, admissible structures. To illustrate this, the first theorem we will head towards is Spector-Gandy, which, in its simplest form, states that the Π^1_1 subsets of ω are precisely those Σ_1 definable over the least admissible structure. The specific theorems such as Spector-Gandy are less important than the method, which is to reduce questions about Borel and projective compexity of sets of reals to set theoretical calculations over inner models. In the end we will reproduce all the basic uniformization and separation theorems of "classical" descriptive set theory in about a dozen pages, but this will be at the cost of not repeating proofs which closely resemble earlier arguments and treating all set theoretical calculations involving quantifier manipulation as "routine" exercises.

Fundamentally these exercises probably are routine calculations, but they are not necessarily at all routine for someone new to the area. I apologize, but it is partly an inevitable consequence of trying to give a full treatment of the entire body of theory in a concise form.

2 Prerequisites

Some knowledge of recursion theory is necessary, but probably not much more than one would learn as part of a course covering the Gödel incompleteness theorems, and certainly any one semester course in recursion theory should be amply sufficient. The reader also needs to be able to manipulate quantifiers, in the sense of placing formulas in some kind of normal form with the minimal number of alternations of $\exists$'s and $\forall$'s in the front; I leave all those manipulations as exercises, but it is possible that anyone who has gone through the proof of the completeness theorem will already have the basic idea. Finally, I assume an ability to work with set theoretical operations like transfinite recursion. Optimistically one might suppose this is not much beyond what one would learn in going through the proof $L \models \mathrm{CH}$.

3 Comparison with the bold faced version

I will start with some purely classical theorems. Even for people who have seen these before, there are some points I want to make about the proofs.

Definition $T \subset \omega^{<\omega} \times \omega^{<\omega}$ is said to be a *tree* if

1. $(u, v) \in T$ implies $lh(u) = lh(v)$ (they have the same length as finite sequences); and
2. T is closed under subsequences, in the sense that if $(u, v) \in T$ and $\ell < lh(u)$ then $(u|_\ell, v|_\ell) \in T$.

We then let $[T]$, the set of branches through T, be the set of all $(x, y) \in \omega^\omega$ such that

$$\forall n (x|_n, y|_n) \in T,$$

where here $x|_n = (x(0), x(1), ..., x(n-1))$. We let $p[T]$ be the projection of $[T]$ – that is to say the set of x for which there exists a y with $(x, y) \in [T]$. We then say that a set is $\underset{\sim}{\Sigma}^1_1$, or *analytic*, if it equals $p[T]$ for some tree T.

Notation For $(u, v), (u', v') \in T$, I will write $(u, v) < (u', v')$ if u is strictly extended by u' and v is strictly extended by v'. I write $(u, v) \perp (u', v')$ if neither extends the other.

Theorem 3.1 *$p[T]$ is empty if and only if T has a ranking function. That is to say, some $\rho : T \to \delta$, some ordinal δ, with $\rho(u, v) > \rho(u|_\ell, v|_\ell)$ all $(u, v) \in T$, $\ell < lh(u)$.*

Proof This is a routine application of transfinite recursion.

We define a transfinite increasing sequence of subsets, $T_0 = \emptyset \subset T_1 \subset ... \subset T_\omega \subset T_{\omega+1}...$, and a corresponding sequence of functions

$$\rho_\alpha : T_\alpha \to \alpha,$$

with $T_{\alpha+1} = \{(u, v) \in T : \forall (u', v') \in T((u, v) < (u', v') \Rightarrow (u', v') \in T_\alpha)\}$, and for $(u, v) \in T_{\alpha+1}$

$$\rho_{\alpha+1}(u, v) = \sup\{\rho_\alpha(u', v') + 1 : (u', v') \in T, (u, v) < (u', v')\}.$$

At limit stages we take unions.

If, after exhausting this process, we end up with some $T_\delta = T$, and a total ranking function on T, then clearly any infinite branch through T will give an infinite descending sequence of ordinals, and a contradiction.

Conversely, if we finally finish with $T_\delta \neq T$ and there is no hope to extend the process further (basically, $T_{\delta+1} = T_\delta$), then for any $(u, v) \in T \setminus T_\delta$ we can find a $(u', v') \in T \setminus T_\delta$ with $(u, v) < (u', v')$. □

Theorem 3.2 *(Perfect set theorem for $\underset{\sim}{\Sigma}^1_1$.) Let T be a tree. Then either $p[T]$ is countable, or it contains a homeomorphic copy of 2^ω, and hence has cardinality $2^{\aleph_0}$.*

Proof This is actually similar to the last proof.

We define an increasing transfinite sequence of subsets, (T_α), but the specifics of the definition changes slightly. We let $T_{\alpha+1}$ be the set of $(u, v) \in T \setminus T_\alpha$ such that there do not exist extensions which are incompatible in the first coordinate:

$$(u, v) < (u_0, v_0), (u_1, v_1),$$

$$u_0 \perp u_1.$$

In the case that the process terminates with some $T_\delta \neq T$ we have that every node in $T_\delta \neq T$ admits extensions, again in $T_\delta \neq T$, which are incompatible. From this it is routine to construct an array $(u_s, v_s)_{s \in 2^{<\omega}}$ such that $(u_s, v_s) < (u_t, v_t)$ when $s < t$ and $u_s \perp u_t$ when $s \perp t$. It is relatively routine to verify that

$$\varphi : 2^\omega \to p[T]$$

$$z \mapsto \bigcup u_{z|_n}$$

provides a homeomorphic embedding.

On the other hand, if the process terminates with $T_\delta = T$ then first of all note that δ is a countable ordinal, since T is countable and it is impossible to find a strictly increasing ω_1 sequence of countable subsets of a countable set. Note more over that each $(u, v) \in T$ has a unique $\alpha_{u,v}$ such that $(u, v) \in T_{\alpha+1} \setminus T_\alpha$, and moreover $(u, v) < (u', v')$ implies $\alpha_{u,v} \geq \alpha_{u',v,}$.

Thus for every $(x, y) \in [T]$ we can find a smallest α such that at every n

$$(x|_n, y|_n) \notin T_\alpha.$$

Fixing this α and some n such that for all $\ell \geq n$ we have $(x|_n, y|_n) \in T_{\alpha+1} \setminus T_\alpha$, and letting $(u, v) = (x|_n, y|_n)$, we see that x has a privileged position in T_α: It is the only remotely plausible $x' \supset u$ for which there might exist a $y' \supset v$ with $(x', y') \notin T_\alpha$.

Thus, every $x \in p[T]$ will be definable from some finite pair of sequences (u, v) over some T_α. Since there are only countably many possible choices for (u, v) and α, we indeed obtain the projection of T countable. $\square$

To keep things simple, we just started out with the definition of $\underset{\sim}{\Sigma}^1_1$ sets for subsets of ω^ω, but it naturally extends.

Definition $T \subset (\omega^{<\omega})^{n+1}$ is said to be a *tree* if

1. $(u^0, u^1, ..., u^n) \in T$ implies $lh(u^0) = lh(u^1) = ... = lh(u^n)$;
2. is closed under subsequences, in the sense that if $(u^0, u^1, ...) \in T$ and $\ell < lh(u^0)$ then $(u^0|_\ell, u^1|_\ell, ...) \in T$.

We then let $[T]$, the set of branches through T, be the set of all $(x^0, x^1, ..., x^n) \in (\omega^\omega)^{n+1}$ such that

$$\forall \ell (x^0|_\ell, x^1|_\ell, ...) \in T.$$

We let $p[T]$ be the projection of $[T]$: the set of $(x^0, x^1, ..., x^{n-1})$ for which there exists a y with $(x^0, ..., x^{n-1}, y) \in [T]$. We then say that a subset of $(\omega^\omega)^n$ is $\underset{\sim}{\Sigma}^1_1$, or *analytic*, if it equals $p[T]$ for some tree T.

Theorem 3.3 *(Kunen-Martin for $\underset{\sim}{\Sigma}^1_1$) A analytic well founded relation on 2^ω has countable rank. In other words, if $T \subset (\omega^{<\omega})^3$ is a tree and $R = p[T]$ is well founded relation in the sense that there is no infinite sequence $(x_n)_{n \in \omega}$ such that at each n*

$$x_{n+1} R x_n,$$

then we can find a countable ordinal δ and

$$\rho : 2^\omega \to \delta$$

with $\rho(x) = \sup\{\rho(y) + 1 : yRx\}$.

