Life in Logic

Valedictory Lecture

Peter Koepke

31 January 2020

- Logic

- Logic
- Set Theory

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- Models of Set Theory

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- Formal Mathematics

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- Mathematical Language

- Logos ≈ Language, Argumentation

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- Logic ≈ about Logos, the science of Logos

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- Me: But grandfather, stones cannot be cooked soft.

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- Me: But grandfather, stones cannot be cooked soft.
- Grandfather: Exactly! So I am right.

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- If A then B.

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- If A then B.
- $A \rightarrow B$

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- Abstract Boolean logic (George Boole, 1815-1864)

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- Truth table:

A	B	$A \to B$
F	F	?
F	T	?
T	F	?
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Dr. Buchholz (right)

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- $\text{size}(\mathbb{N}) = \text{size}(\{0, 1, 2, \ldots\}) = \aleph_0$
- The property $\text{size}(\mathbb{R}) = \aleph_1$ is Cantor's continuum hypothesis.

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- The property "$\text{size}(\mathbb{R}) = \aleph_1$" is *independent* of the usual assumptions of mathematics.
- How can one prove that one cannot prove something?

- Euclid's Proof of the infinity of primes

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- For any finite set $\{p_1, \ldots, p_r\}$ of primes, consider the number $n = p_1 p_2 \cdots p_r + 1$. This n has a prime divisor p. But p is not one of the p_i; otherwise p would be a divisor of n and of the product $p_1 p_2 \cdots p_r$, and thus also of the difference $n - p_1 p_2 \cdots p_r = 1$, which is impossible. So a finite set $\{p_1, \ldots, p_r\}$ cannot be the collection of *all* prime numbers.

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- The proof uses natural argumentation based on properties of division and prime numbers.

-

	×		:		=	18
×		+		+		
	+		×		=	63
+		×		×		
	:		:		=	4
=		=		=		
44		11		20		

a	×	*b*	:	*c*	=	18
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- Lemma: $h = 1$.

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- Gödel's *Completeness Theorem*: Every valid mathematical statement can be generated from the underlying assumptions by using the following proof rules:

A complete proof calculus:

$$\frac{\Gamma \quad \varphi}{\Gamma \cup \{\psi\} \quad \varphi} \qquad \frac{}{\Gamma \quad \varphi}\text{, if } \varphi \in \Gamma$$

$$\frac{\Gamma \cup \{\varphi\} \quad \psi}{\Gamma \quad \varphi \rightarrow \psi} \qquad \frac{\begin{array}{l}\Gamma \quad \varphi \\ \Gamma \quad \varphi \rightarrow \psi\end{array}}{\Gamma \quad \psi} \qquad \frac{\begin{array}{l}\Gamma \quad \varphi \\ \Gamma \quad \neg\varphi\end{array}}{\Gamma \quad \bot} \qquad \frac{\Gamma \cup \{\neg\varphi\} \quad \bot}{\Gamma \quad \varphi}$$

$$\frac{\Gamma \quad \varphi\frac{y}{x}}{\Gamma \quad \forall x \varphi}\text{, if } y \notin \text{free}(\Gamma \cup \{\forall x \varphi\}); \qquad \frac{\Gamma \quad \forall x \varphi}{\Gamma \quad \varphi\frac{t}{x}}$$

$\Gamma \varphi$ means that the formula φ holds under the assumptions in the set of formulas Γ.

Consequences of the completeness theorem:

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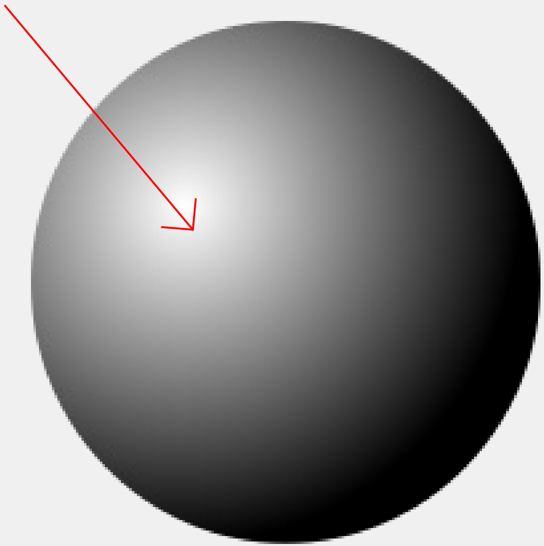
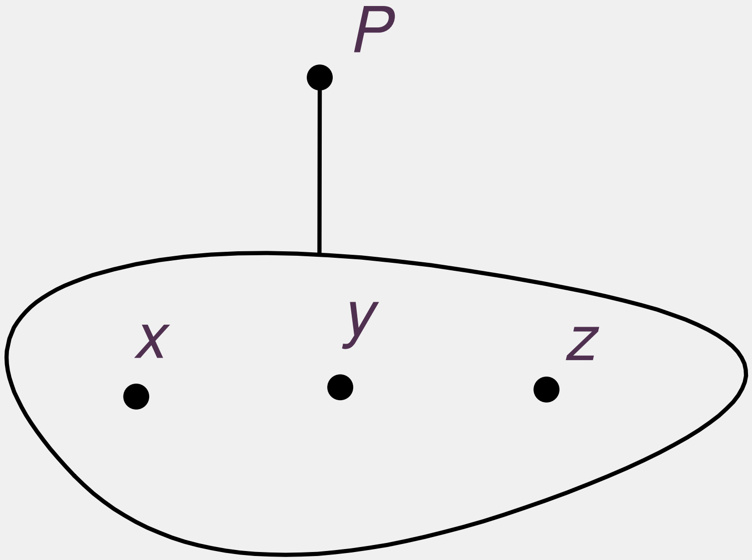
- It is *in principle* possible to generate all valid statements by enumerating all such calculations.
- This mechanical task can be carried out by computer.
- Automatic Theorem Proving (ATP) is in principle possible.

