How proofs are told.
Linguistic aspects of proof texts

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The origins of Naproche

- Originally mainly sentence-based translation of a controlled natural language into a formal language
- Builds on dynamic semantic approaches:
  - Conditional structures
  - Dynamic quantifying
  - Anaphoric relations

$(\exists x \ldots) \rightarrow \ldots x \ldots \quad \Leftrightarrow \quad \forall x(\ldots \rightarrow \ldots x \ldots)$

$(\exists x \ldots) \land \ldots x \ldots \quad \Leftrightarrow \quad \exists x(\ldots \land \ldots x \ldots)$

Controlled natural language (CNL)

- A CNL is a subset of NL
  - Restricting grammar and vocabulary
  - Reducing or eliminating ambiguity and complexity

  - Human readability
  - Reliability of automatic semantic interpretation

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  - **Reliability of automatic semantic interpretation**

Naproche-CNL

- Logical vocabulary
  - Connectives
  - Quantifiers
  - NP-structures
- Mainly sentence-based
- “Transsentential” phenomena:
  - Conditional nesting
  - Anaphora
  - Macro-structures (e.g. theorem, proof)
- Pragmatics? (everything we need to understand beyond semantics in order to get the interpretation right)

Proof texts

■ **Third Proof.** Suppose $\mathbb{P}$ is finite and p is the largest prime. We consider the so-called *Mersenne number* $2^p - 1$ and show that any prime factor q of $2^p - 1$ is bigger than p, which will yield the desired conclusion. Let q be a prime dividing $2^p - 1$, so we have $2^p \equiv 1 \pmod{q}$. Since p is prime, this means that the element 2 has order p in the multiplicative group $\mathbb{Z}_q \backslash \{0\}$ of the field $\mathbb{Z}_q$. This group has $q - 1$ elements. By Lagrange's theorem (see the box) we know that the order of every element divides the size of the group, that is, we have $p \mid q - 1$, and hence $p < q$. □

- Based on textbook proofs
- Features:
  - Proof texts are characterized by **recursive nested structures** (conditionals, case distrinctions, subproofs).
  - Most proof texts make vast use of **formal notation**.
- Main theses:
  - Proof texts exhibit (nearly) all features of texts in other domains that make automatic interpretation hard, differing in degree.
  - Proof texts resemble narratives in small worlds.

Proof texts, outline of the talk

- Vagueness
- Argumentative gaps
- Metaphor of time
- Nested structure
- Ambiguity
- Explicature, implicature
- Presuppositions

- Referential structure
- Frames

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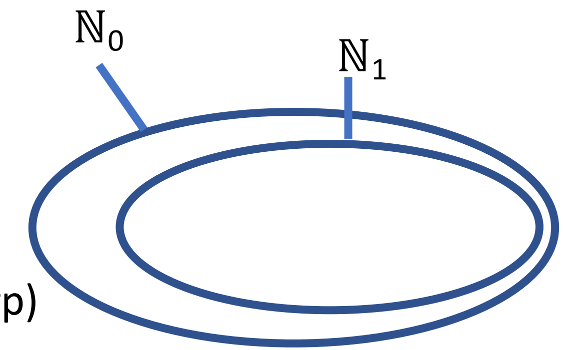
Sharpness

- Vagueness is almost absent in mathematical texts (except some comments in proof texts).

- Vagueness: gradable property (child, tall, rich, …)

- Polysemy (form of ambiguity):
  - Child: young human (vague), offspring of someone (sharp)
  - “natural numbers” starting with 0 or 1



Vagueness in comments

Before we start with the proof, we show that the induction scheme is easy to generalize. In the classical form the induction step requires that one derive a statement for $n + 1$ out of a statement depending on n.

(Reformulation of the proof in Kowalski, 2016, 92f by a master student in physics, data collected by Deniz Sarikaya)

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Argmentative gaps

General relevance principle:

Tell exactly what is necessary for the recipient to get the message.

☛ Implicature

The metaphor of time

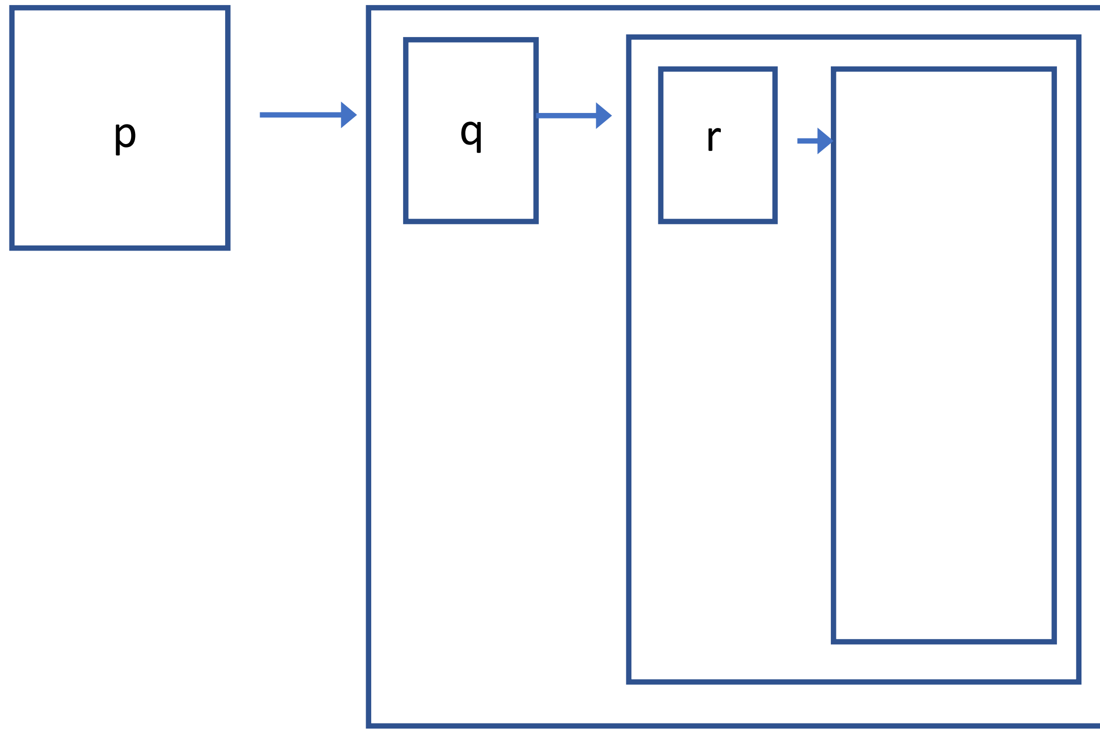
- Proof texts make wide use of the metaphor of time, derivations are conceptualized as successions in time:
  - Terminology: “follow”, “antecedent”, “conclusion”
  - Tense, temporal expression:
    - “as we have proven”
    - “as we will show”
    - “now”, ”first”, “next”
- The temporal structure is essentially the linear text structure, but may be some abstract structure, too, if referring to omitted parts.
- Techniques from narrative texts
- Macro-structures mostly have a (conventional) linear order.
  Conventional order can be overridden, esp. locally:
  - Postponed conditionals: …, if n is even; postponed quantifiers: … for all natural number

Nested structures: Conditionals

Assume p …

Suppose q …

If r …



Conditionals and existential quantification

(1) Suppose there are natural numbers n, m.

(2) Let g be a group.

- Historically these constructions go back to an ancient greek imperative of the 3rd person.
  ("The situation should be such that …")

- Cf.: Let's imagine there are pink unicorns. People would chase them and keep them in zoos. They would be an attraction.

Anaphora

Coreference to an object mentioned previously.

- Pronouns
- Definite NPs
- Proper nouns

- Mathematics: Variables

Anaphora

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Anaphora

Since $K^r \not\subseteq G$, each of the sets V_i is independent, and they partition $V(G)$.

Diestel, Reinhard (2006). *Graph Theory*. Heidelberg: Springer, 165f

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Ambiguity

- Perceived ambiguity is a rare phenomenon.
  - Interpretation may be only partial.
  - We are mostly able to find the intended reading by heuristics, semantic plausibility, and context information.

- Syntactic ambiguities (abstracting from semantics)

- Semantic ambiguities (abstracting from plausibility, background knowledge, and context)

- Ambiguity of notation (lexical, syntactical):
  - *0*: number, vector, zero element of a ring
  - X^2: square, repetition
  - *x(a+b)*: function, product

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Notions of a. relevant for machine interpretation

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- Ambiguities are pervasive.
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⇒ “Avoid ambiguities!” in the sense of is not a feasible strategy in proof writing.

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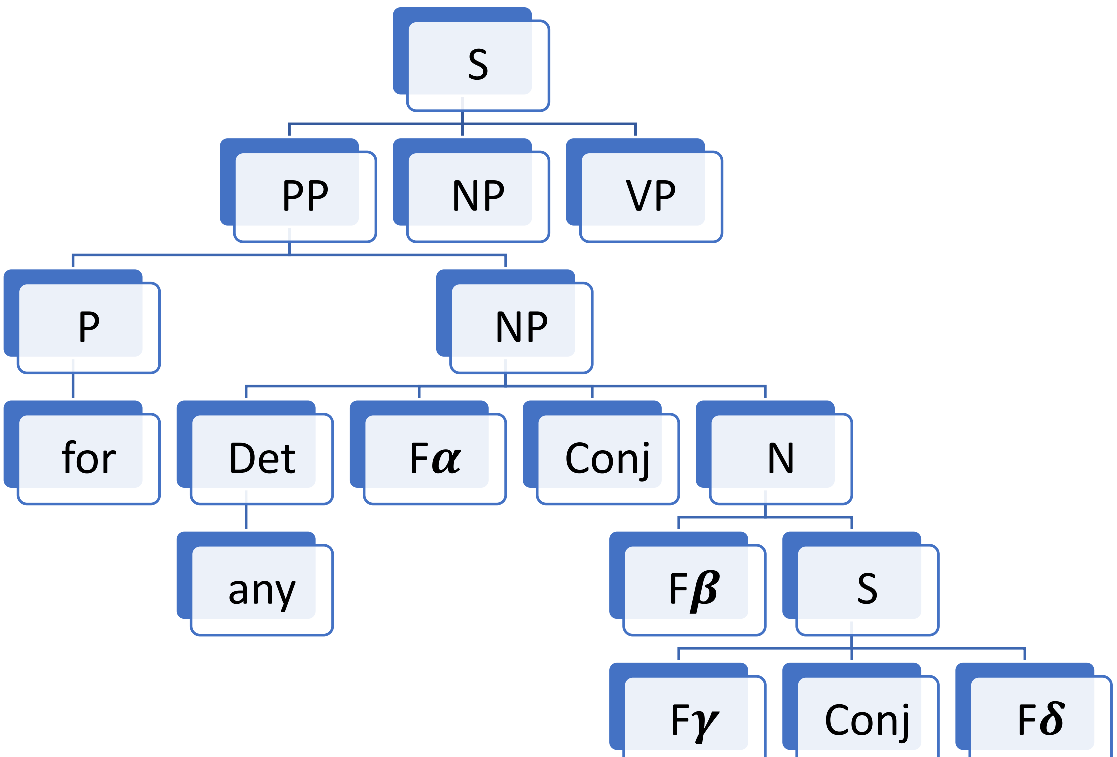