Proof First begin with the canonical ranking function

$$\rho : 2^\omega \to \kappa,$$

$$\rho(x) = \sup\{\rho(z) + 1 : zRx\}.$$

Our whole task comes down to showing the range of ρ is bounded below ω_1.

Define the relation R^* on

$$\bigcup_{n\in\omega} (\omega^n)^{2n+1},$$

with

$$(u_0, s_1, u_2, ..., u_{2n})R^*(v_0, t_1, v_2, ..., v_{2m})$$

provided:

(i) $n = m + 1$;
(ii) $u_{2k}|_m = v_{2k}$ for $k \leq m$;
(iii) $s_{2k-1}|_m = t_{2k-1}$ for $k \leq m$;
(iv) at each $k < n$ we have

$$(u_{2k+2}, u_{2k}, s_{2k+1}) \in T.$$

Note then that (iv) along with the earlier conditions also gives that at each $k < m$, $(v_{2k+2}, v_{2k}, t_{2k+1}) \in T$.

We then obtain that R^* is also wellfounded, since in some sense R^* is the tree of partial attempts to find an infinite descending chain through R. Since it is a well founded relation on a countable set, we can find a ranking function ρ^* with range some countable ordinal δ.

If we have a descending R-chain, $x^0, x^1, ...x^{n+1}$, with corresponding witnesses $y^0, ..., y^n$, each $(x^{i+1}, x^i, y^i) \in [T]$, then we have at each $\ell \leq n$ that

$$(x^0|_{\ell+1}, y^0|_{\ell+1}, x^1|_{\ell+1}, ..., x^{\ell+1}|_{\ell+1})R^*(x^0|_\ell, y^0|_\ell, x^1|_\ell, ..., x^{n+1}|_\ell).$$

Let R^+ be the well founded relation on finite sequences of reals defined by

$$(x^0, y^0, ..., x^n, y^n, x^{n+1})R^+(x^0, y^0, ..., y^{n-1}, x^n)$$

if at each $\ell \leq n$

$$(x^{\ell+1}, x^\ell, y^\ell) \in T.$$

Let ρ^+ be the canonical ranking function for R^+. It then follows that

$$\rho^+(x^0, y^0, ..., x^n, y^n, x^{n+1}) \leq \rho^*(x^0|_n, y^0|_n, ..., x^n|_n, y^n|_n, x^{n+1}|_n).$$

But finally, if we have a sequence $(x^0, y^0, ..., y^{n-1}, x^n)$ and at each $\ell < n$

$$(x^{\ell+1}, x^\ell, y^\ell) \in T,$$

then $\rho(x^n) = \rho^+(x^0, y^0, ..., y^{n-1}, x^n)$. □

There is a key thing about all these proofs: The main driving force is transfinite recursion, and the legitimacy of the transfinite recursion is based on Σ_1 collection. Here a formula in set theory is Σ_0 if it only contains *bounded quantifiers* of the form $\exists x \in y$ or $\forall x \in y$. It is Σ_1 if it has the form

$$\exists x_1 \exists x_2 ... \exists x_n \varphi(\vec{x}, \vec{y}),$$

for φ Σ_0. Σ_1 collection is the statement that whenever we have a Σ_1 φ and X is a set such that

$$\forall x \in X \exists y \varphi(x, y),$$

then we can find a set Y such that

$$\forall x \in X \exists y \in Y \varphi(x, y).$$

(In actual fact, under even very modest background set theoretical assumptions, Σ_1 collection follows from Σ_0 reflection, but I am following the usual nomenclature.) I do want to allow parameters in my notion of Σ_1 formula in this definition of Σ_1 collection.

In all the above arguments, we could try to build the relevant ranking function or ascending sequences of subsets $T_0 \subset T_1 \subset ...T_\omega....$ inside a model containing T and satisfying Σ_1 collection. I claim that in the case that the ranking function exists, in 3.1, 3.3, or that there is finally a $T_\delta = T$ in 3.2, that would be same inside any reasonable inner model of Σ_1 collection.

Let us just consider the structure of the argument in theorem 3.1. Suppose the tree T is indeed well founded, and let ρ be the canonical ranking function in V. Suppose M is a transitive model of Σ_1 reflection and some basic set theoretical operations like Σ_0 separation. We can go ahead and try to define the partial sequence of approximations

$$\rho_\alpha : T_\alpha \to \alpha$$

inside M. Those definitions are very transparent, and at limit stages it is a straightforward application of collection for M to likewise take the unions as we do in V. The whole question mark here, however, is whether M has enough ordinals to complete the transfinite recursion to the point that it exhausts the entire tree.

For a contradiction, let (u, v) in T have minimal ρ value subject to the requirement that there is no $(\rho_\alpha, T_\alpha) \in M$ with $(u, v) \in T_\alpha$ – in other words, let $\rho(u, v)$ be minimized subject to the requirement that M cannot find a partial ranking function which has (u, v) in its domain.

It actually follows that $\rho(u, v)$ equals the supremum of the ordinals in M, though this is not a critical point for us at this stage.

Now the minimality of $\rho(u, v)$ implies that for all $(u', v') \in T$ with $(u, v) < (u', v')$ we have some $(\rho_\alpha, T_\alpha) \in M$ with $(u', v') \in T_\alpha$. But the set of proper extensions of (u, v) in T wil be a set in M. So at this stage we can apply Σ_1 collection to that set inside M and the ranking functions. Hence we can find a set $Y \in M$ such that for all $(u', v') \in T$ with $(u, v) < (u', v')$ we have some $(\rho_\alpha, T_\alpha) \in Y$ with $(u', v') \in T_\alpha$. But M can take the union of all the partial ranking functions of the form (ρ_α, T_α) inside Y. This will give a *single* ranking function which exists inside M and ranks all proper extensions of (u, v) – and then there is nothing to stop M taking the process one further step and capturing (u, v) in its domain.

Before we push on, I want to make a couple of technical comments. At this stage they may seem like splitting hairs, since I have as yet offered absolutely no evidence at all that we should have a special affection for the theory of Σ_0 separation and Σ_1 collection. Later on, though, they will be totally critical for a whole sequence of calculations.

First, a slight caution. What this argument shows is that from Σ_0 separation and Σ_1 collection the existence of a ranking function in V will pass down to M. It does *not* show that the existence of an infinite branch through T will pass down to M. The problem is that in the case that T is illfounded and M tries to set up the transfinite recursion $(\rho_\alpha, T_\alpha)_\alpha$ of partial rankings, they might continue out through the ordinal height of M – and then without Σ_1 separation, we will not be able to mimic the infinite branch argument which works inside V.

On the other hand, in the case that T really is well founded, and there is a ranking function on all of T, we saw above that each point inside T is ranked at some stage inside M. That in particular means that the empty sequence is ranked inside M, and thus the rank of the tree is less than the ordinal height of M.

4 Recursive ordinals

Definition An ordinal α is said to be *recursive* if there is a recursive well order $<$ of ω with

$$(\alpha; \in) \cong (\omega; <).$$

We then let ω_1^{ck} be the supremum of the recursive well orders.

Lemma 4.1 *A well founded recursive relation on ω has rank*[1] *less than ω_1^{ck}.*

Note that a recursive well order of an infinite recursive set will be isomorphic to a recursive well order of ω and hence have order type less than ω_1^{ck}.

Proof Let $S = \{(n_0, n_1, ...n_\ell) : \forall i < \ell(n_{i+1} R n_i)\}$. Set $\vec{n} <_R \vec{m}$ if either $\vec{n}$ strictly extends $\vec{m}$ or for i least with $n_i \neq m_i$ we have $n_i < m_i$. The relation $<_R$ now well orders S and hence has rank less than ω_1^{ck}. □

Definition A tree is said to be *recursive* if the set of Gödel codes for its elements is recursive as a subset of ω. A subset of $(\omega^\omega)^n$ is said to be Σ^1_1 if it is the projection of a recursive tree; it is Π^1_1 if its complement is Σ^1_1; and it is Δ^1_1 if it is both Σ^1_1 and Π^1_1.

There is another equally good definition of recursive when talking about subset of the hereditarily finite sets: $A \subset HF$ is *recursive* if it is Δ_1 definable over $(HF, \in)$. This agrees with the definition we have used above, though I will not take the time out to prove that.

We will also need these notions in the context of subsets of ω.

