A Few Issues Regarding Sets

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Outline

Tarski Example in Naproche-SAD

Part of the Naproche-SAD `tarski.ftl` example:

```
Signature ElmSort.  An element is a notion.
...
Axiom EOfElem.  Every element of S is an
element.
...
Signature LessRel.  x <= y is an atom.
...
Definition DefLB. Let S be a subset of T.
A lower bound of S in T is an element u of T
such that for every (x << S) u <= x.
```

Challenge: Translate to a set theory system's library.

Tarski Example in Egal

- The definitions and theorems in the article need to make sense outside the article, so all dependencies must be made explicit.
- The `tarski.ftl` was translated by hand to Egal in two ways.
- Version 1 tries to follow the Naproche version closely.
- Version 2 tries to be more natural for Egal.
- See the `ForSet` repo for the full files.
- Consider the definition of `a_lower_bound` in both versions.

Tarski Example in Egal Version 1

- Version 1 definition of `a_lower_bound`

  ```
  Definition a_lower_bound: set -> prop
  := fun u ⇒ u ∈ T ∧ ∀x ∈ S, u ≤ x.
  ```
- Dependencies in context:

  ```
  Variable Elt: set -> prop.
  Variable Leq: set -> set -> prop.
  Infix ≤ 400 := Leq.
  Variable S T:set.
  ```
- Mathematically: Given a class of elements E, a binary relation $\leq$ and two sets S and T, `a_lower_bound` is a predicate recognizing elements of T that give a lower bound for S relative to $\leq$.
- Theorems in this context also have hypotheses that would be exported as explicit: S and T should only contain members from E, $\leq$ should be a partial order on E and we should have $S \subseteq T$.

Tarski Example in Egal Version 2

- In Version 2 the definition is the same, but the dependency on a class E of elements is removed along with the relevant hypotheses.
- Instead of assuming $\leq$ is a partial order on E, we assume it is globally a partial order.
- This version is more natural for a mathematical library since a class E would not need to be fixed in order to use the definitions and theorems outside the article.
- However it is noticably different from the original Naproche-SAD version.
- Which of these versions should an autotranslator from Naproche-SAD to Egal target?

Outline

Issue 1: Translating Naproche to Egal

Issue 2: Type Theory vs. Set Theory

Groups and Subgroups

- There are many different ways to formalize what "subgroup" means.
- Set theoretically: given two groups G and H, H is a subgroup of G if...*left to reader*.
- Type theoretically there are different possibilities. One approach is:
  - "Group" is a structure type with a carrier and some other information.
  - For a group G, "subgroup of G" is a structure type giving a predicate on the carrier of G and some other information.
- Note that "subgroup of G" and "group" are different types.
- Also "subgroup" is not a relation between groups, so asking if the relation is transitive makes no sense.

Type Theory Subgroups

- Given a fixed "ambient" group G, then one can define a "subgroup relation" on "subgroups of G" in an obvious way.
- Let's write $K \leq H$ if K and H are of type "subgroup of G" and K and H are in this "subgroup relation".
- Transitivity of this relation is now a proposition about one group G and three subgroups of G:
  - For every ambient group G, and subgroups M, K and H of G, if $M \leq K$ and $K \leq H$, then $M \leq H$.
- It's tempting to hide the dependency on G and just say: If $M \leq K$ and $K \leq H$, then $M \leq H$.

Type Theory Normal Subgroups

- Let's write $K \trianglelefteq H$ if K and H are of type "subgroup of G" and K is a "normal subgroup" of H defined in an obvious way.
- We could ask if $\trianglelefteq$ is transitive.
- Ignoring the ambient group G, the proposition looks like $\forall M, K, H. M \trianglelefteq K \wedge K \trianglelefteq H \Rightarrow M \trianglelefteq K$.
- The answer seems to be no, but technically this depends on the G. For small G, $\trianglelefteq$ is transitive.
- The false proposition is the one with G explicitly universally quantified.

Set Theory Normal Subgroups

- In a set theory formalization, these issues do not arise.
- Groups are sets coding some information (a carrier and operations).
- Subgroup and normal subgroup are relations between sets (where the related sets are always groups).
- The subgroup relation is transitive, but the normal subgroup relation is not.
- An Egal formalization of the example can be found in the `ForSet` repo.