Syntactic ambiguity

- for any $\boldsymbol{\alpha}$ and [$\boldsymbol{\beta}$ such that [$\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$]] NP VP
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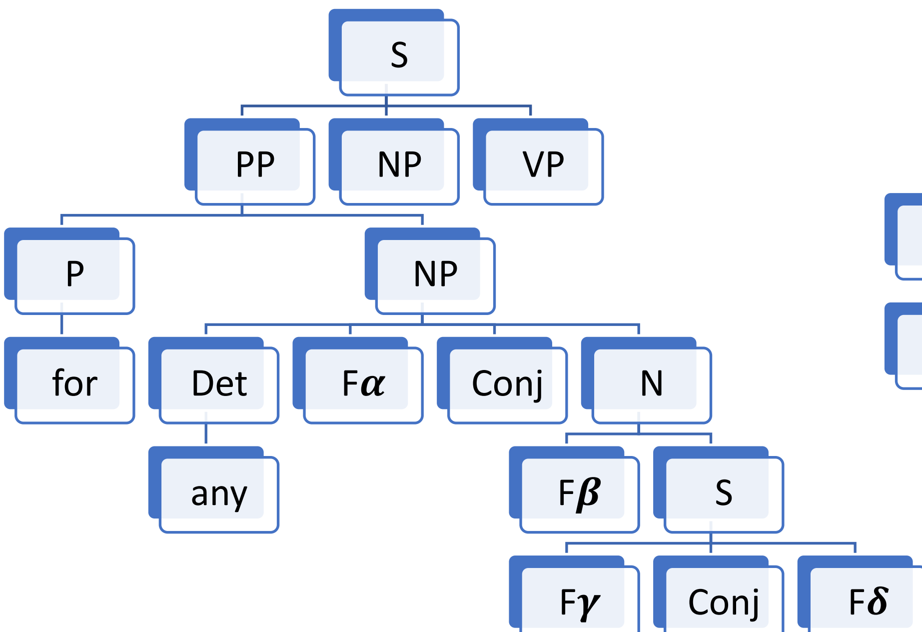
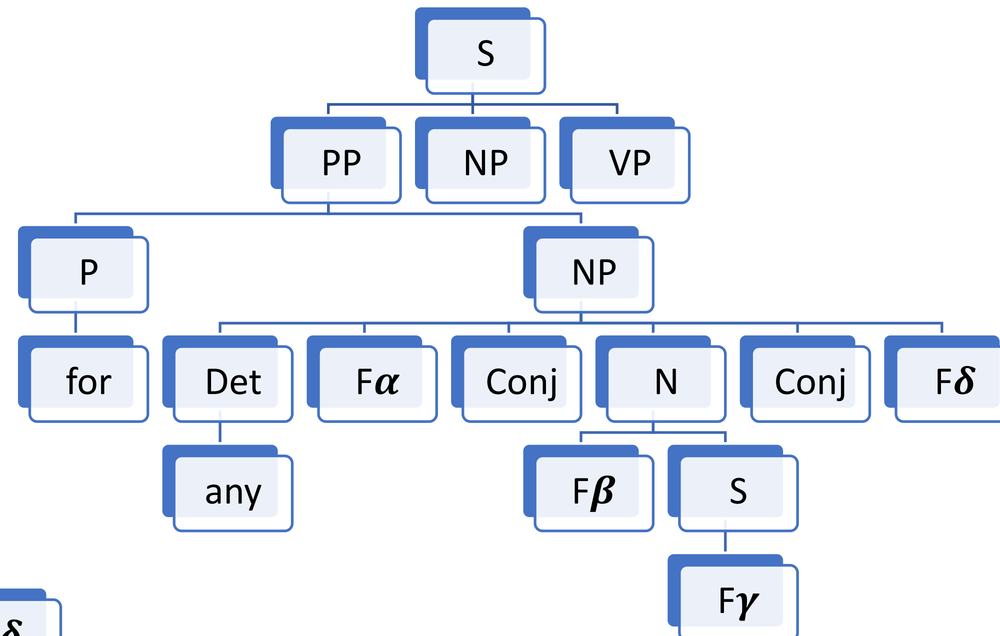
  (formulas can be N(P) or S)





If it is shown that for any vector $v \neq 0$ in V and $k \geq 0$ such that $f^k(v) = 0$ and $f^{k-1}(v) \neq 0$ the vectors $(v, f(v), \ldots, f^{k-1}(v))$ are linearly independent, then $k \leq n$, because $(v, f(v), \ldots, f^{k-1}(v))$ are k linearly independent vectors and there are at most n linearly independent vectors in V.

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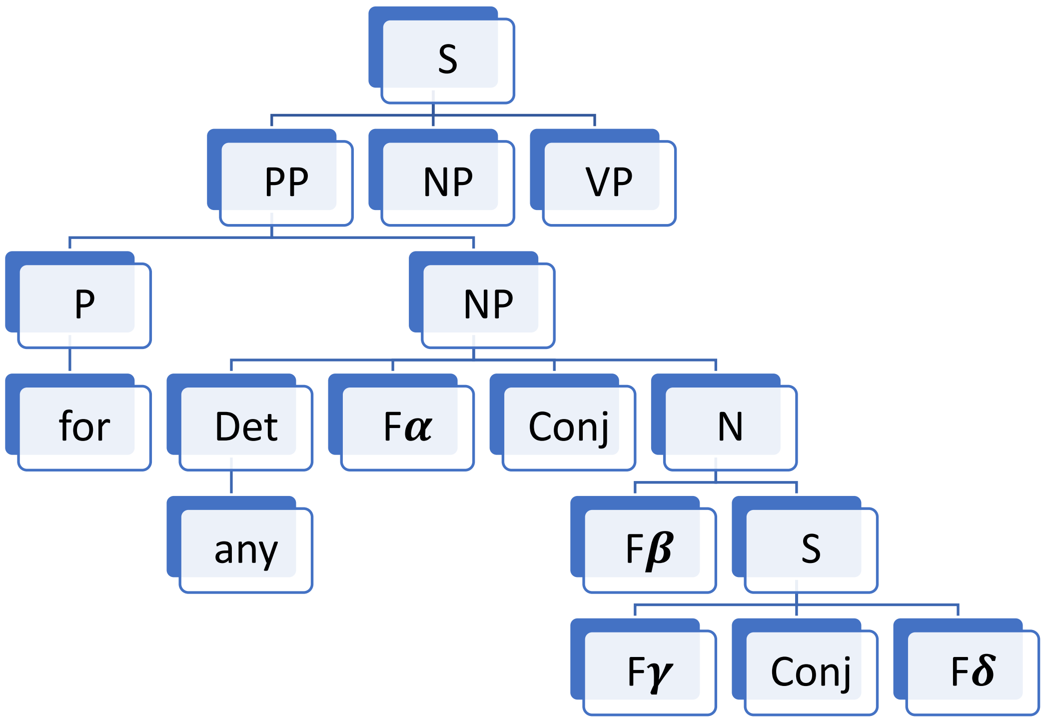

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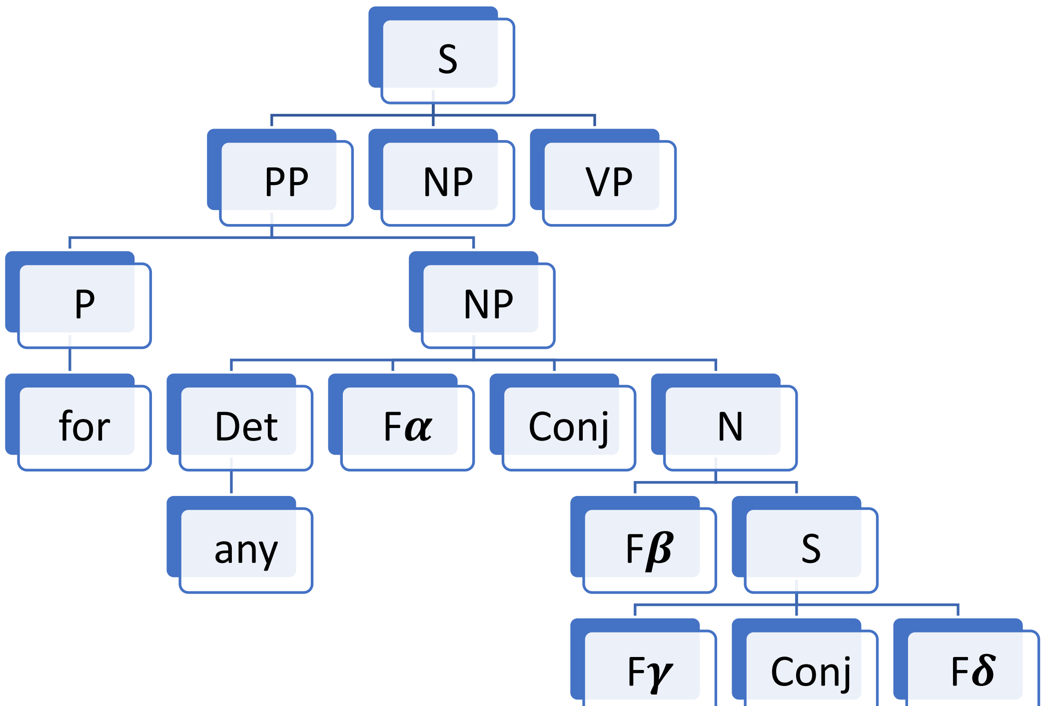


Semantic ambiguities

Scope ambiguities:

- Scope of assumptions
- Scope of connectives
- Quantifier scope
- Distributive and collective readings (plurals, NP conjunctions)

Semantic ambiguities: quantifier scope

(3) All students of our university should read a book.
$\forall\exists$, $\exists\forall$?

(4) Some element of any nonempty set S is not a subset of S. (Andrei Paskevitch)

(5) Any points belong to some line. (Andrei Paskevitch)

Semantic ambiguities: scope

(3) All students of our university should read a book.

(6) At the UDE *Olivias Garten* by Alina Bremer was chosen in the program of the Stifterverband “Eine Uni Ein Buch”.

- $\exists\forall$!

Semantic ambiguities: scope

- Scope ambiguities are everywhere

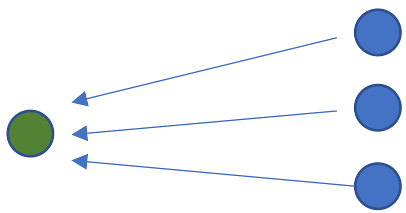
Scope and plurals

(Plurals in the Naproche CNL, cf. Cramer, Schröder, 2010)

- Three musicians of the chamber orchestra played a string instrument.

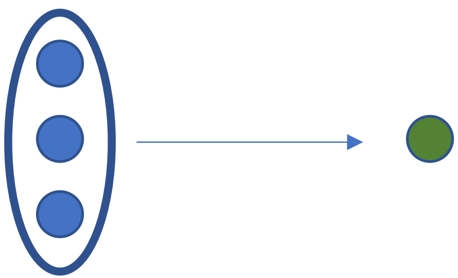
Scope and plurals

- Three men carried a piano.



Scope and plurals

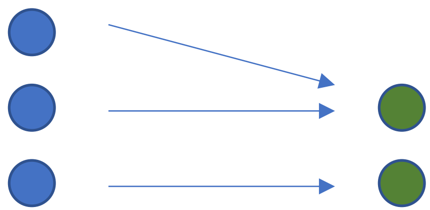
- Three men carried a piano.
  (plural entity, **collective** reading)



Let p_1, p_2, p_3, … be a sequence of primes in increasing order …

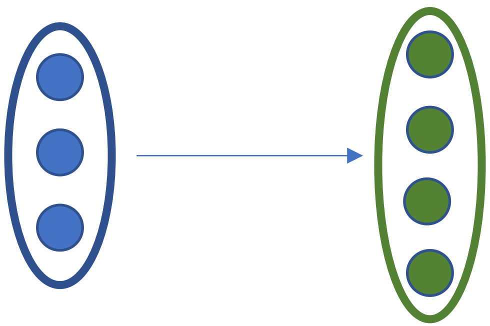
Scope and plurals

- Three musicians of the chamber orchestra played a string instrument. (**distributive** reading)



Scope and plurals

- Three of the guests drank four bottles of wine.
  (**cumulative** reading, often used in statistics)



Plural readings and ambiguity

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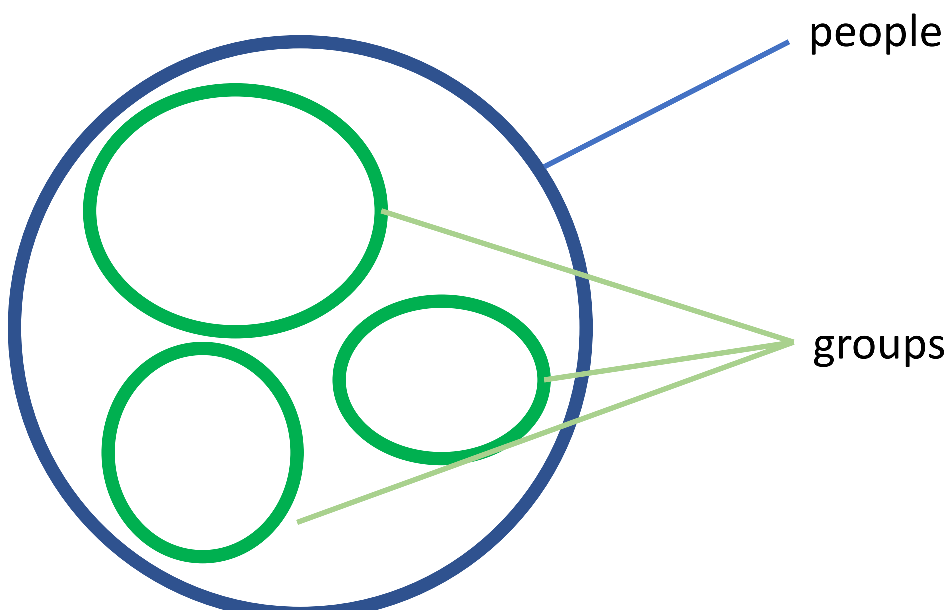