Definition $A \subset \omega^m \times (\omega^\omega)^n$ is Σ^1_1 if there is a recursive $T \subset \omega^m \times (\omega^{<\omega})^n$ such that

1. $(\vec{k}, u^0, u^1, ..., u^n) \in T$ implies $lh(u^0) = lh(u^1) = ... = lh(u^n)$;
2. T is closed under subsequences, in the sense that if $(\vec{k}, u^0, u^1, ...) \in T$ and $\ell < lh(u^0)$ then $(\vec{k}, u^0|_\ell, u^1|_\ell, ...) \in T$;
3. A equals the set of all $(\vec{k}, x^0, x^1, ..., x^{n-1}) \in \omega^m \times (\omega^\omega)^n$ such that there exists y with
$$\forall \ell(\vec{k}, x^0|_\ell, x^1|_\ell, ..., y|_\ell) \in T,$$
.

Exercise

1. If T is a recursive tree on $(\omega^{<\omega})^{n+2}$, then there is a recursive tree S on $(\omega^{<\omega})^{n+1}$ such that
$$p[S] = pp[T](=_{\mathrm{df}} \{\vec{x} : \exists y, z((\vec{x}, y, z) \in [T]\}).$$
2. The intersection of two Σ^1_1 sets is Σ^1_1.
3. The union of two Σ^1_1 sets is Σ^1_1.

[1]For R a well founded relation on a set S, there will be a canonical ranking function

$$\rho : S \to \delta$$

with $\rho(a) = \sup\{\rho(b) + 1 : bRa\}$. The sup of the image of ρ is said to be the *rank* of R.

Exercise In any reasonable sense, isomorphism is Σ^1_1.

For instance, if we view each $x \in \omega^\omega$ as coding a binary relation $\in^x$ by

$$n \in^x m \Leftrightarrow x(2^n 3^m) = 1,$$

then there is a recursive tree $T \in (\omega^{<\omega})^3$ such that $(x, y, f) \in [T]$ if and only if f is injective and at each n, m

$$n \in^x m \Leftrightarrow f(n) \in^y f(m).$$

Then as in the previous exercises we can find a recursive tree S such that $(x, y, f, g) \in [S]$ if and only if

$$(x, y, f) \in [T],$$

$$(y, x, g) \in [T],$$

and

$$f = g^{-1},$$

and conclude that $pp[S]$ is Σ^1_1.

Exercise Ill foundedness is Σ^1_1. (I.e. the set of x for which there is an infinite sequence $(n_k)_k$ with

$$\forall k(x(2^{n_{k+1}} 3^{n_k}) = 1)$$

is Σ^1_1).

Exercise Let $T \subset (\omega^{<\omega})^n$ be a recursively enumerable tree. Show there is a recursive tree S with $p[S] = p[T]$. (Hint: Write T as something of the form $\{(u, v) : \exists k R(u, v, k)\}$, where R is suitably recursive. Now make S consist of (u, w) where w encodes a sequence $(v(0), v(1), ..., v(\ell - 1), k_0, k_1, ..., k_{\ell-1})$ such that each $(u|_i, (v(0), ..., v(i-1)), k_i) \in R$.)

Exercise $\Sigma^1_1 \neq \Pi^1_1$. (Hint: This is a variation on the kinds of diagonalization arguments one would use to say show that not every recursively enumerable set is recursive. One starts by building a *universal* Σ^1_1 subset of $\omega \times \omega$, and for that one needs the previous exercise.)

Exercise Σ^1_1 is closed under number quantifiers. That is to say, if $A \subset \omega^{m+1} \times (\omega^\omega)^n$ is Σ^1_1, then so is the set of $(\vec{k}, \vec{x})$ such that there exists ℓ with $(\vec{k}, \ell, \vec{x}) \in A$ and so is the set $(\vec{k}, \vec{x})$ such that for every ℓ we have $(\vec{k}, \ell, \vec{x}) \in A$. Similarly then for Π^1_1 and Δ^1_1.

Exercise Truth for countable structures is Σ^1_1. (I.e. if we define, as before, $\in^x$ to be the set of all (n, m) such that $x(2^n 3^m) = 1$, then for any formula ϕ in the language of set theory, $\{(\vec{k}, x) : (\omega; \in^x) \models \phi(\vec{k})\}$ is Σ^1_1.)

Exercise For ψ a formula of set theory, the set of $n \in \omega$ such that there exists an ω-model M with

$$M \models \psi(n)$$

is Σ^1_1. (Here an ω-model is one whose ω is isomorphic to true ω.)

Note for future reference, that if there is some set of natural numbers with a first order definition which is correctly calculated by all ω-models, then it is necessarily Δ^1_1.

Definition We say that $a \in \omega^\omega$ is Σ^1_1 (respectively, Π^1_1, Δ^1_1) if the set $\{(n, m) : a(n) = m\}$ is Σ^1_1 (respectively, Π^1_1, Δ^1_1).

Exercise For $a \in \omega^\omega$ the following are equivalent:

1. a is Δ^1_1;
2. $\{a\}$ is Δ^1_1;
3. $\{a\}$ is Σ^1_1 (as a subset of ω^ω).

(Warning: These are *not* equivalent to a being Σ^1_1 – and that non-equivalence follows from $\Sigma^1_1 \neq \Pi^1_1$ for subsets of ω.)

Theorem 4.2 *If T is a recursive tree with $[T] = 0$, then it has rank less than ω_1^{ck}.*

Proof This follows from 4.1. $\square$

Theorem 4.3 *(The effective perfect set theorem for Σ^1_1.) If $A \subset \omega^\omega$ is Σ^1_1, then either it contains a homeomorphic copy of 2^ω or $A \subset \Delta^1_1$.*

A quick clarification: When I say $A \subset \Delta^1_1$, I mean that every element of A is Δ^1_1 as *an element of ω^ω*.

Proof This follows from what is going on in the proof of 3.2 using 4.1, but I am going to be rather cavalier with the details.

First of all we try to define a tree of partial attempts to build a copy of 2^ω inside $p[T]$, where T is a recursive tree giving rise to A.

Let S consist of all arrays $(u_s, v_s)_{s\in 2^{\leq m}}$, some m, where for $s, t \in 2^{\leq m}$ we have, $s, t \in T$, $s < t$ implies $(u_s, v_s) < (u_t, v_t)$, and $s \perp t$ implies $u_s \perp u_t$. We define R to be the relation of extension on S – where we say one array extends another when it has larger domain and agrees on the domain in common. If A does not contain a perfect set, then the proof of 3.2 shows that R is well founded. Then by 4.1 we get that there is a bound $\delta < \omega_1^{\mathrm{ck}}$ for the rank of R.

Now as before we let $T_{\alpha+1}$ be the set of $(u, v) \in T \setminus T_\alpha$ such that there exist extensions which are incompatible in the first coordinate:

$$(u, v) < (u_0, v_0), (u_1, v_1),$$

$$u_0 \perp u_1.$$

It then turns out, and this can be seen by careful inspection of the definitions, that the transfinite recursion finishes by stage δ. That is to say, $T_\delta = T$.

Then the proof of 3.2 gives that each $x \in p[T]$ will be uniformly first order definable over

$$(T \setminus T_\alpha; <)$$

from some (u, v) in T and some $\alpha \leq \delta$. But for e encoding a recursive well order, $<_e$, of order type $\alpha \leq \delta$, it is a relatively routine calculation to show that T_α will be uniformly Δ^1_1 definable from e, and then x will be uniformly definable from $<_e$ and (u, v). (Alternately, over any ω-model M, we will have that the transfinite recursion $\beta \mapsto T_\beta$ will be correctly calculated for all ordinals in its well founded part. However, $<_e$ is correctly calculated by M, and hence so is T_α. Then x will be uniformly definable over from (u, v) and $<_e$ over M, and we can therefore apply the exercise on existence of ω-models with a certain belief being Σ^1_1.) $\square$

Theorem 4.4 *(Kunen-Martin for Σ^1_1) Every Σ^1_1 well founded relation has rank less than ω_1^{ck}.*

Proof This follows from the proof of 3.3 using 4.1. $\square$

5 Admissible sets

Definition A set of the form L_α is *admissible* if α is a limit and L_α satisfies Σ_1 collection. We then also say that α is an *admissible ordinal.*

Exercise (i) Any infinite cardinal is admissible.
(ii) $\aleph_\omega$ is admissible, but $L_{\aleph_\omega}$ fails Σ_2 collection.