Plural readings and ambiguity

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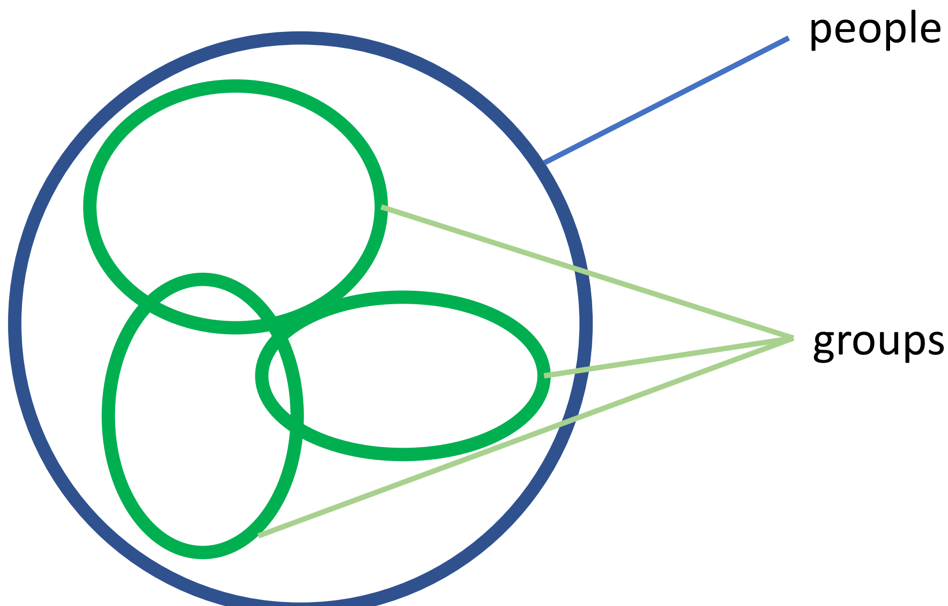
- Negation of: *In each group were the same people.*

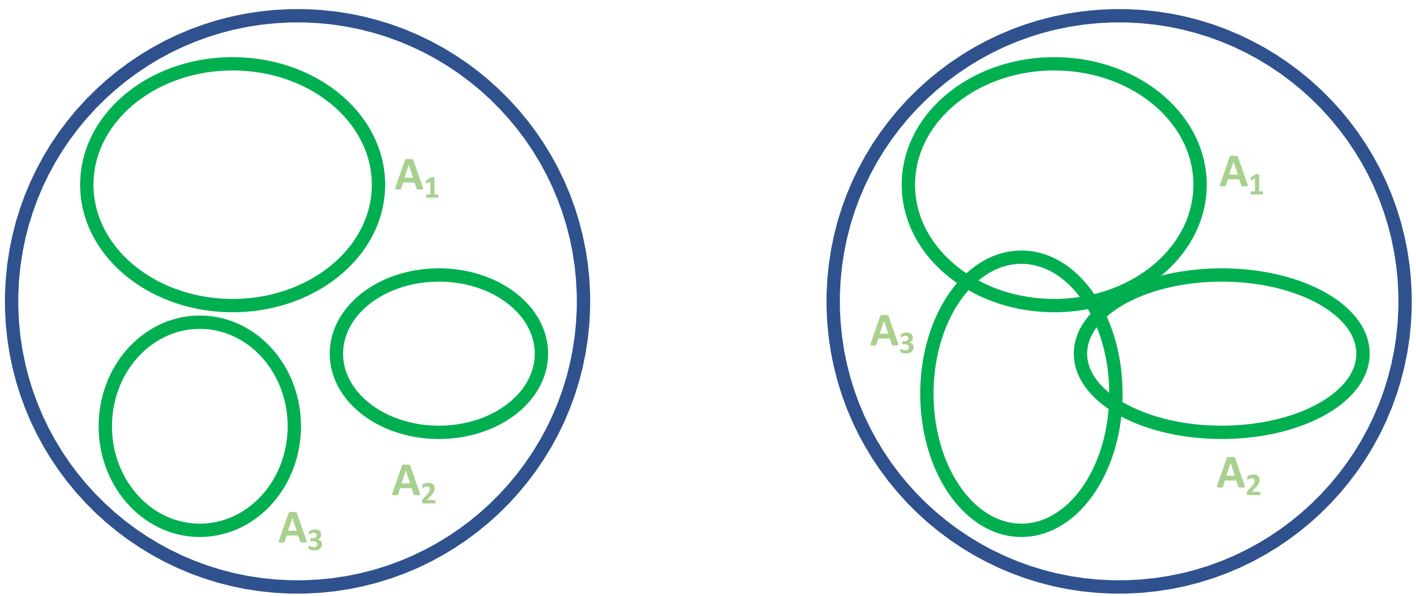
(8) Each group consisted of different people.



Plural readings and ambiguity

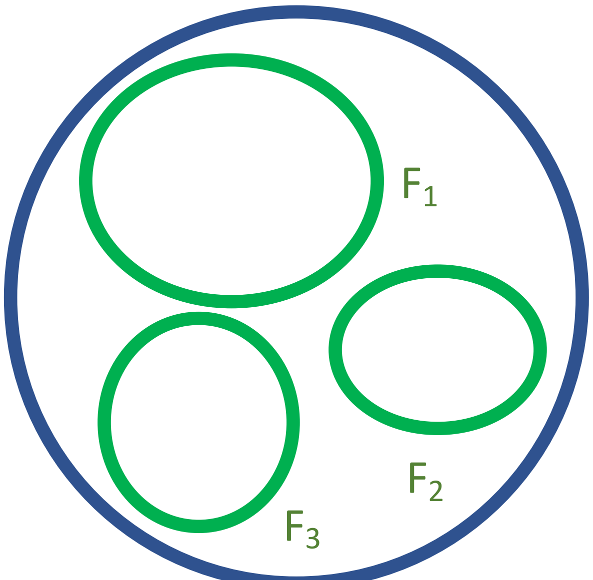
(9) The sets A_1, A_2, A_3 consist of different members.

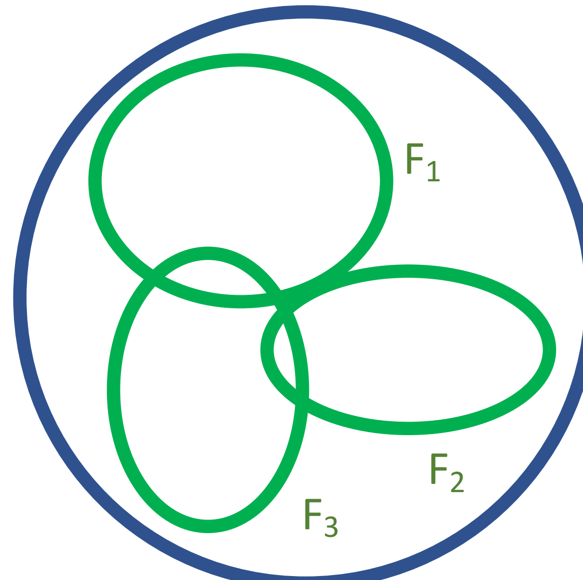
(10) A_1, A_2, A_3 are sets with different members.



Plural readings and ambiguity

(11) Every a_n is thus a product of *different* small primes […]





F_n: (aggregate of) factors of the product a_n

Plural readings and ambiguity

(11) Every a_n is thus a product of *different* small primes [...]

- The aggregates (sequences) s_n of the factors of a_n are pairwise distinct:

  $\forall n$ (*different'*$(\{s_n | \prod s_n = a_n\}))$

Plural readings and ambiguity

Let us now look at N_s. We write every $n \leq N$ which has only small prime divisors in the form $n = a_n b_n^2$, where a_n is the square-free part. Every a_n is thus a product of *different* small primes, and we conclude that there are precisely 2^k different square-free parts. Furthermore, as $b_n \leq \sqrt{n} \leq \sqrt{N}$, we find that there are at most $\sqrt{N}$ different square parts, and so

$$N_s \;\leq\; 2^k \sqrt{N}.$$

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  different'$(\{s_n \mid \prod s_n = a_n\})$

- The members of the sequence of factors of a_n are pairwise different for each n.

  $\forall n(\textit{different'}(s_n))$

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  $\forall n($*different'*$(s_n))$

- From symmetric relations R properties R' can be derived and applied to plural entities. R' means that the members of these entities are pairwise R.
- xRy ∧ R is symmetric ⇒ R' = λP(∀x,y ∈ P ∧ x≢y → xRy)

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  for non-reflexive R

- xRy ∧ R is symmetric ⇒ R' = λP(∀x,y ∈ P ∧ x≢y → xRy)

Plural readings and ambiguity

(7) In each group there were different people.

(8) Each group consisted of different people.

- The plural expression *people* introduces a plural entity P.
- For each group g_n a plural entity P_n is implicitly introduced.
- Therefore the plural entity $\mathcal{P}$ of all P_ns becomes semantically available.
- *different'* can be applied to every P_n or to $\mathcal{P}$.
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- $\boldsymbol{\lambda}P(\forall x,y \in P \wedge x \not\equiv y \rightarrow xRy)$: $x \not\equiv y$ could mean: "elements at different positions in the sequence"

Presuppositions

(Cramer, Schröder, Kühlwein, 2010)

- Types, definitions ranges of functions are usually presupposed.
- Ambiguity between a local and a global presupposition.

(12) Assume $\frac{1}{x}$.

- x ≠ 0:
  - Part of the assumption (local presupposition).
  - Proven, stated before (global presupposition).

Explicature

Most sentences in NL do not allow a direct truth conditional evaluation.

(13) It is raining.

Where? When?

(14) Max: How was the party? Did it go well?

(15) Amy: There wasn't enough drink and everyone left early.
(Carston/Hall, 2012)

Enough? What kind of drink? Everyone of which group? What did they leave?

Explicature

(16) By Lagrange's theorem [...] we know that the order of every element divides the size of the group, that is, we have $p \mid q - 1$, and hence $p < q$.

Which element? Which group?

Implicature

- What is meant, but is not content of the compositional truth-conditional meaning and of explicature.

(17) She injured her leg and she fell to the floor.

(18) She fell to the floor, and she injured her leg.

Implicated: Succession in time, causation

- Implicature in comments:

(19) First, the second statement is indeed more precise than the first:
  ...

Implicature

Strengthening:

p if q	$\rightsquigarrow$	p iff q
a is P (exhaustivation)	$\rightsquigarrow$	a and only a is P

Relevance implicature:

What is said is relevant for understanding the proof (choice of frames, slots, non-trivial step, relevant for subsequent steps).
(applicable in premise selection)

Thoughts on a CNL

- A CNL avoiding ambiguous NL constructions would be a too reduced fragment to represent proof in an efficient manner.
- CNL as an unambiguous language fragment: simple rules for canonical readings are needed:
  - Quantifier scope: sequence + depth first
  - Plurals: strictly type dependent on predicates
  - Bracketing constructions in NL (e.g. *thus* releasing an assumption)
  - Longest/shortest match of embedded constructions
  - …

  Increasing deviation from naturalness.
- Alternative approach: more ambitious disambiguation heuristics, visualisation of disambiguation

Formal notation

- Constants ≈ proper names
- Variables ≈ ?
- Constants can be used as variables:
  - We introduce a binary operation + …
- Complex notation

Variables

- Variables (compared to proper names)
  - Act of naming contained in the text
  - Presupposition of ***relative*** **uniqueness**
  - **Locality**

- Proper names have a presupposition of existence and uniqueness.

Absolute uniqueness

(20) On three successive days Ms Smith taught a different class. Each day she asked Greta the first question.

- *the respective Greta* or *the Greta in the respective class* or *the girl called ‚Greta‘*

Relative uniqueness of variables:

(21) On three successive days Ms Smith taught a different class. Each day she asked the pupil A the first question.

Variables as nouns?

- Use with and without determiner: *an x, the x, x.*
- Type restriction comparable to the property expressed by noun? ambiguous, domain-dependent
- Usually several variables for each type: *i, j, k, …* **(referentially differentiated)**
- Coreferential substitutions (by pronouns, NP) unusual.

Complex notation

- A unique feature of the formal sciences.
- Compositionality combined with
  - Referential relativity
  - Locality
  - Referential differentiation

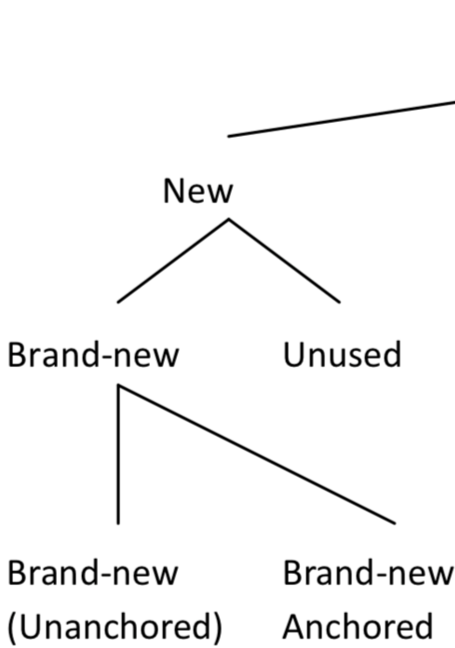
$$\mathbb{Z} \setminus \bigcup_{i=1}^{b-1} N_{a+i,b},$$

Givenness and activation

- Limited working memory, attention
- Referring expressions (indef. NP, def. NP, pronouns, zero reference)

- Givenness (Prince, 1981)
- Centering (Grosz et al., 1995, Walker et al. 1998)
- Landscape model of reading (van den Broek et al., 1996, van den Broek, 1999), „activation“

Prince, 1981


Assumed Familiarity
New
Inferrable
Evoked
Brand-new
Unused
(Noncontaining) Inferrable
Containing Inferrable
(Textually) Evoked
Situationally Evoked
Brand-new (Unanchored)
Brand-new Anchored

Centering

a. Harry suppressed a snort with difficulty.

b. The Dursleys really were astonishingly stupid about their son, Dudley.

c. They had swallowed all his dim-witted lies about having tea with a different member of his gang every night of the summer holidays.

d. Harry knew perfectly well that Dudley had not been to tea anywhere;

e. he and his gang spent every evening vandalising the play park, […]

Example taken from Gundel/Hedberg: Reference: Interdisciplinary Perspective. 2008

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d. Harry knew perfectly well that Dudley had not been to tea anywhere;

e. he and his gang spent every evening vandalising the play park, [...]

Cf: Harry, snort
Cp: Harry
Cb: Ø

Cf: forward-looking center list
Cb: backward-looking center
Cp: preferred center

Centering

a. Harry suppressed a snort with difficulty.

b. The Dursleys really were astonishingly stupid about their son, Dudley.

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Cf: Dursleys, Dudley, lies, tea, member, gang, night, holiday
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Cb: Dursleys

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Cf: Harry, Dudley, tea
Cp: Harry
Cb: Dudley

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Centering

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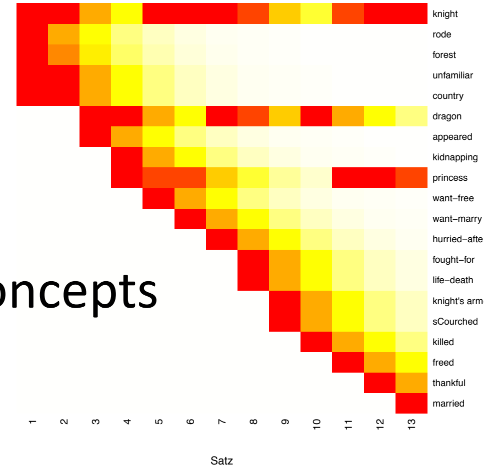
Cf: Dudley, gang, evening, park
Cp: Dudley
Cb: Dudley

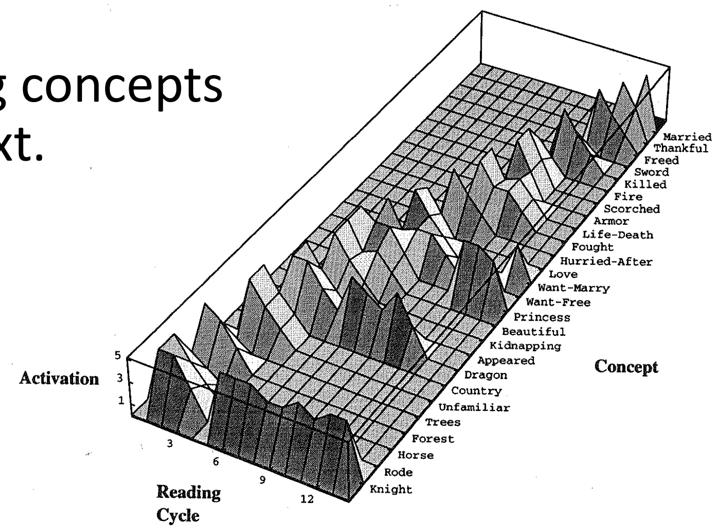
Cf: forward-looking center list
Cb: backward-looking center
Cp: preferred center

Landscape model of reading

- Mentions of, references to concepts activate these concepts
- Activation fades out after mention.



- Strong correlation between memorizing concepts and overall activation in a text.
- Strong correlation between co-memorizing concepts and similarity of activation patterns in a text.



The knight story

A young knight rode through the forest.
The knight was unfamiliar with the country.
Suddenly, a dragon appeared.
The dragon was kidnapping a beautiful princess.
The knight wanted to free her.
The knight wanted to marry her.
The knight hurried after the dragon.
They fought for life and death.
Soon, the knight's armor was completely scorched.
At last, the knight killed the dragon.
He freed the princess.
The princess was very thankful to the knight.
She married the knight.

(van den Broek et al., 1996, 170)

Activation of concepts

- 5: Explicit mention
- 4: pronominal anaphor, needed for coherence
- 2: inferred from the context

- Activation halves in subsequent sentences without a renewal of the concept.

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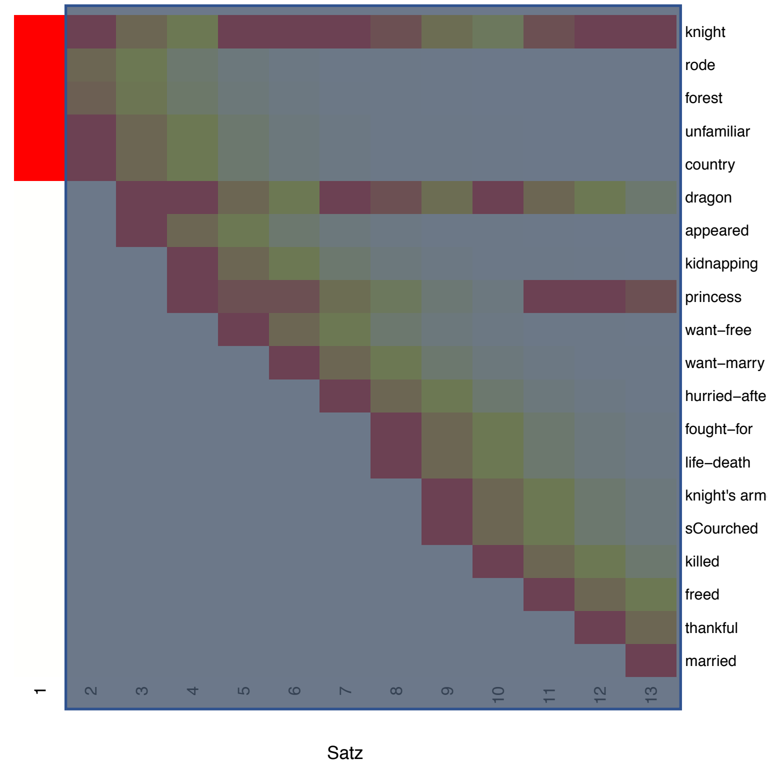

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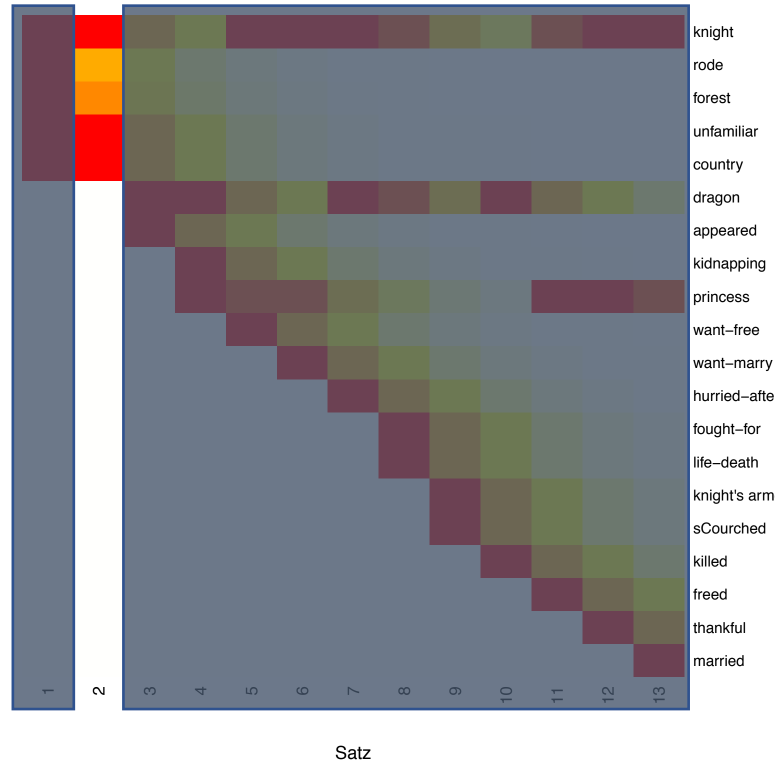

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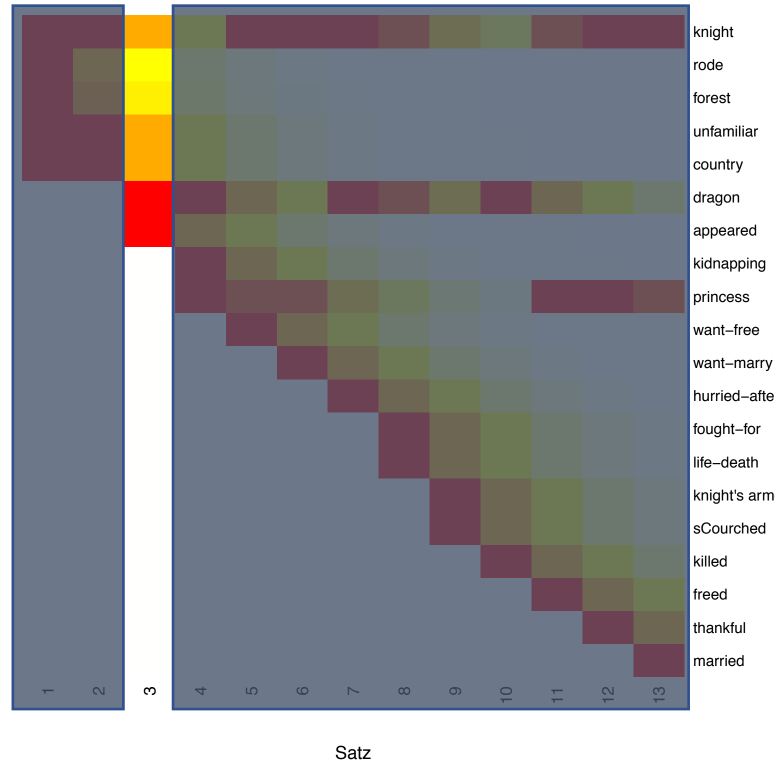

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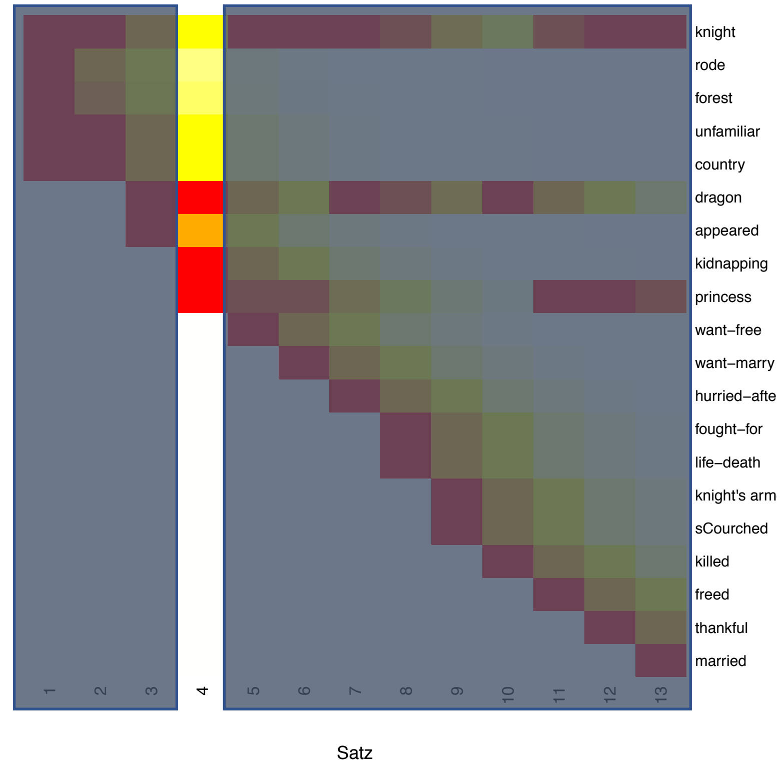
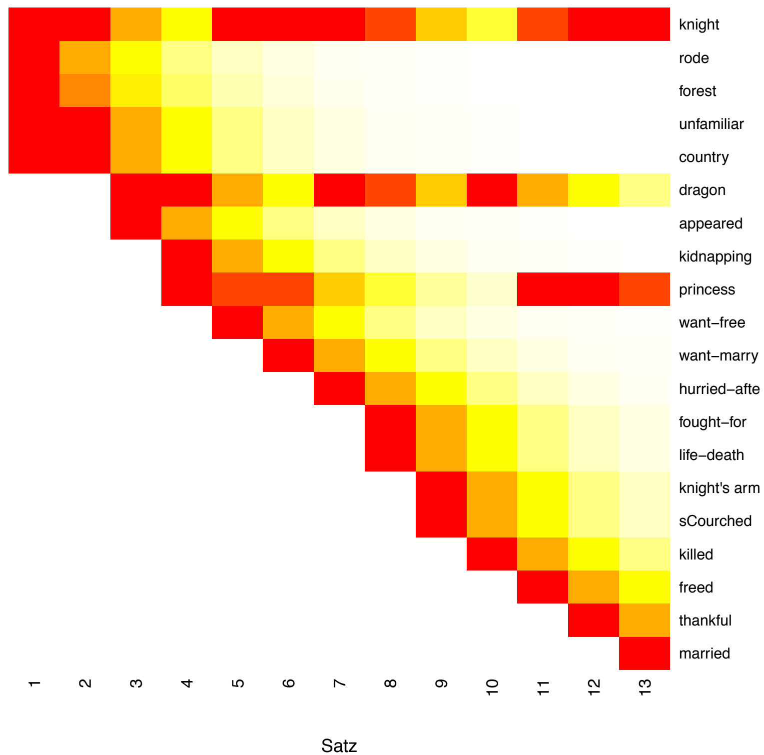