Definition For M a model with of a binary relation, $\in^M$, we let $O(M)$ be the sup of the ordinals in the well founded part. More precisely, $O(M)$ is the sup of those ordinals α such that there exists some $a \in M$ with

$$(\alpha, \in) \cong (\{b : b \in^M a\}, \in^M).$$

Lemma 5.1 *If M is a model of ZFC, then $O(M)$ is admissible.*

Proof First of all we may assume the well founded part of M is actually a transitive set. If M is well founded, then there is nothing to prove, so assume M is ill founded.

If the lemma fails, then there would be some set X in the well founded part of M and some Σ_1 formula ψ such that at each $x \in X$ there is a least ordinal $\beta_x < O(M)$ such $\psi(x, \beta_x)$ and the β_x's are unbounded in $O(M)$. But then M would have the have the ability to define the cut corresponding $O(M)$ and would be confronted by its own ill foundedness. □

Of course assuming $M \models$ ZFC is something of an over kill in the assumptions of the lemma.

Theorem 5.2 ω_1^{ck} *is admissible.*

Proof It is not hard to see that for any set $Y \in L_{\omega_1^{\mathrm{ck}}}$ there is a $\Delta_1^{L_{\omega_1^{\mathrm{ck}}}}$ surjection from ω onto Y. Thus it suffices for us to consider the case that $\psi \in \Sigma_1$ and for all $n \in \omega$ there is some $\alpha < \omega_1^{\mathrm{ck}}$ such that

$$L_\alpha \models \psi(n).$$

Let B be the set of pairs (n, e) such that:

1. $n, e \in \omega$;
2. e is a code for a recursive linear $<_e$ such that there is a model M whose ordinals are isomorphic to $<_e$ and satisfies the statements:
   (a) $V = L$,
   (b) $\psi(n)$, and
   (c) for all α, $L_\alpha \models \neg\psi(n)$;

   moreover $<_e$ also has the property that
3. for all other recursive linear orderings $<$ we have either:
   (a) $<$ is ill founded; or
   (b) $<_e$ is isomorphic to an initial segment of $< e$;
   (c) there exists a model N whose ordinals are isomorphic to $<$ and satisfies $\neg\psi(n)$.

Since isomorphism and ill foundedness are both Σ^1_1, and we previously observed the closure of Σ^1_1 under number quantifiers, this is a Σ^1_1 set of pairs of natural numbers. Moreover our assumptions on ψ imply that $(n, e) \in B$ will always have e coding a *well ordering* of ω. So we can define a well founded relation on B by $(n, e)R(n', e')$ if $<_e$ is isomorphic to a proper initial segment of $<_{e'}$. The rank of R must be less than ω_1^{ck} by 4.4. □

There is an alternative proof of admissibility of ω_1^{ck} based on ill founded models and 5.1. It is not hard to see that every model of a sufficiently rich fragment of ZFC containing ω must *include* ω_1^{ck} in its well founded part. We only need to show that there is some ω model which does not *contain* the ordinal ω_1^{ck}.

But for any ill founded recursive tree T we know from 4.2 that there is an infinite branch below a node (u, v) if and only if the rank of the set of extensions of (u, v) is some recursive ordinal. Any ω model will be able to correctly calculate which nodes have rank some given $\alpha < \omega_1^{\mathrm{ck}}$. Hence if it actually has the ordinal ω_1^{ck} it will be able to form the set of all nodes which have rank less than ω_1^{ck}. Then it will be able to form the subtree $T' \subset T$ consisting of nodes *not* ranked by an ordinal less than ω_1^{ck} – and this tree will have no terminal nodes.

Thus we have argued that a recursive tree has an infinite branch if and only if every ω model can find a non-empty subtree which has no terminal nodes – and this reduces Σ^1_1 to Π^1_1, with a contradiction.

Theorem 5.3 *(Spector-Gandy) $A \subset \omega$ is Π^1_1 if and only if it is Σ_1 over $L_{\omega_1^{\mathrm{ck}}}$.*

Proof If A is Π^1_1 then membership in A is uniformly reduced to the well foundedness of a recursive tree, which by the proof of 3.1 and 4.2 is Σ_1 over any admissible model containing the tree.

Conversely, if ψ is a Σ_1 formula defining A, then $n \in A$ if and only if there is a recursive linear ordering $<_e$ such that:

1. $<_e$ is well ordering, and
2. every $M \models V = L$ whose ordinals are isomorphic to $<_e$ satisfies $\psi(n)$.

Since both these conditions are Π^1_1, A will be Π^1_1. □

Corollary 5.4 *$L_{\omega_1^{\mathrm{ck}}}$ fails Σ_1 separation.*

Proof Otherwise the argument of 3.1 would show that a recursive tree is non-empty if and only if there is an infinite branch in $L_{\omega_1^{\mathrm{ck}}}$, which by 5.3 would reduce Σ^1_1 to Π^1_1. □

The above steps allow certain generalizations by relativizing to a real.

Definition For $x \in \omega^\omega$, ω_1^x is the sup of the recursive in x ordinals – i.e. those ordinals for which there is a recursive in x well ordering of the same order type.

Definition A structure of the form $L_\alpha[x]$ is *admissible* if α is a limit ordinal and $L_\alpha[x]$ satisfies Σ_1 reflection.

As well as all the previous results generalizing in a straight forward way, we also obtain a more powerful form of Spector-Gandy:

Theorem 5.5 *Let $A \subset \omega^\omega$. Then A is Π^1_1 if and only if there is a Σ_1 formula such that*

$$x \in A \Leftrightarrow L_{\omega_1^x}[x] \models \psi(x).$$

Theorem 5.6 *Every countable Σ^1_1 set is included in $L_{\omega_1^{\mathrm{ck}}}$.*

Proof In the case that the Σ^1_1 set is countable, admissibility allows us to complete the derivation process of 3.2 inside $L_{\omega_1^{\mathrm{ck}}}$. □

Exercise $L_{\omega_1^{\mathrm{ck}}} \cap \omega^\omega$ equals the Δ^1_1 elements of ω^ω.

6 Separation and uniformization

Theorem 6.1 *Any two disjoint Σ^1_1 sets can be separated by a Δ^1_1 set.*

In other words, if we have A, B Σ^1_1 subsets of a space such as ω^ω, then there is a Δ^1_1 D which includes A and avoids B.

Proof This follows from Spector-Gandy, using the uniformization principles which hold for Σ_1 inside the constructible and relatively constructible universes.

Let φ be a Σ_1 formula such that $x \notin A$ if and only if there is some $\alpha < \omega_1^x$ such

$$L_\alpha[x] \models \varphi(x).$$

Let ψ be a similar formula for $x \notin B$.

The assumption of disjointness imply that the complements of A and B cover ω^ω, and thus for each x that there is some $\alpha < \omega_1^x$ with

$$L_\alpha[x] \models \psi(x) \vee \varphi(x).$$

Now let C be the set of pairs (x, e) such that the e^{th} recursive in x linear order is a recursive well order of order type α with α least such that

$$L_\alpha[x] \models \psi(x) \vee \varphi(x).$$

Exactly the same calculation as in the proof of 5.2 shows that this is a Σ^1_1 set. But then likewise appealing to the proof of the effective version of Kunen-Martin shows that there is a bound $\delta < \omega_1^{\text{ck}}$ such that for each (x, e) in C the order type of the e^{th} recursive in x linear order is equal to some ordinal less than δ. Thus, for each x we have

$$L_\delta[x] \models \psi(x) \vee \varphi(x).$$

Then we let D be the set of x such that

$$L_\delta[x] \models \neg\psi(x).$$

$\square$

Definition We equip ω^ω with the product topology obtained by its natural identification with $\prod_\omega \omega$. We then equip the product spaces of the form $(\omega^\omega)^n$ with the resulting product topology. We say that a set in one of these space is *Borel* if it appears in the smallest σ-algebra containing the open sets.

Exercise If α is a countable ordinal and ψ is a formula in the language of set theory, then the set of x for which

$$L_\alpha[x] \models \psi(x)$$

is Borel.

Thus the proof of 6.1 shows a bit more:

Theorem 6.2 *Disjoint Σ^1_1 sets can be separated by Borel sets, and hence every Δ^1_1 set is Borel.*

Definition A function from ω^ω to ω^ω is Δ^1_1 if it is Δ^1_1 as a subset of $\omega^\omega \times \omega^\omega$.