The knight story

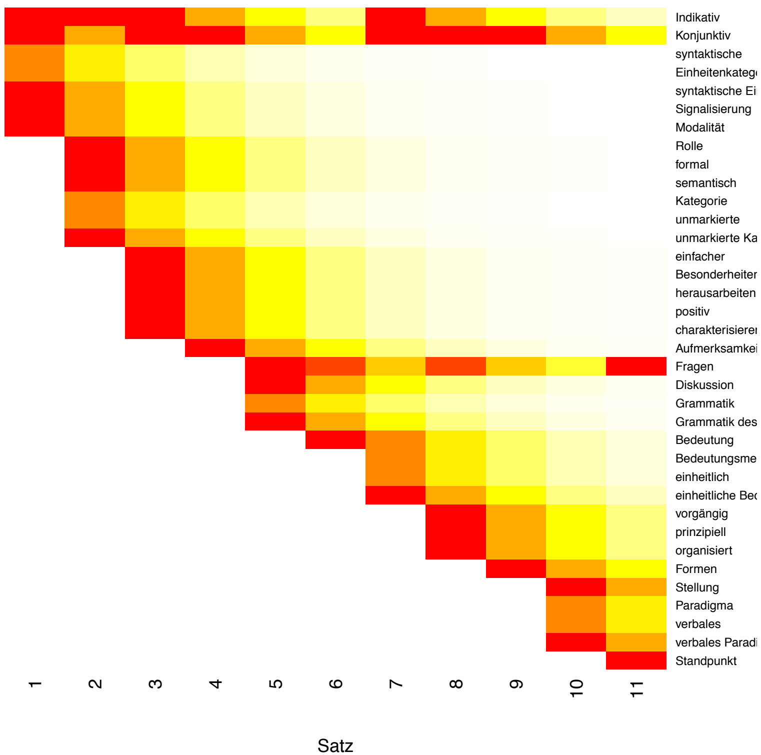


Activation of concepts, extension

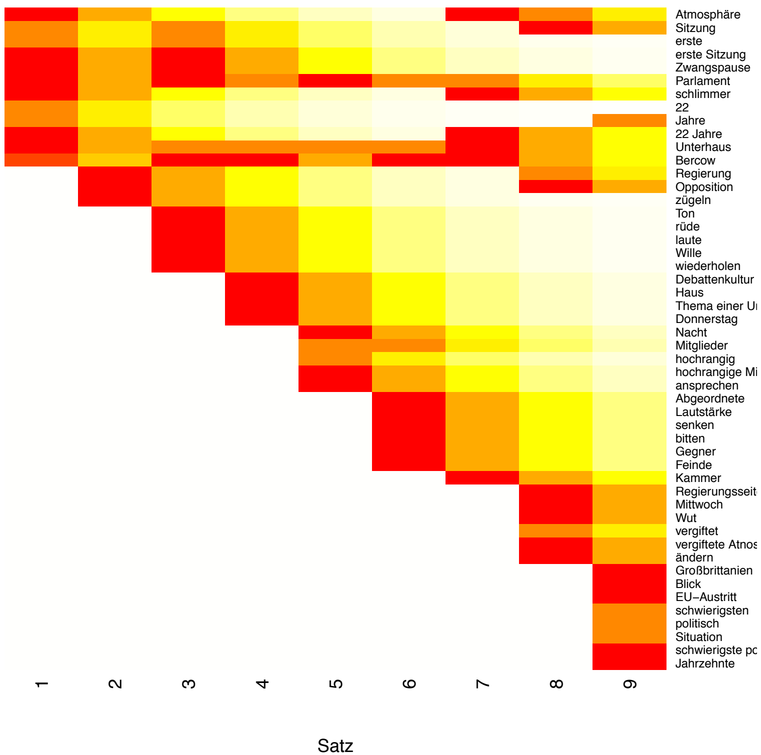
- 5: Explicit mention; ***objects referenced by compound nouns and CN***
- 4: pronominal anaphor, needed for coherence
- ***3: objects referenced by constituents of compound nouns and CN***
- 2: inferred from the context

- Activation halves in subsequent sentences without a renewal of the concept.

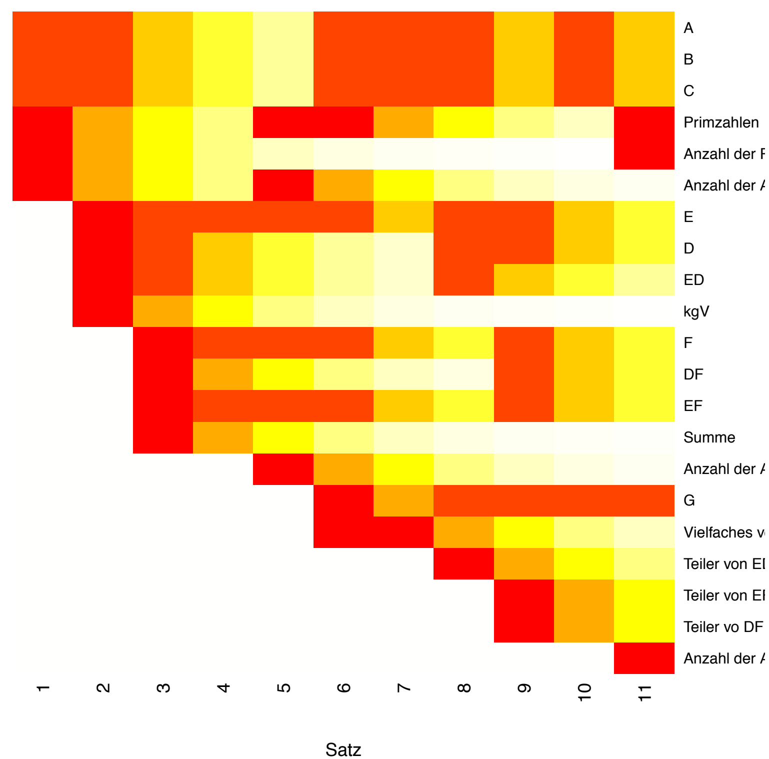
A linguistic text (Eisenberg: Der Satz)



A newspaper article
(British parliament debate about Brexit in the FAZ)



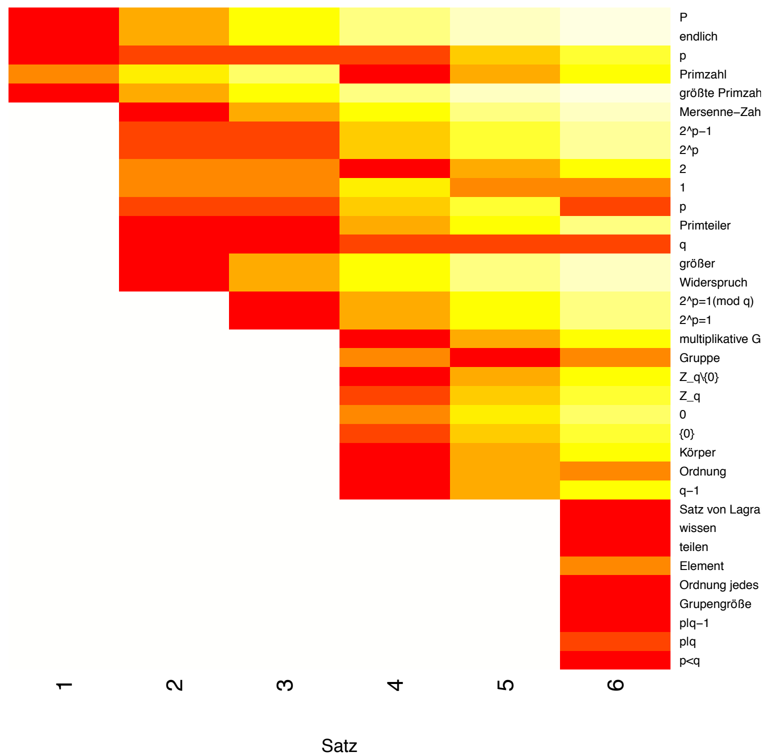
Euclid’s proof of the infinity of prime numbers



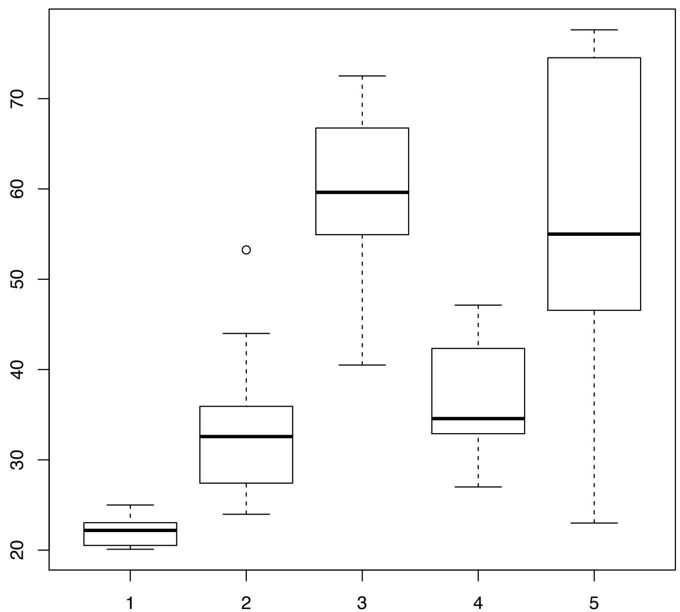
From (Aigner/Ziegler 2010, 4)

■ **Third Proof.** Suppose $\mathbb{P}$ is finite and p is the largest prime. We consider the so-called *Mersenne number* $2^p - 1$ and show that any prime factor q of $2^p - 1$ is bigger than p, which will yield the desired conclusion. Let q be a prime dividing $2^p - 1$, so we have $2^p \equiv 1 \pmod{q}$. Since p is prime, this means that the element 2 has order p in the multiplicative group $\mathbb{Z}_q \backslash \{0\}$ of the field $\mathbb{Z}_q$. This group has $q - 1$ elements. By Lagrange's theorem (see the box) we know that the order of every element divides the size of the group, that is, we have $p \mid q - 1$, and hence $p < q$. □

Aigner/Ziegler 2010, 4th proof of the infinity of prime numbers

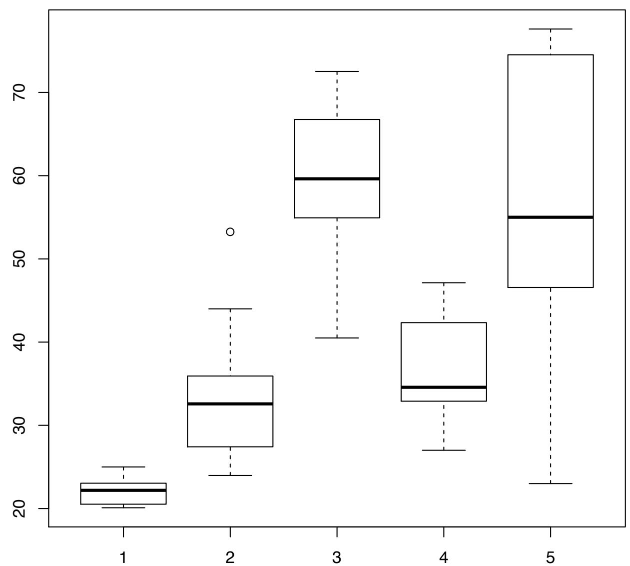


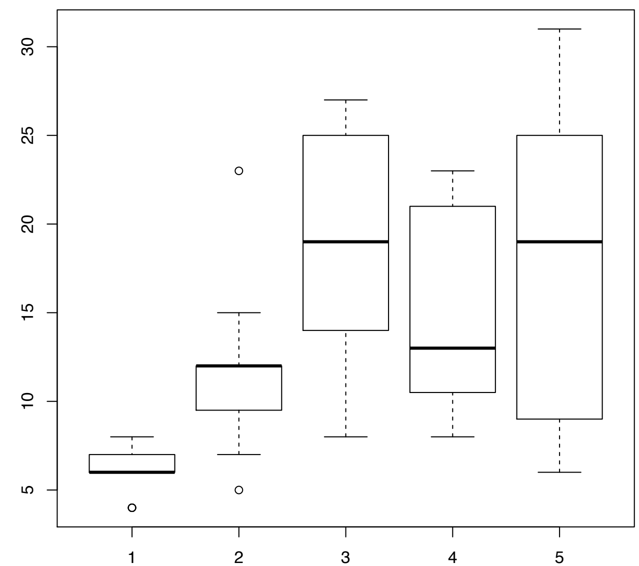
Activation sums per sentence



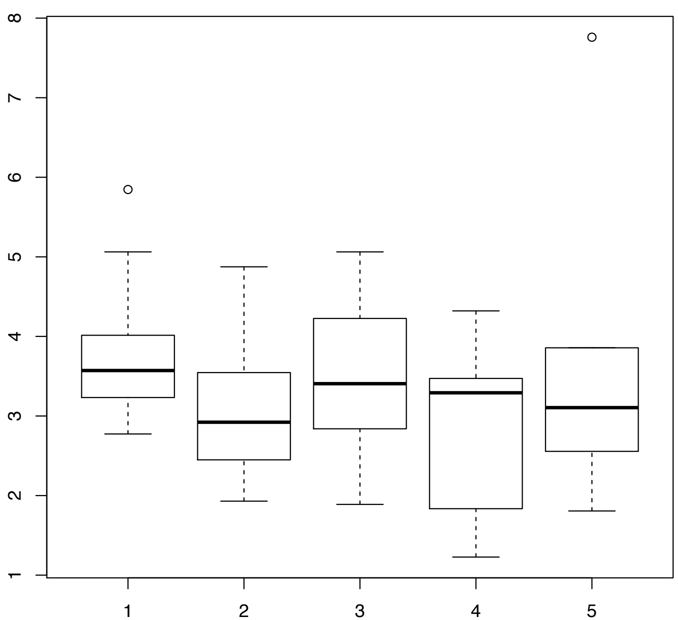
1 Knight story
2 Linguistics
3 Newspaper
4 Euclid
5 Aigner/Ziegler

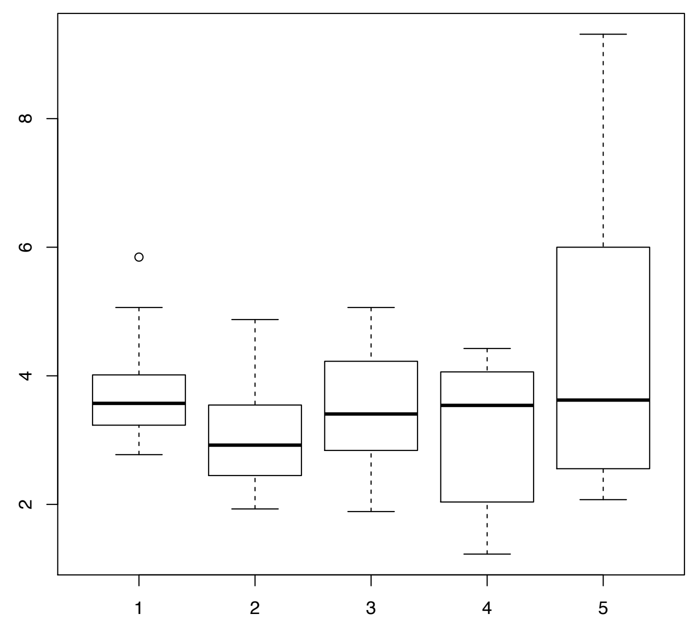
Activation sums per sentence/sentence lengths



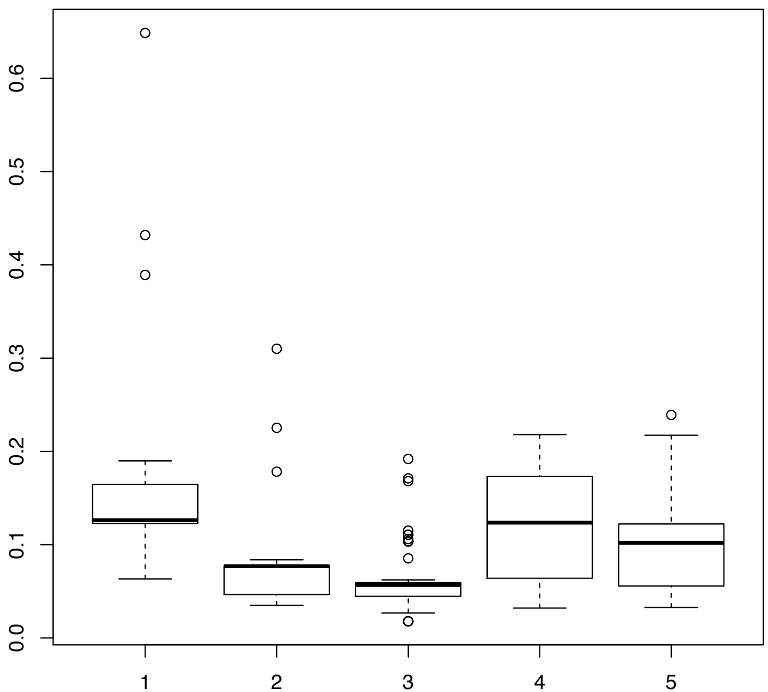


Activation sums relative to sentence lengths
Complex notations counted as several words/one word





Activation sums of concepts (relative to text length) (distribution of the line sums of the heatmaps / length)



1 Knight story
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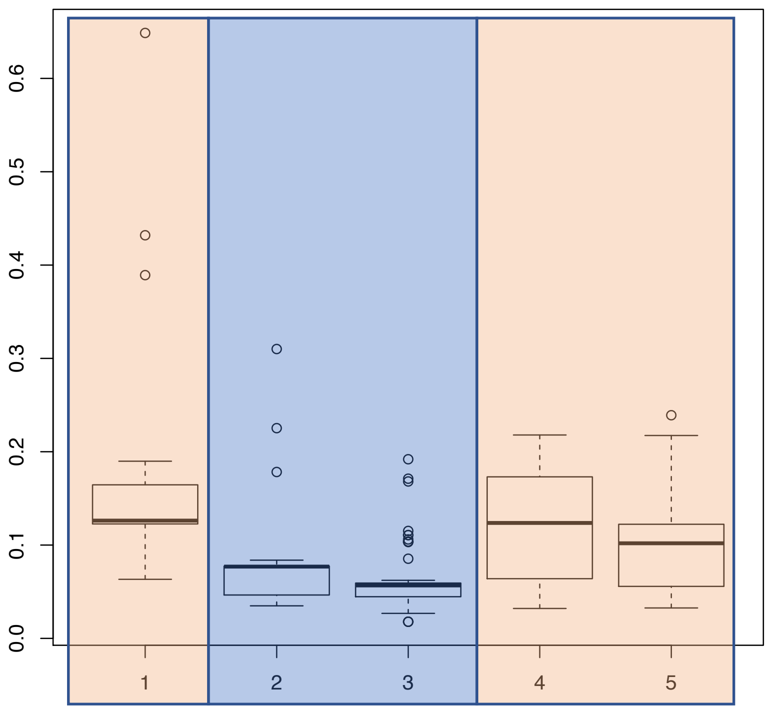

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Referential features of mathematical texts

- Activation relative to sentence length is comparable to other genres.
- Formal notation causes greater density (relative to character tokens).

- Activation focuses on less concepts than in most other genres, comparable to stories in small worlds.

- Proofs make use of a manageable number of objects (discourse referents). Similar activation pattern hint to arrangement in frames.

„mathematics describes a small world situation of facts about mathematical object in a time-less self-contained environment“
(Peter Koepke: An Brief Tutorial on Mathematical Formalizations in Naproche-SAD, 2019)

Frames in the Language of Mathematics

Joint work with

- Marcos Cramer
- Bernhard Fisseni
- Deniz Sarikaya
- Martin Schmitt

Frame (Script, Schema)

A *frame* is a data-structure for representing a stereotyped situation, like being in a certain kind of living room, or going to a child's birthday party. Attached to each frame are several kinds of information. Some of this information is about how to use the frame. Some is about what one can expect to happen next. Some is about what to do if these expectations are not confirmed. (Minsky, 1974)

Frame (Script, Schema)

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A frame with standard slot values

Children's birthday party:

- Inviting child
- Invited children [n ≈ age]
- Birthday cake
- Decoration [balloons]
- Events [Partial ordering: Welcome, Games, …, Goodbye]
- Period [Start, End, Duration [approx. 3h]]

Frame slots often have standard values.

Frame-based Disambiguation

(14) I remember, how we went to Dirk’s birthday last year.

(15) It was a really cold winter day, and the tram was overcrowded.

(16) They had a lot of balloons.

(17) And my son told me later that the kids romped around in the house, the balloons burst, and the table with the cake tipped over.

Frame-based Disambiguation

(14) I remember, how we went to Dirk's birthday last year.

- Birthday frame activated

(15) It was a really cold winter day, and the tram was overcrowded.

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Frame-based Disambiguation

(14) I remember, how we went to Dirk's birthday last year.

- Birthday frame activated

(15) It was a really cold winter day, and the tram was overcrowded.

- Other frames activated

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- Fits best into the birthday frame, *they* is not related to people on the tram

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(17) And my son told me later that the kids romped around in the house, the balloons burst, and the table with the cake tipped over.
- Cake = Birthday cake

Frame-based disambiguation

- I remember, how we went to Dirk's birthday last year.
  - Birthday frame activated
- It was a really cold winter day, and the tram was overcrowded.
  - Other frames activated.
- They had a lot of balloons.
  - Fits best into the birthday frame, they is not related to people on the tram.
- And my son told me later that the kids romped around in the house, the balloons burst, and the table with the cake tipped over.
  - Cake: birthday cake

- Coherence, presuppositions of definite NPs, …

The Buy-Frame

$$\begin{bmatrix} \textit{buy} & \\ \textsc{Buyer!} & [\![\textit{John}]\!] \\ \textsc{Goods!} & [\![\textit{a beautiful medieval book}]\!] \\ \textsc{Time} & [\![\textit{yesterday}]\!] \\ \textsc{Seller} & \textit{person} \\ \textsc{Money} & \textit{money} \\ \textsc{Purpose} & \textit{purpose} \\ \ldots & \end{bmatrix} = \begin{bmatrix} \textit{buy} & \\ \textsc{Buyer!} & j \\ \textsc{Goods!} & b \\ \textsc{Time} & \begin{bmatrix} \textit{point-in-time} & \\ \textsc{year} & \textit{2018} \\ \textsc{month} & \textit{02} \\ \textsc{day} & \textit{28} \\ \textsc{hour} & \{1,\ldots,24\} \\ \textsc{minute} & \{0,\ldots,60\} \\ \ldots & \end{bmatrix} \\ \textsc{Seller} & \textit{person} \\ \textsc{Money} & \textit{money} \\ \textsc{Purpose} & \textit{purpose} \\ \ldots & \end{bmatrix}$$

A textbook proof in linear algebra: the proposition

PROPOSITION 4.4.6. Let V be a finite-dimensional $\mathbf{K}$-vector space and let f be a nilpotent endomorphism of V. Let $n = \dim(V)$. Then $f^n = 0$. More precisely, for any vector $v \neq 0$ in V, and $k \geq 0$ such that $f^k(v) = 0$ but $f^{k-1}(v) \neq 0$,[10] the vectors

$$(v, f(v), \ldots, f^{k-1}(v))$$

are linearly independent.

KOWALSKI, Emmanuel (Sept. 15, 2016). *Linear Algebra*. Lecture Notes, ETH Zurich, published at `https://people.math.ethz.ch/~kowalski/script-la.pdf`.