It is an easy manipulation of quantifiers to see that the pullback of a basic open set, or even a Δ^1_1 set, under a Δ^1_1 function is Δ^1_1. Hence a Δ^1_1 function will be *Borel measurable*, in the sense of pulling back open sets to Borel.

Theorem 6.3 *Let $A \subset \omega^\omega \times \omega^\omega$ be Σ^1_1 such that for each x there are only countably many y with $(x, y) \in A$. Then we can find countably many Δ^1_1 functions, $(f_n)_n$, such that for each x the set $\{f_n(x) : n \in \omega\}$ includes A_x, the set of y such that $(x, y) \in A$.*

Proof For each x there will be a uniformly recursive in x tree T^x such that $p[T^x]$ equals A_x. Inside each admissible structure containing x we will be able to form the sequence of trees appearing in 3.2 or 4.3, and come to some $\delta < \omega^x_1$ and subsets $(T^x_\alpha)_{\alpha\leq\delta}$ – and by analogy with the situation from before, T^x_δ will be empty, $T^x_{\alpha+1}$ will be set of nodes which admit a future splitting in the first coordinate inside T^x_α, and at limit stages we are taking intersections. The assignment of $(T^x_\alpha)_{\alpha\leq\delta}$ to x will be uniformly Σ_1 (and hence Δ_1)[2] over any admissible containing x, and moreover, harking back to the earlier arguments, we will have $A_x \subset L_{\delta+\omega}[x]$.

$L_{\omega^x_1}[x]$ realizes that all its initial segments are countable. Thus we can effectively enumerate the reals in $L_{\delta+\omega}[x]$ in a manner that is uniformly Δ_1 over $L_{\omega^x_1}[x]$. But thus by Spector-Gandy we uniformly obtain a Δ^1_1 enumeration. □

In the case that the set A is Δ^1_1 we obtain a strengthening.

Theorem 6.4 *(Effective version of Lusin-Novikov) Let $A \subset \omega^\omega \times \omega^\omega$ be a Δ^1_1 set with every A_x countable and non-empty. Then we can find a countable sequence of Δ^1_1 functions, $(g_n)_n$ such that at each x,*

$$A_x = \{g_n(x) : n \in \omega\}.$$

Proof In the case that each $|A_x| = \aleph_0$, we get Borel functions $(f_n)_n$ as in the last theorem with each $A_x \subset \{f_n(x) : n \in \omega\}$ and then the fact that A is Borel allows us to set

$$g_n(x) = f_m(x),$$

where m least such that $f_m(x) \neq g_0(x), g_1(x), ..., g_{n-1}(x)$. The case with some sections finite, but with the complication that we may have to cater for some n with $g_k(x) = g_n(x)$ all $k \geq n$. □

Exercise ("Boundedness") Define, as before, $\in^x$ to be the set of all (n, m) such that $x(2^n3^m) = 1$. Let WO be the set of $x \in \omega^\omega$ such that $\in^x$ well orders ω. Show that if $A \subset$ WO is Σ^1_1, then there is $\delta < \omega^{\text{ck}}_1$ such that for every $x \in A$ there exists $\alpha < \delta$ with

$$(\alpha; \in) \cong (\omega; \in^x).$$

Many times people will use term *boundedness* in a loose and sweeping way to describe any of the myriad arguments where some collection of ordinals get bounded below ω^{ck}_1 or ω_1. The arguments in which we used Kunen-Martin could have been reorganized to directly follow from this exercise, which in turn can be given an independent proof – roughly speaking, if we had a Σ^1_1 class C of well ordered structures with unbounded rank, then we could reduce well foundedness of a recursive trees into the existence of a ranking function into some element of C, with a reduction of Π^1_1 to Σ^1_1.

7 Extended aside: These theorems in the bold faced case

It probably makes sense to spend a bit of time showing how the theorems of classical descriptive set theory, in the sense of Lebesgue, Lusin, and Suslin from the early 1900's, can at this point be derived for general Polish spaces.

[2] A set is Δ_1 if both it and its complement are Σ_1. It is a standard manipulation of quantifiers argument to see that at Σ_1 function from a Δ_1 set must be Δ_1

Notation A set is said to be $\Sigma^1_1(x)$ if it is the projection of a tree which is recursive in x. We then say a set is $\Pi^1_1(x)$ if its complement is $\Sigma^1_1(x)$, and $\Delta^1_1(x)$ if it is in the union of the two classes. Similarly $\utilde{\Pi}^1_1$ is the complement of $\utilde{\Sigma}^1_1$, and $\utilde{\Delta}^1_1 = \utilde{\Pi}^1_1 \cap \utilde{\Sigma}^1_1$.

Since every subset of ω is recursive in some element of ω^ω we obtain

$$\utilde{\Sigma}^1_1 = \bigcup_x \Sigma^1_1(x).$$

We also obtain obvious parallels of theorems from the last two sections by relativizing Σ^1_1 to $\Sigma^1_1(x)$, ω_1^{ck} to ω_1^x, $L_{\omega_1^{\mathrm{ck}}}$ to $L_{\omega_1^x}[x]$, and so on.

Notation For $\vec{s} = (s_0, ..., s_{n-1}) \in (\omega^{<\omega})^n$, we let $N_{\vec{s}}$ be the set of $\vec{x} \in (\omega^\omega)$ such that each $x_i \supset s_i$.

We have equipped $(\omega^\omega)^n$ with the product topology, and the sets of the form $N_{\vec{s}}$ form a basis for this topology.

Lemma 7.1 *Every closed is $\utilde{\Sigma}^1_1$. Every open set is $\utilde{\Sigma}^1_1$.*

Proof For notational simplicity, let us work just for the space ω^ω.

First for C closed, let S be the collection of all $s \in \omega^{<\omega}$ such that N_s is disjoint from C. Then let T be the collection of of all $u \in \omega^{<\omega}$ such that u does not extend and $s \in S$. It follows that

$$C = [T] =_{\mathrm{df}} \{x \in \omega^\omega : \forall \ell (x|_\ell \in T)\},$$

and it is trivial then to find a tree T' with $p[T'] = [T]$.

Now for open sets, since $\utilde{\Sigma}^1_1$ is closed under number quantification, it suffices to show to prove the lemma for basic open sets. However since these are all clopen, it reduces to the last paragraph. □

Lemma 7.2 *Every Borel set is $\utilde{\Sigma}^1_1$.*

Proof This follows from the last lemma and the closure of $\utilde{\Sigma}^1_1$ under number quantification in light of the fact that the Borel sets can be generated from the open and closed sets by the operations of countable union and intersection. □

Corollary 7.3 *$\utilde{\Delta}^1_1$ equals Borel.*

Proof Using the last lemma and 6.2. □

Definition A separable topological space is said to be *Polish* if it admits a complete metric. The *Borel* subsets of the space are those which appear in the smallest σ-algebra containing the open sets.

Theorem 7.4 *Any uncountable Polish space contains a homeomorphic copy of 2^ω.*

Proof Let $\mathcal{B}$ be a countable basis for the Polish space X and let d be a complete metric. Let

$$U = \bigcup \{V \in \mathcal{B} : |V| \leq \aleph_0\}.$$

U is countable and open, so by the assumptions on X the complement C is non-empty, and the definition of U implies that it will be without isolated points.

That absence of isolated points makes it possible to find an array of open sets, $(V_s)_{s \in 2^{<\omega}}$ with the following properties:

1. $V_\emptyset = X$;
2. each $V_s \cap C \neq \emptyset$;
3. $d(V_s) \to 0$ as $lh(s) \to \infty$;
4. $s \perp t \Rightarrow V_s \cap V_t = \emptyset$;
5. $s < t \Rightarrow \overline{V_t} \subset V_s$.

Then for each $y \in 2^\omega$ there is a unique $x_y \in \bigcap_\ell V_{y|_\ell}$. The resulting function

$$y \mapsto x_y$$

will be a continuous embedding of 2^ω into X. The compactness of 2^ω implies it will in fact provide a homeomorphism with its image. □

Definition A function between two Polish spaces is said to be *Borel* if the pullback of any open set is Borel.

Exercise If $f : X \to Y$ is Borel, then the pullback of any Borel set in Y is Borel.

Definition Given Polish spaces X, Y and Borel subsets $B \subset X$, $C \subset Y$, a bijection

$$f : B \to C$$

is said to be a *Borel isomorphism* if the image of any Borel subset of B is Borel and the pullback of any Borel subset of C is Borel.

Lemma 7.5 *Any uncountable Polish space has a Borel subset which is Borel isomorphic to ω^ω.*

Proof In light of the last lemma, it suffices to prove this for $X = 2^\omega$. But if we let

$$f(x) = (0^{x(0)}10^{x(1)}10^{x(2)}....)$$

then f provides a Borel isomorphism of ω^ω with the set of elements in 2^ω that have infinitely many 1's. □

Lemma 7.6 *Any Polish space is Borel isomorphic to a Borel subset of ω^ω.*

Proof Fix Polish X with complete metric d. Let $(V_s)_{s\in\omega^{<\omega}}$ be an array of open sets such that:

1. $V_\emptyset = X$;
2. $d(V_s) \to 0$ as $lh(s) \to \infty$;
3. $V_s = \bigcup_{n\in\omega} V_{s^\frown n}$ (where $s^\frown n$ is the result of adjoining n to the end of the sequence s).

For each $x \in X$ we can then let $f(x)$ be the union of all the lexicographically minimal sequences s for which $x \in V_s$. In other words, $s \subset f(x)$ if and only if for all t with $lh(t) \leq lh(s)$ and for all $i < lh(s)$ we have either:

1. $s|_i \neq t|_i$, or
2. $s(i) \leq t(i)$, or
3. $x \notin V_t$.

It is a relatively routine manipulation of quantifiers type argument to see that $f[X]$ is Borel and f provides a Borel isomorphism with this set. $\square$

Theorem 7.7 *Any uncountable Polish space is Borel isomorphic to ω^ω.*

Proof This follows from the last two lemmas by a Borel version of the Schroeder-Bernstein argument. $\square$

Thus every theorem we have proved in the Borel context for ω^ω and its products holds as well for arbitrary uncountable Polish spaces. The final act of reframing is to correctly generalize the definition of $\underset{\sim}{\Sigma}^1_1$.

Theorem 7.8 *Let X be a Polish space, $B \subset X$ a Borel set, and*

$$f : X \to (\omega^\omega)^n$$

a Borel function. Then $f[B]$ is $\underset{\sim}{\Sigma}^1_1$.

Proof Without loss of generality $X = \omega^\omega$ and $n = 1$. Then let C be the collection of $(x, y) \in B \times \omega^\omega$ with $f(y) = x$. C is Borel and hence certainly $\underset{\sim}{\Sigma}^1_1$, so fix some T with $p[T] = C$. Note that $pp[T] = f[X]$, and thus it suffices to observe that there exists a tree T' with $p[T'] = pp[T]$. $\square$

Definition For X a Polish space, we say that $A \subset X$ is $\underset{\sim}{\Sigma}^1_1$, or *analytic*, if there is a Polish space Y, a Borel set $B \subset Y$, and a Borel function $f : Y \to X$ with

$$f[B] = Y.$$

By the last theorem, this extends our original definition of $\underset{\sim}{\Sigma}^1_1$ for product spaces of the form $\omega^m \times (\omega^\omega)^n$.

8 Reduction and reflection

Theorem 8.1 *(The recursion theorem) Let $(\varphi_e)_e$ be one of the standard enumerations of partial recursive functions. Let $f : \omega \to \omega$ be recursive. Then there is an e with*

$$\varphi_e = \varphi_{f(e)}.$$

Proof We use about the sequence $(\varphi_e)_e$ that

$$(e, n) \mapsto \varphi_e(n)$$

is partially recursive and that there is an enumeration of $(\psi_e)_e$ of partial recursive functions from $\omega \times \omega \to \omega$ such that

$$(e, m, n) \to \psi_e(m, n)$$

is partially recursive and that there is an associated total recursive function $S : \omega \times \omega \to \omega$ such that at all (e, m, n)

$$\psi_e(m, n) = \phi_{S(e,m)}(n).$$

Find an index k such that for all m, n

$$\phi_{f(S(m,m))}(n) = \psi_k(m, n).$$

Then for $e = S(k, k)$ we have

$$\phi_e(n) = \psi_k(k, n) = \phi_{f(S(k,k))}(n) = \phi_{f(e)}(n).$$

$\square$

Theorem 8.2 *(Π^1_1 reflection) Let $A \subset \omega \times \omega^\omega$ be some canonical universal Π^1_1 set. Suppose $B \subset \omega$ is a Π^1_1 set with the property that for all n, m*

$$A_n = A_m \Rightarrow (n \in B \Leftrightarrow m \in B).$$

Then for every $n \in C$ there exist some other k in C with $A_k \in \Delta^1_1$ and $A_k \subset A_n$.

Here the statement of the theorem basically says that if we have a property of Π^1_1 sets which is Π^1_1 in the indices, then every Π^1_1 set with the property contains a Δ^1_1 set with that property. The requirement that the sequence be canonical is simply that it arise from some sufficiently canonical partial recursive enumeration of trees. For instance if T_n is equal to the set of (u, v) of the same length and all $\ell < lh(u)$ having $\phi_n(\langle u, v\rangle) = 1$ (where $\langle\cdot\rangle$ is some system of Gödel coding), then $A = \{(n, x) : x \notin p[T_n]\}$ would be sufficient for our purposes.

Proof Fix some $C = A_k$ with $k \in B$. Let φ be Σ_1 and such that

$$n \in B \Leftrightarrow L_{\omega_1^{\mathrm{ck}}} \models \varphi(n).$$

Then similarly Σ_1 θ such that

$$x \in C \Leftrightarrow L_{\omega_1^x}[x] \models \theta(x).$$

We can first of all find a recursive partial function f such that

$$x \in A_{f(n)}$$

if and only if $x \in A$ and for α least with

$$L_\alpha[x] \models \theta(x)$$

we have

$$L_\alpha \models \neg\varphi(n).$$

Applying the recursion theorem, we obtain some e with

$$A_e = A_{f(e)}.$$

If $e \notin B$ then the definitions give $A_{f(e)} = C = A_k$, but then a contradiction breaks out since $k \in B$ and we already assumed that two indices given rising to the same Π^1_1 set must be equal. So $e \in B$. But then we fix some least $\delta < \omega_1^{\mathrm{ck}}$ with

$$L_\delta \models \varphi(e).$$

Then

$$A_{f(e)} = \{x : \exists\alpha < \delta L_\delta[x] \models \theta(x),$$

which is Δ^1_1. □

Corollary 8.3 *Let A, B be as above. Then the set of x such that for all x meets every A_n with $n \in B$ is Σ^1_1.*

Proof From the proof above and the effectiveness of the recursion theorem, we recursively obtain from each k some indices e_k, f_k which in the event $k \in B$ have the properties

1. $A_{e_k} \in \Delta^1_1$,
2. $A_{e_k} = \omega^\omega \setminus A_{f_k}$,
3. $A_{e_k} \subset A_k$,

4. $e_k \in B$.

Then the set in question is the collection of x such that for all k we have either:

1. $k \notin B$, or

2. $x \notin A_{f_k}$.

□

Exercise Show there are $A, B \subset \omega \times \omega^\omega$ and $C \subset \omega$ such that both are Π^1_1 and so that $\{A_n : n \in C\}$ enumerates the Δ^1_1 subsets of ω^ω and $n \in C \to A_n = \omega^\omega \setminus B_n$. (One way: Let C be the set of pairs of codes for $(<, \psi)$ such that $<$ is a recursive well order of ω and ψ is a formula of set theory. Then let $A_{(<,\psi)}$ be the set of x such that for α the order type of $<$ we have

$$L_\alpha[x] \models \psi(x).)$$

Exercise (Π^1_1 reduction) Let A and B be Π^1_1. Show there are Π^1_1 sets $A_0 \subset A$, $B_0 \subset B$ with $A_0 \cap B_0 = \emptyset$ and $A_0 \cup B_0 = A \cup B$. (For instance when the Σ_1 formulas ψ, θ define A, B uniformly over admissible sets, let A_0 be the set of x for which there exists $\alpha < \omega_1^x$ with $L_\alpha[x] \models \psi(x) \wedge \neg\theta(x)$ and B_0 be the set of x for which there exists $\alpha < \omega_1^x$ with $L_\alpha[x] \models \theta(x)$ and for all $\beta < \alpha$ $L_\beta[x] \models \neg\psi(x)$.)

9 Barwise compactness

Definition Given a language $\mathcal{L}$ we use $\mathcal{L}_{\infty,\omega}$ to to denote the result of closing the class of atomic formulas under the first order operations of negation and existential and universal quantifiers, as well as conjunctions and disjunctions of arbitrary size.