A textbook proof in linear algebra: the proof

Proof. First, the second statement is indeed more precise than the first: let $k \geq 1$ be such that $f^k = 0$ but $f^{k-1} \neq 0$; there exists $v \neq 0$ such that $f^{k-1}(v) \neq 0$, and we obtain $k \leq n$ by applying the second result to this vector v. We now prove the second claim. Assume therefore that $v \neq 0$ and that $f^k(v) = 0$ but $f^{k-1}(v) \neq 0$. Let $t_0, \ldots, t_{k-1}$ be elements of $\mathbf{K}$ such that

$$t_1 v + \cdots + t_{k-1} f^{k-1[sic!]}(v) = 0.$$

Apply f^{k-1} to this relation; since $f^k(v) = \ldots = f^{2k-2[sic!]}(v) = 0$, we get

$$t_1 f^{k-1}(v) = t_1 f^{k-1}(v) + t_2 f^k(v) + \cdots + t_{k-1} f^{2k-2[sic!]}(v) = 0,$$

and therefore $t_1 f^{k-1}(v) = 0$. Since $f^{k-1}(v)$ was assumed to be non-zero, it follows that $t_1 = 0$. Now repeating this argument, but applying f^{k-2} to the linear relation (and using the fact that $t_1 = 0$), we get $t_2 = 0$.
Then similarly we derive by induction that $t_i = 0$ for all i, proving the linear independence stated.

induction

- INDUCTION-DOMAIN $\boxed{d}$
  - *inductive-type*
  - BASE-CONSTRUCTOR $\boxed{bc}$ *base-constructor*
  - RECURSIVE-CONSTRUCTOR $\boxed{rc}$ *recursive-constructor*
- INDUCTION-VARIABLE
  - *variable*
  - NAME $\boxed{x}$ *symbolic*
  - TYPE $\boxed{d}$
- <u>ASSERTION</u> $\forall \boxed{x} . \boxed{s}$
- <u>PROOF</u>
  - *induction-proof*
  - INDUCTION-SIGNATURE
    - *induction-signature*
    - INDUCTION-HYPOTHESIS $\boxed{ih}$ *sentence*
    - STEP-FUNCTION (?!) $\boxed{rc}$
    - BASE-CONDITION $\boxed{bcc}$ (?!) $\boxed{x} = \boxed{bc}$
    - INDUCTION-CONDITION $\boxed{icc}$ (?!) $\boxed{x} = \boxed{rc}(\ldots)$
  - BASE-CASE
    - *proved-under-hypothesis*
    - HYPOTHESIS $\boxed{bcc}$
    - THESIS $\boxed{s}$
    - ASSERTION $\boxed{bcc} \Rightarrow \boxed{s}$
    - PROOF *list(proof-step* $\vee$ *assumption* $\vee$ *definition* $\vee$ *goal)*
  - INDUCTION-STEP
    - *proved-under-hypothesis*
    - HYPOTHESIS $\boxed{icond} : (\boxed{icc} \wedge \boxed{ih})$
    - THESIS $\boxed{s}$
    - ASSERTION $\boxed{icond} \Rightarrow \boxed{s}$
    - PROOF *list(proof-step* $\vee$ *assumption* $\vee$ *definition* $\vee$ *goal)*

Proof. First, the second statement is indeed more precise than the first: let $k \geq 1$ be such that $f^k = 0$ but $f^{k-1} \neq 0$; there exists $v \neq 0$ such that $f^{k-1}(v) \neq 0$, and we obtain $k \leq n$ by applying the second result to this vector v. We now prove the second claim. Assume therefore that $v \neq 0$ and that $f^k(v) = 0$ but $f^{k-1}(v) \neq 0$. Let $t_0, \ldots, t_{k-1}$ be elements of $\mathbf{K}$ such that