Infinitary logic differs from traditional first order logic in that the notions of proof theoretical and semantical consistency may diverge. We clarify the meaning that *consistency* will take for this paper.

Definition A proposition $\psi \in \mathcal{L}_{\infty,\omega}$ is *consistent* if in some generic extension there exists a model for ψ.

It is an easy absoluteness argument to see that any two transitive models satisfying a sufficiently large fragment of ZFC in which ψ is hereditarily countable will agree as to whether it has a model. Thus consistency is Δ_1 in the Levy hierarchy.

For later purposes we need a sharper result: Consistency in our sense is uniformly Π_1 over any admissible structure containing ψ. I will present a game theoretical explication, though one could also work with forcing.

Definition For $\psi \in \mathcal{L}_{\infty,\omega}$ let $F_\psi \subset \mathcal{L}_{\infty,\omega}$ be the fragment generated by ψ. For technical purposes, involving the usual variants of Henkenization, we will assume that $\mathcal{L}$ contains infinitely many constants not appearing in ψ. Let G_ψ be the following closed[3] game of length ω: I and II alternate playing elements of F_ψ. II can play any element at all of F_ψ, but I's moves are tightly constrained, and if I is ever unable to make a legal move then II wins.

1. I must begin with the move ψ;

2. if II plays $\phi \vee \neg\phi$, then I must at the next turn play ϕ or $\neg\phi$;

3. if I has previously played $\bigwedge_\alpha \phi_\alpha$ and II plays some ϕ_β, then I must immediately respond with ϕ_β;

[3]Here *closed* means that the closed player wins if the game continues forever, and loses if at some finite stage the other side has already reached a winning position

4. if I has previously played $\bigvee_\alpha \phi_\alpha$ and II plays $\bigvee_\alpha \phi_\alpha$ then I must respond with some ϕ_β;
5. if I has previously played $\neg \bigwedge_\alpha \phi_\alpha$ and II plays $\neg \bigwedge_\alpha \phi_\alpha$ then I must respond with $\bigvee_\alpha \neg\phi_\alpha$;
6. if I has previously played $\neg \bigvee_\alpha \phi_\alpha$ and II plays $\neg \bigvee_\alpha \phi_\alpha$ then I must respond with $\bigwedge_\alpha \neg\phi_\alpha$;
7. if I has previously played $\neg\neg\phi$ and II plays $\neg\neg\phi$, then I must play ϕ;
8. if I has previously played $\exists x\phi(x)$ and II plays $\exists x\phi(x)$, then I must respond with $\phi(c)$ for some constant c.
9. at no stage can two of I's previous moves consist at one move in ϕ and at another move $\neg\phi$.

II wins if I is ever unable to make a next move in accordance with these requirements. I wins if the game keeps going for infinitely many turns.

One should think of this game as a interrogation between II and I. I begins by asserting ψ, and then II steadily asks questions about how I might imagine a model of this proposition to be formed. I wins if the story continues indefinitely without contradiction; II wins if I's account is shown at some stage to be absurd.

The next couple of lemmas are well known and standard.

Lemma 9.1 *Let ψ, G_ψ be as above. Assume ψ is hereditarily countable.*
Then ψ has a model if and only if I wins G_ψ.

Proof First assume that ψ has a model. Then I wins by simply by responding to each of II's "questions" with the answer found in the model.

Conversely, let I have a winning strategy in G_ψ. Our assumptions give that F_ψ is countable, and so we can let II make a maximal play which mentions every element of F_ψ infinitely often. We take the model which consists of the collection of atomic propositions asserted at some stage by I. It is then a routine induction on logical complexity to show that this model satisfies some $\phi \in F_\psi$ if and only if I has at some point played ψ. □

Lemma 9.2 *Let P_ψ be the collection of legal positions in the game G_ψ. Then I wins if and only if there is no*

$$\rho : P_\psi \to \gamma \cup \{\infty\},$$

for some ordinal γ, such that at each position $p \in P$:

1. *if I is to move then $\rho(p) = \sup\{\rho(p^\frown\phi) : p^\frown\phi \in P_\psi\}$;*
2. *if II is to move then $\rho(p) = \inf\{\rho(p^\frown\phi) + 1 : \phi \in F_\psi\}$;*
3. *$\rho(\langle\phi\rangle) \neq \infty$.*

Proof First suppose we have such function $\rho : P_\psi \to \gamma$. Note that our first two assumptions give $\rho(p) = 0$ if and only if it is I's turn to move and there is no legal move to be made. Thus II can fashion a winning strategy by driving the ordinal down until it hits zero.

Conversely, let us just attempt to define by induction ρ with the initial requirement that $\rho(p) = 0$ if it is I's move and there is no legal response, and then recursively continuing with

1. if I is to move then $\rho(p) = \sup\{\rho(p^\frown\phi) : p^\frown\phi \in P_\psi\}$ provided all such $\rho(p^\frown\phi)$ have been given ordinal values;

2. if II is to move then $\rho(p) = \inf\{\rho(p^\frown\phi) + 1 : \phi \in F_\psi\}$ provided at least one such $\rho(p^\frown\phi)$ has been given an ordinal value.

This resembles the earlier ranking argument at trees, but with a slight additional complication because the other side gets to move. At stage α we will have defined $\rho^{-1}[\alpha+1] = \{p : \rho(p) \leq \alpha\}$.

Once this has been continued as far as it will permit, we set $\rho(p) = \infty$ if p has not previously been assigned an ordinal value. By assumption, $\rho(\langle\psi\rangle) = \infty$, and I's winning strategy is to always play to keep $\rho(p) = \infty$. $\square$

Lemma 9.3 *There is a Π_1 formula Φ such that for any $\psi \in \mathcal{L}_{\infty,\omega}$ and admissible $\mathcal{M}$ containing ψ,*

$$\mathcal{M} \models \Phi(\psi)$$

if and only if ψ is consistent.

Proof By 9.1, it suffices to see I winning G_ψ is uniformly Π_1 over $\mathcal{M}$, and this in turn follows from the proof of 9.2. Inside the admissible we can recursively define

$$\alpha \mapsto \rho^{-1}[\alpha+1]$$

using the description from before. This will be a Δ_1 recursion, and hence defined by a Δ_1 formula over $\mathcal{M}$, and hence total and calculated the same in $\mathcal{M}$ as in V. It follows from the definition of admissibility that $\rho(p) = \infty$ if and only if there is no $\alpha \in \mathcal{M}$ with $\rho(p) = \alpha$, and this statement itself is Π_1 over $\mathcal{M}$. $\square$

Lemma 9.4 *Let $\mathcal{M}$ be admissible and $T \subset \mathcal{L}_{\infty,\omega} \cap \mathcal{M}$ be Σ_1 over $\mathcal{M}$. Suppose that every $X \in \mathcal{M}$ with $X \subset T$ is consistent.*

Then T is consistent.

Proof Since our definition of consistency is absolute between generic extensions, we may assume $\mathcal{M}$ countable.

We can give a direct game theoretic proof, where the strategy of the first player is to always maintain that the run of moves so far along with T preserves the property that any subset inside $\mathcal{M}$ is consistent in our sense. Using that inconsistency is Σ_1, if I ever reaches a position where every legal move p has some corresponding $X_p \subset T$, $X_p \in \mathcal{M}$ which gives rise to an inconsistent collection, then we can collect them together to show that there was already a single $X \in \mathcal{M}$ which witnesses inconsistency. $\square$

There are other results which we can obtain with this technique.

Lemma 9.5 *Let $\mathcal{L}, \mathcal{L}', \mathcal{L}''$ be non-empty languages with $\mathcal{L} = \mathcal{L}' \cap \mathcal{L}''$. Let $\phi' \in \mathcal{L}'_{\infty,\omega}, \phi'' \in \mathcal{L}''_{\infty,\omega}$ with $\phi' \Rightarrow \phi''$. Then there is $\phi \in \mathcal{L}_{\infty,\omega}$ with*

$$\phi' \Rightarrow \phi \Rightarrow \phi''$$

(i.e. $\phi' \wedge \neg\phi$ and $\phi \wedge \neg\phi''$ are both inconsistent).

Proof Suppose instead there is no such interpolant.

We define a game $G_{\phi',\neg\phi''}$, a variant of G_ψ given earlier. At each turn II can plays some $\tau' \in \mathcal{L}'_{\infty,\omega}$ and some $\tau'' \in \mathcal{L}''_{\infty,\omega}$. I then responds with some $\sigma' \in \mathcal{L}'_{\infty,\omega}$ and some $\sigma'' \in \mathcal{L}''_{\infty,\omega}$. Again we stay inside the fragments generated by ϕ', ϕ''. In analogy to before, I begins with $\phi', \neg\phi''$. The conditions from the last argument are transplanted in the obvious way:-

1. if $\tau' = \phi \vee \neg\phi$, then I must at the next turn play ϕ or $\neg\phi$;

2. if I has previously played $\bigwedge_\alpha \phi_\alpha$ and $\tau' = \phi_\beta$, then I must immediately respond with $\sigma' = \phi_\beta$;
3. if I has previously played $\bigvee_\alpha \phi_\alpha$ and $\tau' = \bigvee_\alpha \phi_\alpha$ then I must respond with some $\phi_\beta = \sigma'$;
4. if I has previously played $\neg \bigwedge_\alpha \phi_\alpha$ and $\tau' = \neg \bigwedge_\alpha \phi_\alpha$ then I must respond with $\sigma' = \bigvee_\alpha \neg\phi_\alpha$;
5. if I has previously played $\exists x \phi(x)$ and $\tau' = \exists x \phi(x)$ then I must respond with $\phi(c)$;
6. if I has previously played $\neg \bigvee_\alpha \phi_\alpha$ and $\tau' = \neg \bigvee_\alpha \phi_\alpha$ then I must respond with $\sigma' \bigwedge_\alpha \neg\phi_\alpha$;
7. if I has previously played $\neg\neg\phi$ and II plays $\tau' = \neg\neg\phi$, then I must play $\sigma' = \phi$.

Similarly there will conditions which have τ'' for τ' and σ'' for σ'.

We have also included the requirement that in the union of all the moves played by I, all the τ''s and τ'''s, there is no outright contradiction.

Here one can adapt the usual proof of the Craig interpolation theorem to show the existence of a winning strategy for the first player: I just maintains that if ρ' is the conjunction of the propositions in $\mathcal{L}'_{\infty,\omega}$ asserted by I and ρ'' is the conjunction of the propositions in $\mathcal{L}''_{\infty,\omega}$ asserted by I, then there is no $\rho \in \mathcal{L}_{\infty,\omega}$ such that

$$\rho' \Rightarrow \rho \Rightarrow \neg\rho''.$$

We then go to a generic extension in which ϕ', ϕ'' are hereditarily countable. We let II run the complete play, asking every question in the respective fragments infinitely often. We get a model of $\mathcal{L}' \cup \mathcal{L}''$, since there is no disagreement on the common parts of the language, by setting $\mathcal{M} \models \psi$, for ψ an atomic proposition, if at some point in I's play ψ appears. Following the proof of 9.1 we get

$$\mathcal{M} \models \phi', \neg\phi'',$$

with a contradiction to the hypotheses of the lemma. □

The proof of the last lemma actually gives an interpolant in any admissible structure $\mathcal{M}$ containing ϕ' and ϕ'': The point is that we can adapt the requirement that there be no interpolant *in* $\mathcal{M}$. Note moreover from 9.3 we have that there is a Σ_1 formula, Ψ, such that

$$\mathcal{M} \models \Psi(\phi', \phi'', \phi)$$

if and only if ϕ is an interpolant between ϕ' and ϕ''.

10 Gandy-Harrington forcing

Definition We let $\mathcal{P}$ be the partial ordering of all non-empty *lightfaced* Σ^1_1 sets $A \subset \omega^\omega$ ordered by inclusion.

Lemma 10.1 *For any reasonably generic filter $G \subset \mathcal{P}$ there will be a unique point*

$$x(G) \in \bigcap_{A \in G} A.$$

Proof The point here is that if $A = p[T]$ and if for each $s \in \omega^{<\omega}$ we let

$$A_s = \{x : \exists y \supset s((x, y) \in [T]\},$$

then the set $\mathcal{D}_{T,s}$ consisting of all $B \in \mathcal{P}$ which either are included in some $A_{s^\frown n}$ or avoid A_s will be dense open. If G meets all these kinds of dense open sets, then it will have a unique element in its infinite intersection. □

We can also view this as providing a topology on ω^ω, whose basis consists of the Σ^1_1 sets. This topology falls short of being Polish, but it has other desirable properties, such as satisfying the Baire category theorem.

Definition A point $x \in X$ is *low* if ω_1^x equals ω_1^{ck}.

Lemma 10.2 *x is low if and only if for every $A \subset X$ in Σ^1_1 either $x \in A$ or there is some $C \in \Sigma^1_1$ with $x \in C$ and*

$$A \cap C = \emptyset.$$

Proof At each e, let A_e be the set of z such that the e^{th} recursive in z linear order is *not* a recursive well order. Apply the statement of the lemma to x and we obtain that either $x \in A_e$, or x is in some Σ^1_1 B which avoids A_e. In the latter case, we obtain by boundedness a $\delta < \omega_1^{\mathrm{ck}}$ such that every $z \in B$ has the e^{th} recursive in z well order of rank less than δ. □

Theorem 10.3 *(Gandy) Every non-empty Σ^1_1 set contains a low member.*

Proof Force below our non-empty Σ^1_1 set, and observe that the generic G will have $x(G)$ satisfying the conditions of the last lemma. □

Just for the purposes of illustration of how one might use Gandy-Harrington forcing, let us go through a proof of the perfect set theorem for Σ^1_1 sets.

Theorem 10.4 *Let $A \in \Sigma^1_1$. Then either:*

1. *$|A| \leq \aleph_0$, and is in fact included in Δ^1_1; or*
2. *A contains a homeomorphic copy of 2^ω.*

Proof We let $B = \{x \in X : \{x\} \in \Sigma^1_1\}$, the set arising as the union of all Σ^1_1 singletons.

Claim: $X \setminus B$ is Σ^1_1.

Proof of claim: Here it is a routine calculation to show that $\{x\} \in \Sigma^1_1$ if and only if $\{x\} \in \Delta^1_1$. From this one can show by another routine calculation[4] that $X \setminus B$ is indeed Σ^1_1. (Claim□)

Now there is a split in cases.

Case(I) $A \subset B$.

Then we are immediately finished, since every element of A is a Σ^1_1 singleton, and there are only countably many (lightfaced) Σ^1_1 singletons.

Now we go to the other case. Here we will find a way to build a "perfect set" of generic objects in $A \setminus B$. This will give us a homeomorphic copy of Cantor space.

Case(II) $A \setminus B \neq \emptyset$.

We define an array of elements in $\mathcal{P}$, $(p_s)_{s \in 2^{<\mathbb{N}}}$, indexed by finite binary sequences, such that

(0) each $p_s \subset A \setminus B$; we work below the complicated part of A;

(1) $p_s \leq p_t$ for $s \supset t$; that is to say, as the binary sequences get longer, the Σ^1_1 sets get smaller;

(2) if $s \in 2^n$ (i.e. s has length n), then $p_s \in D_n$;

(3) for $s \neq t$ of the same length, $p_s \cap p_t = 0$; that is to say, incompatible binary sequences are associated with disjoint binary sequences.

We can start the construction with

$$p_{\text{empty sequence}} = A \setminus B.$$

[4]Usually in these kinds of arguments there will be some step where the set of "complicated points" is shown to be Σ^1_1, even though it initially looks more complicated. Here it is straightforward calculation, but in general it can be involved, for instance making use of "Π^1_1 reflection."

It is routine to diagonalize in to meet the n^{th} dense set at stage n. The part of this that might be problematic is (3). But here we know that whenever we have some given $p_s \subset A \setminus B$, it must contain two elements, or it would consist of a Σ^1_1 singleton and hence be in B; and so we can divide it up by two disjoint open sets.

In the end we define for any infinite binary sequence $w \in 2^{\mathbb{N}}$ the filter

$$G_w = \{q : \exists n(p_{w|n} \leq q)\}$$

generated by the conditions associated with the finite initial segments of w.

Each such G_w is "sufficiently generic", and one can show that the function assigning to w the real defined by its generic filter,

$$w \mapsto x(G_w),$$

is continuous. And thus

$$\{x(G_w) : w \in 2^{\mathbb{N}}\}$$

is a homeomorphic copy of Cantor space. □

Of course this application may seem rather precious, since we have a far simpler proof. The interest is the method, since in the context of the theory of equivalence relations, this has been the approach which has been most fruitful.