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```
[induction
 INDUCTION-DOMAIN     [d] [inductive-type
                           BASE-CONSTRUCTOR       [bc] base-constructor
                           RECURSIVE-CONSTRUCTOR  [rc] recursive-constructor]
 INDUCTION-VARIABLE   [variable
                       NAME  [x] symbolic
                       TYPE  [d]]
 ASSERTION            ∀[x] . [s]
 PROOF                [induction-proof
                       INDUCTION-SIGNATURE  [induction-signature
                                             INDUCTION-HYPOTHESIS  [ih] sentence
                                             STEP-FUNCTION         (?!) [rc]
                                             BASE-CONDITION        [bcc] (?!) [x] = [bc]
                                             INDUCTION-CONDITION   [icc] (?!) [x] = [rc](...)]
                       BASE-CASE            [proved-under-hypothesis
                                             HYPOTHESIS  [bcc]
                                             THESIS      [s]
                                             ASSERTION   [bcc] ⇒ [s]
                                             PROOF       list(proof-step ∨ assumption ∨ definition ∨ goal)]
                       INDUCTION-STEP       [proved-under-hypothesis
                                             HYPOTHESIS  [icond] : ([icc] ∧ [ih])
                                             THESIS      [s]
                                             ASSERTION   [icond] ⇒ [s]
                                             PROOF       list(proof-step ∨ assumption ∨ definition ∨ goal)]]]
```

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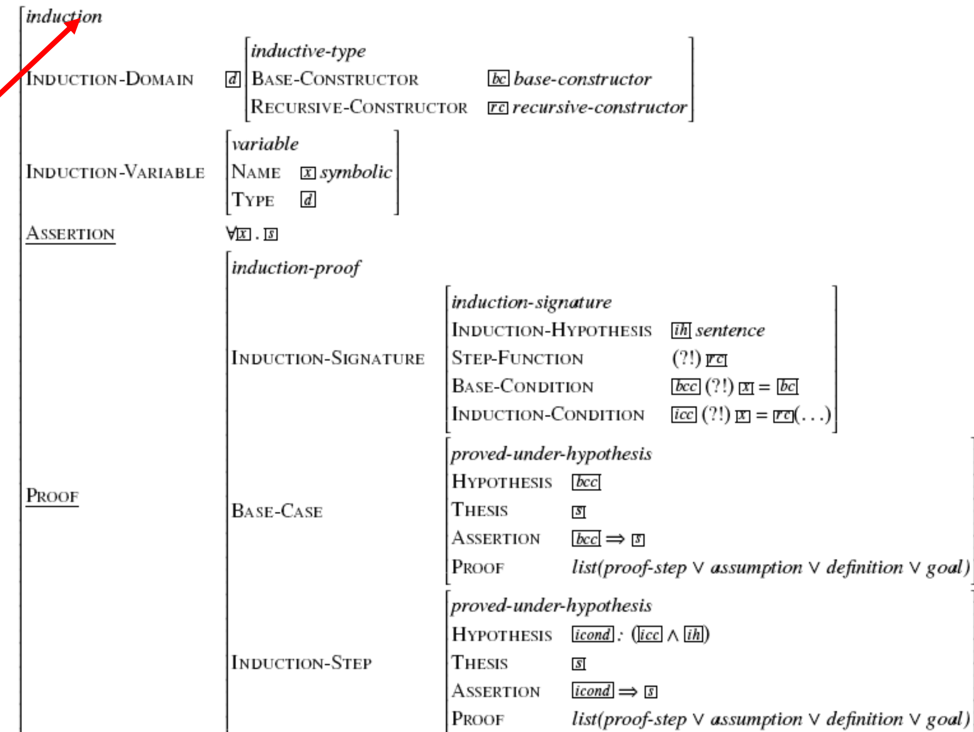

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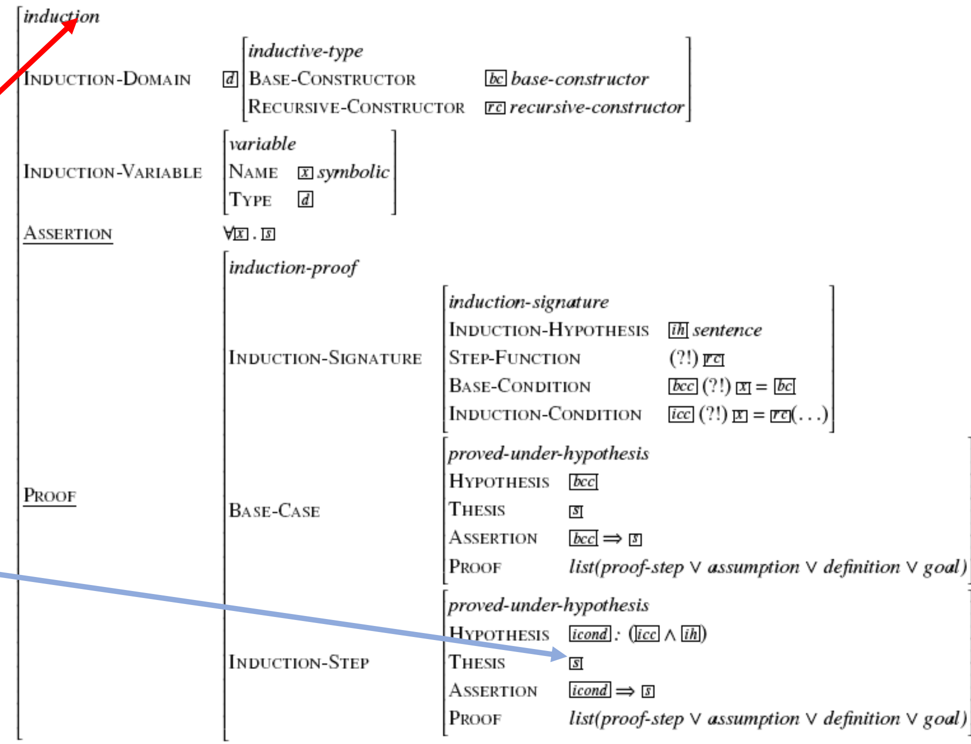
$$t_1 v + \cdots + t_{k-1} f^{k-1[sic!]}(v) = 0.$$

Apply f^{k-1} to this relation; since $f^k(v) = \ldots = f^{2k-2[sic!]}(v) = 0$, we get

$$t_1 f^{k-1}(v) = t_1 f^{k-1}(v) + t_2 f^k(v) + \cdots + t_{k-1} f^{2k-2[sic!]}(v) = 0,$$

and therefore $t_1 f^{k-1}(v) = 0$. Since $f^{k-1}(v)$ was assumed to be non-zero, it follows that $t_1 = 0$. Now repeating this argument, but applying f^{k-2} to the linear relation (and using the fact that $t_1 = 0$), we get $t_2 = 0$.
Then similarly we derive by induction that $t_i = 0$ for all i, proving the linear independence stated.



Proof. First, the second statement is indeed more precise than the first: let $k \geq 1$ be such that $f^k = 0$ but $f^{k-1} \neq 0$; there exists $v \neq 0$ such that $f^{k-1}(v) \neq 0$, and we obtain $k \leq n$ by applying the second result to this vector v. We now prove the second claim. Assume therefore that $v \neq 0$ and that $f^k(v) = 0$ but $f^{k-1}(v) \neq 0$. Let $t_0, \ldots, t_{k-1}$ be elements of $\mathbf{K}$ such that

$$t_1 v + \cdots + t_{k-1} f^{k-1[sic!]}(v) = 0.$$

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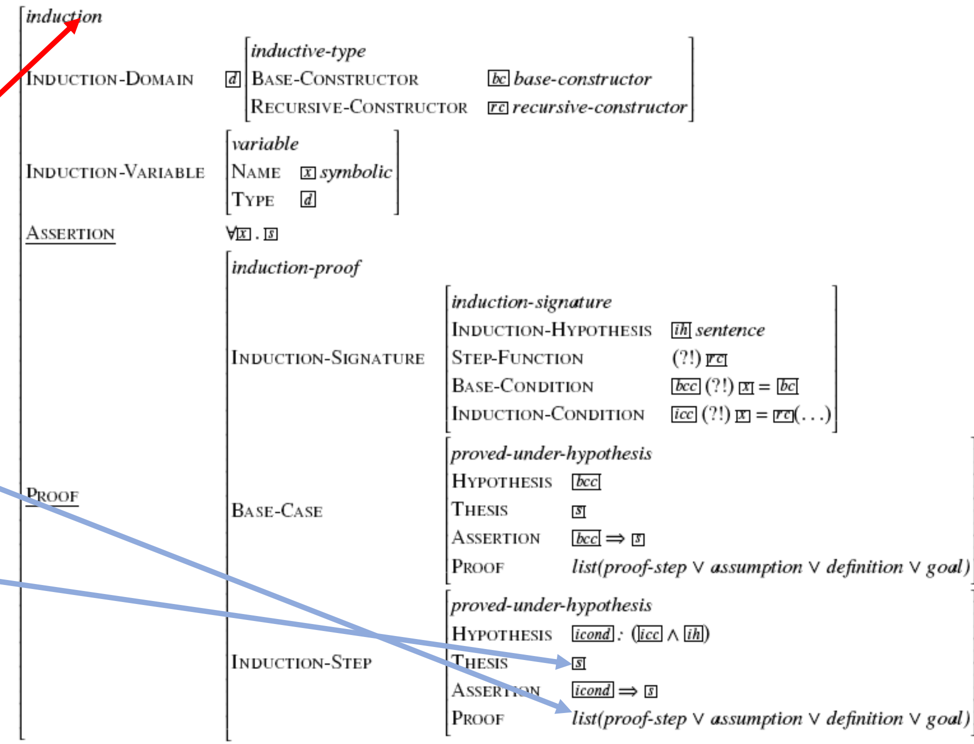


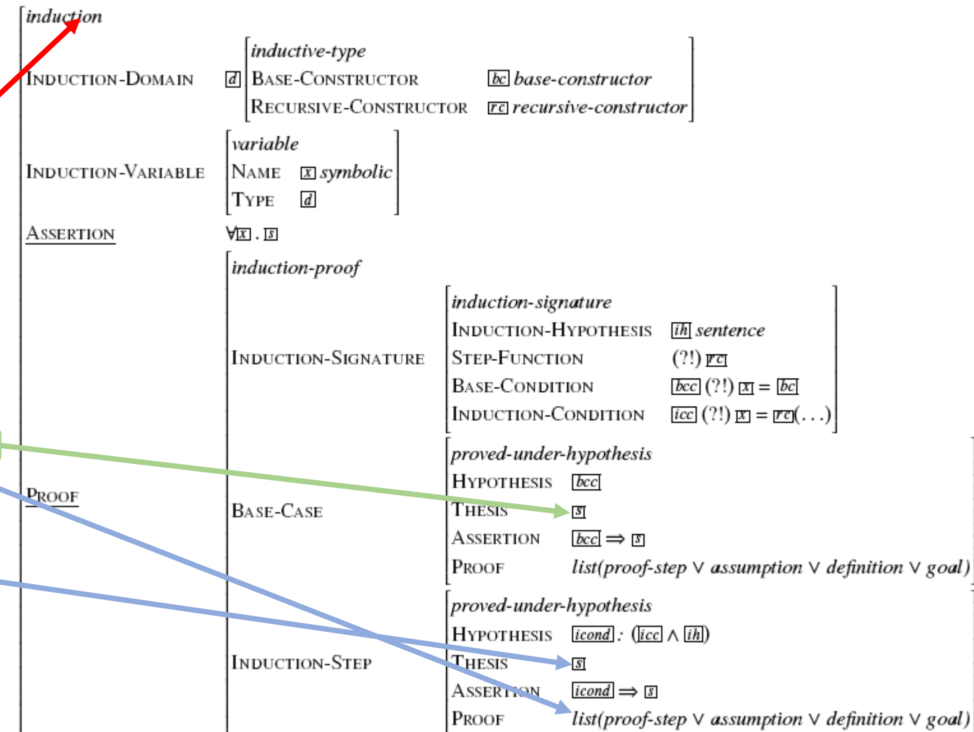
Proof. First, the second statement is indeed more precise than the first: let $k \geq 1$ be such that $f^k = 0$ but $f^{k-1} \neq 0$; there exists $v \neq 0$ such that $f^{k-1}(v) \neq 0$, and we obtain $k \leq n$ by applying the second result to this vector v. We now prove the second claim. Assume therefore that $v \neq 0$ and that $f^k(v) = 0$ but $f^{k-1}(v) \neq 0$. Let $t_0, \ldots, t_{k-1}$ be elements of $\mathbf{K}$ such that

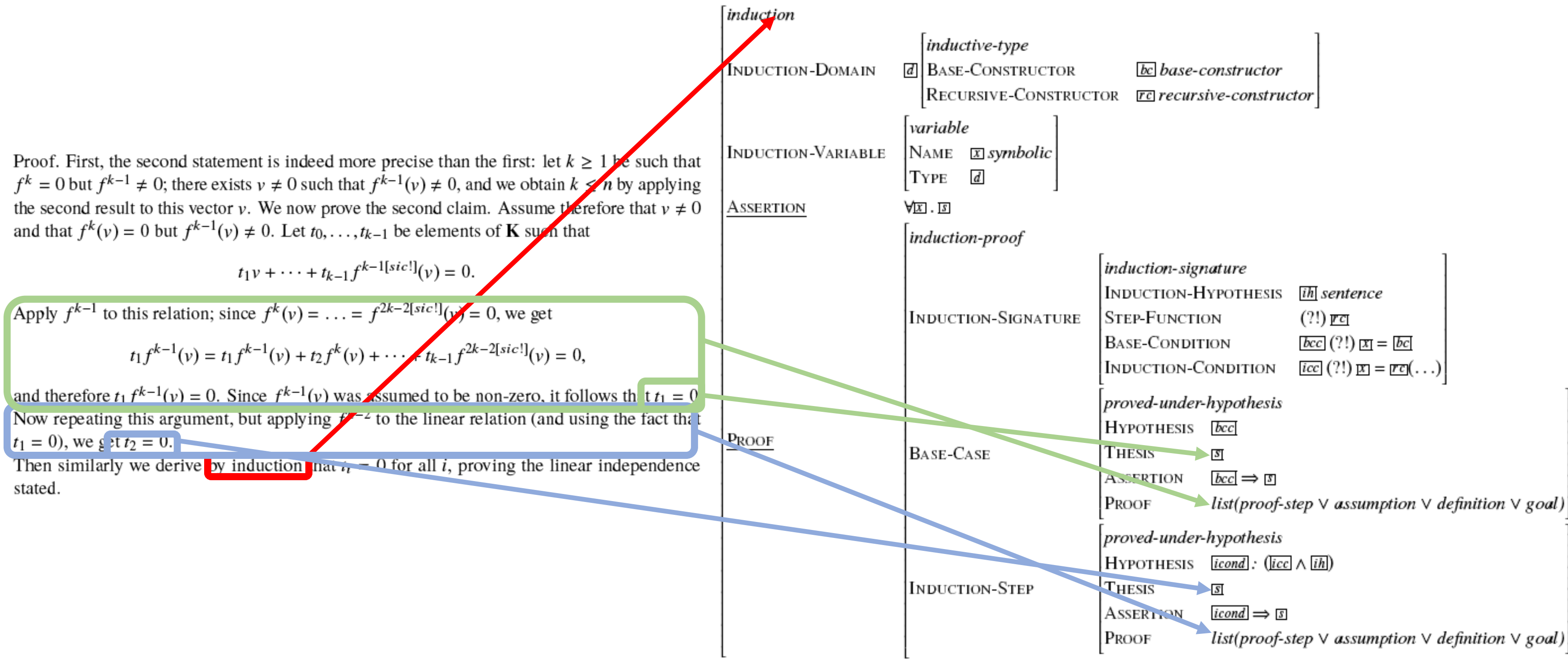
$$t_1 v + \cdots + t_{k-1} f^{k-1[sic!]}(v) = 0.$$

Apply f^{k-1} to this relation; since $f^k(v) = \ldots = f^{2k-2[sic!]}(v) = 0$, we get

$$t_1 f^{k-1}(v) = t_1 f^{k-1}(v) + t_2 f^k(v) + \cdots + t_{k-1} f^{2k-2[sic!]}(v) = 0,$$

and therefore $t_1 f^{k-1}(v) = 0$. Since $f^{k-1}(v)$ was assumed to be non-zero, it follows that $t_1 = 0$. Now repeating this argument, but applying f^{k-2} to the linear relation (and using the fact that $t_1 = 0$), we get $t_2 = 0$. Then similarly we derive by induction that $t_i = 0$ for all i, proving the linear independence stated.




Proof. First, the second statement is indeed more precise than the first: let k ≥ 1 be such that f^k = 0 but f^{k−1} ≠ 0; there exists v ≠ 0 such that f^{k−1}(v) ≠ 0, and we obtain k ≤ n by applying the second result to this vector v. We now prove the second claim. Assume therefore that v ≠ 0 and that f^k(v) = 0 but f^{k−1}(v) ≠ 0. Let t_0, . . . , t_{k−1} be elements of K such that
t_1 v + · · · + t_{k−1} f^{k−1[sic!]}(v) = 0.
Apply f^{k−1} to this relation; since f^k(v) = . . . = f^{2k−2[sic!]}(v) = 0, we get
t_1 f^{k−1}(v) = t_1 f^{k−1}(v) + t_2 f^k(v) + · · · + t_{k−1} f^{2k−2[sic!]}(v) = 0,
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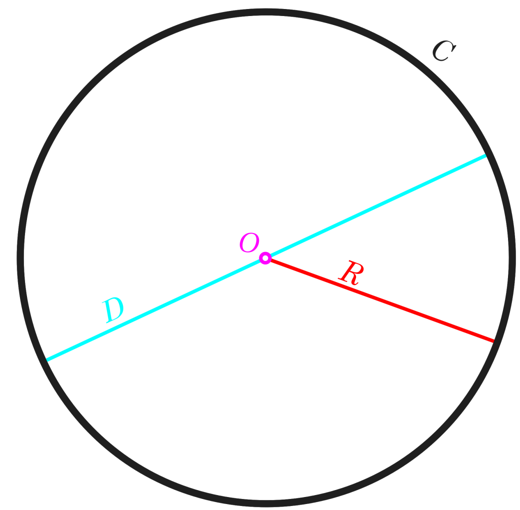
induction
Induction-Domain
d
inductive-type
Base-Constructor
bc base-constructor
Recursive-Constructor
rc recursive-constructor
Induction-Variable
variable
Name
x symbolic
Type
d
Assertion
∀x . s
Proof
induction-proof
Induction-Signature
induction-signature
Induction-Hypothesis
ih sentence
Step-Function
(?!) rc
Base-Condition
bcc (?!) x = bc
Induction-Condition
icc (?!) x = rc(. . .)
Base-Case
proved-under-hypothesis
Hypothesis
bcc
Thesis
s
Assertion
bcc ⟹ s
Proof
list(proof-step ∨ assumption ∨ definition ∨ goal)
Induction-Step
proved-under-hypothesis
Hypothesis
icond : (icc ∧ ih)
Thesis
s
Assertion
icond ⟹ s
Proof
list(proof-step ∨ assumption ∨ definition ∨ goal)

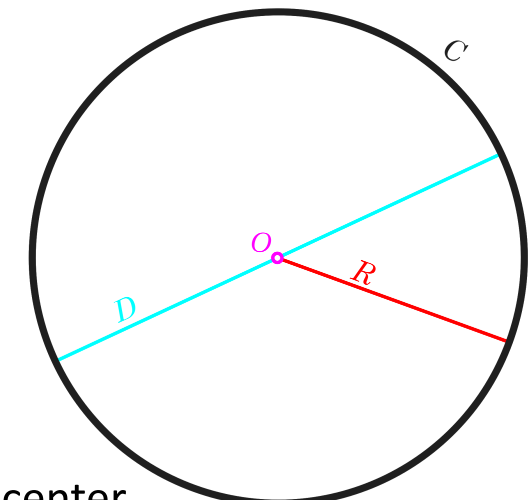
Ontological frames: bridging

- Circle
  - Center
  - Diameter
  - Radius
  - Circumference
  - …



Ontological frames: bridging

- Circle
  - Center
  - Diameter = 2R
  - Radius (R)
  - Circumference = 2πR
  - …



… a circle c … The diameter … the center …

Bridging: Referring expressions get their unique referent by a relation to a previously introduced referent

Ontological frames: bridging

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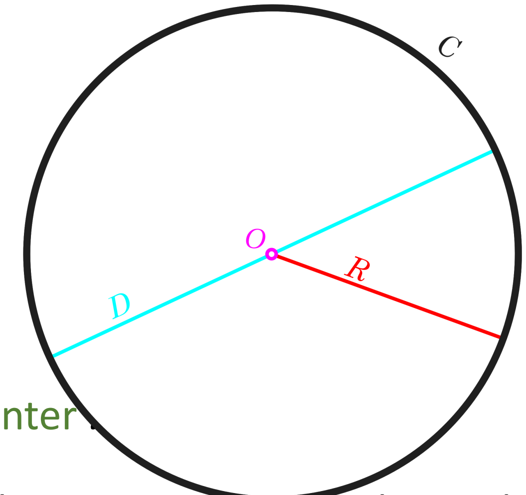

… a circle c … The diameter … the center …

Bridging: Referring expressions get their unique referent by a relation to a previously introduced referent.

Frames also help to get the right explicatures.

Ontological frames

- Make other concepts/referents available (bridging).
- Interact with other frames, e.g. structural frames:
  - Cf. induction on natural numbers / graphs / strings
- Help to get the right explicatures.

Conclusion:
some characteristic features of proof test

- Deeply nested.
- Formal notation
- Small/closed worlds.
- Structurally and ontologically densely structured by frames.
- Very specific dependencies within frames.

- Ambiguity, few vagueness
- Limited use of presupposition
- Explicatures, limited use of implicatures
- Elliptic presentation of arguments