What can one take as general mathematical assumptions?

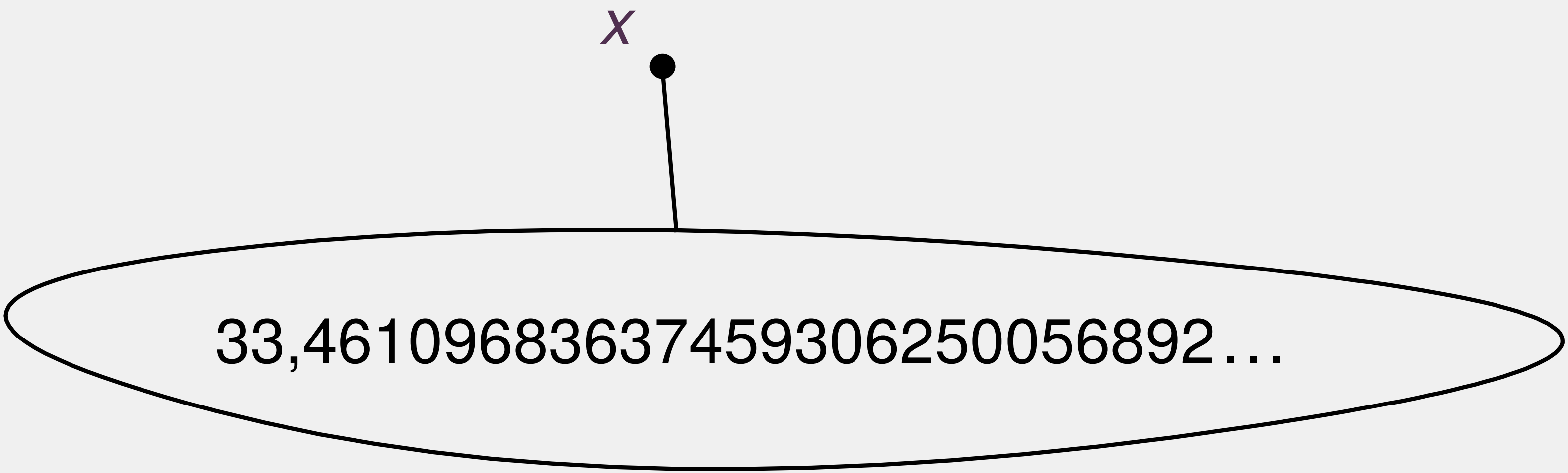
A sphere is built from 3D-points:



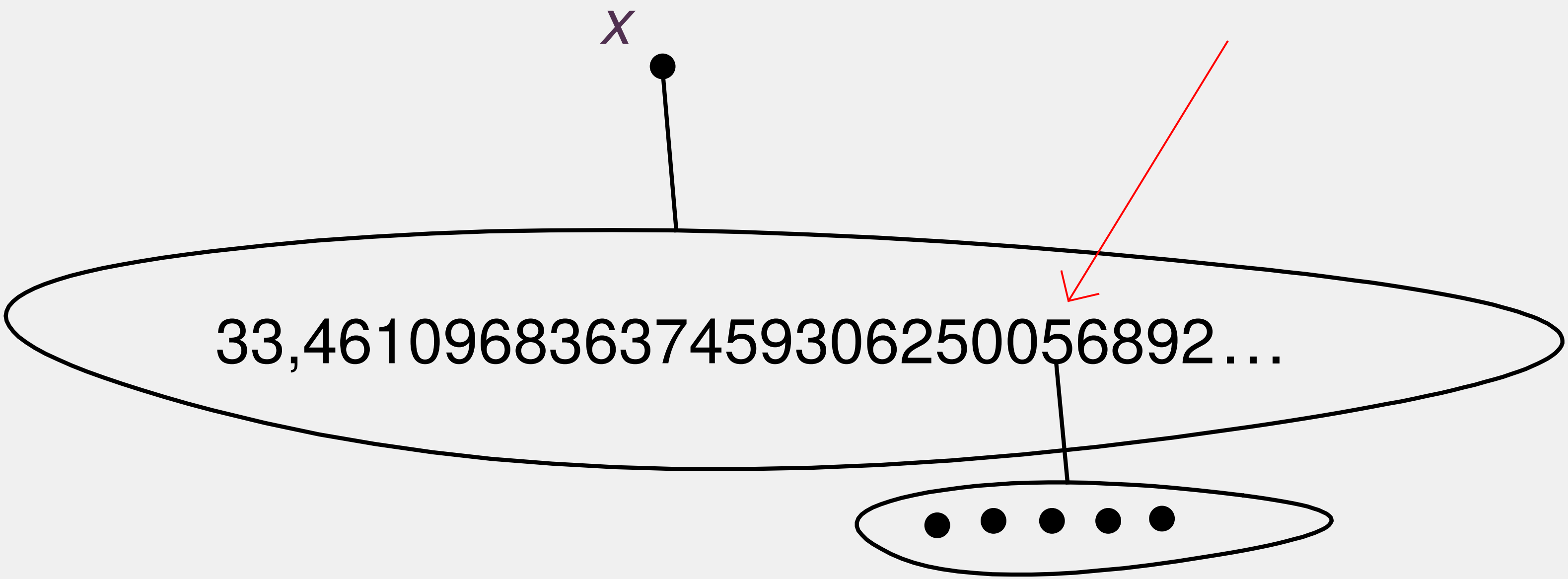
A 3D-point P is built from 3 real numbers x, y, z



A real number x is built from infinitely many decimals



The decimal / number 5 is built from five objects



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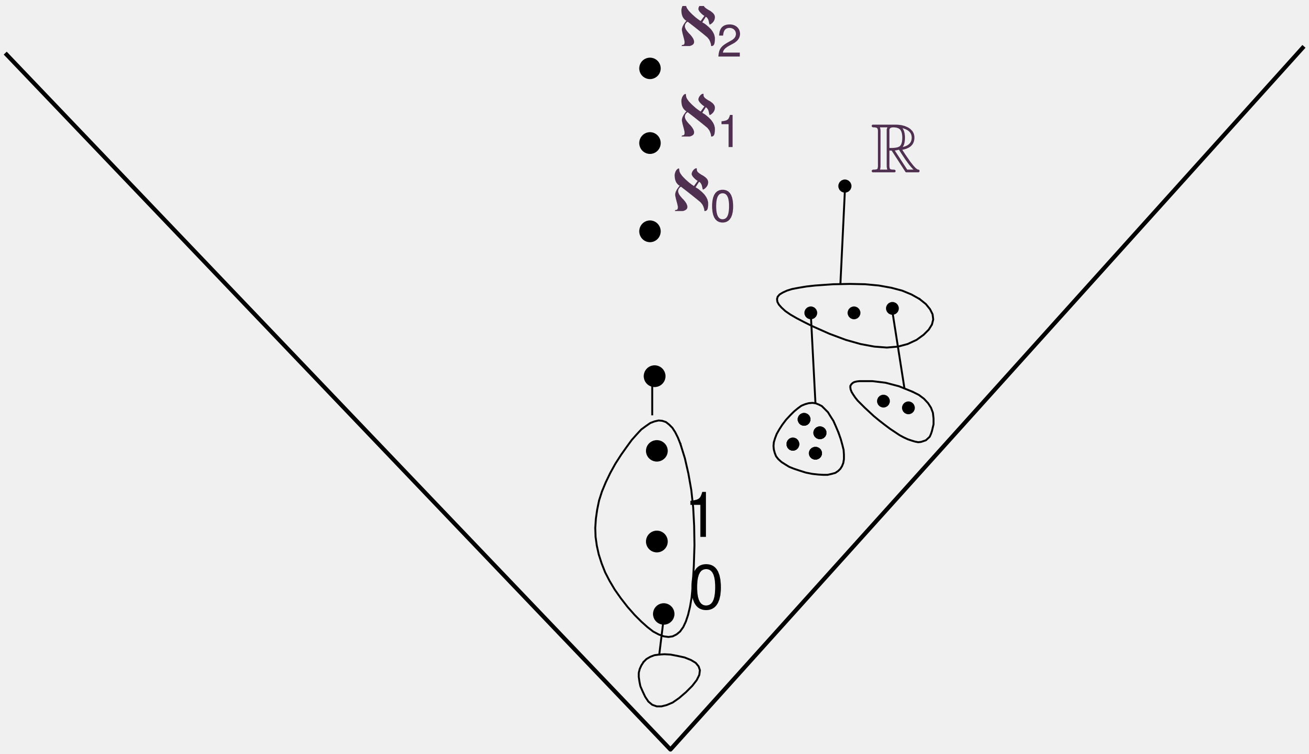
The universe of sets:



The Zermelo-Fraenkel axioms of set theory:

$$x \subseteq y \wedge y \subseteq x \rightarrow x = y$$

$$\{x, y\} \in V$$

$$\bigcup x \in V$$

$$\mathscr{P}(x) \in V$$

$$x \cap A \in V$$

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$x \subseteq y \wedge y \subseteq x \rightarrow x = y \qquad (\forall x, y(\forall u(u \in x \leftrightarrow u \in y) \rightarrow x = y))$

$\{x, y\} \in V \qquad (\forall x, y \exists z \forall u(u \in z \leftrightarrow u = x \vee u = y))$

$\bigcup x \in V \qquad \ldots$

$\mathscr{P}(x) \in V$

$x \cap A \in V$

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A formalistic view of Mathematics:

Valid mathematical statements are exactly those that can be generated by the following (15) proof rules:

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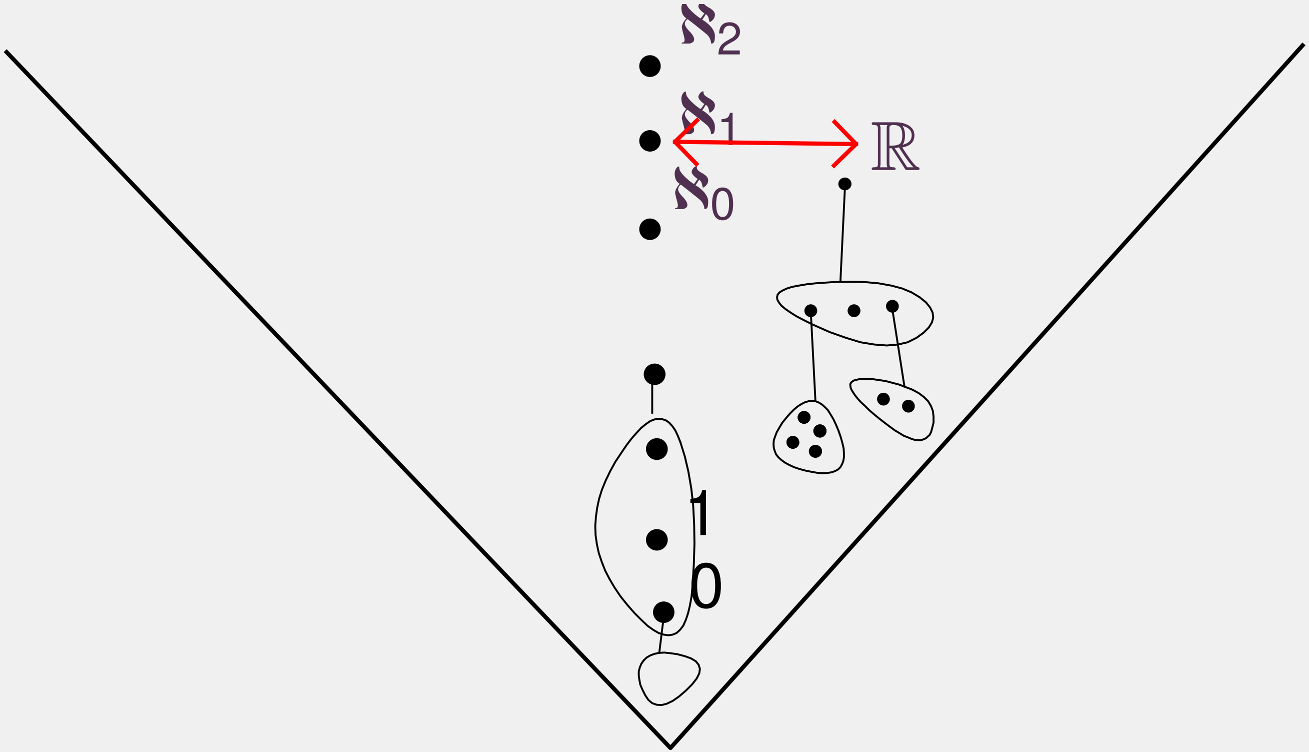
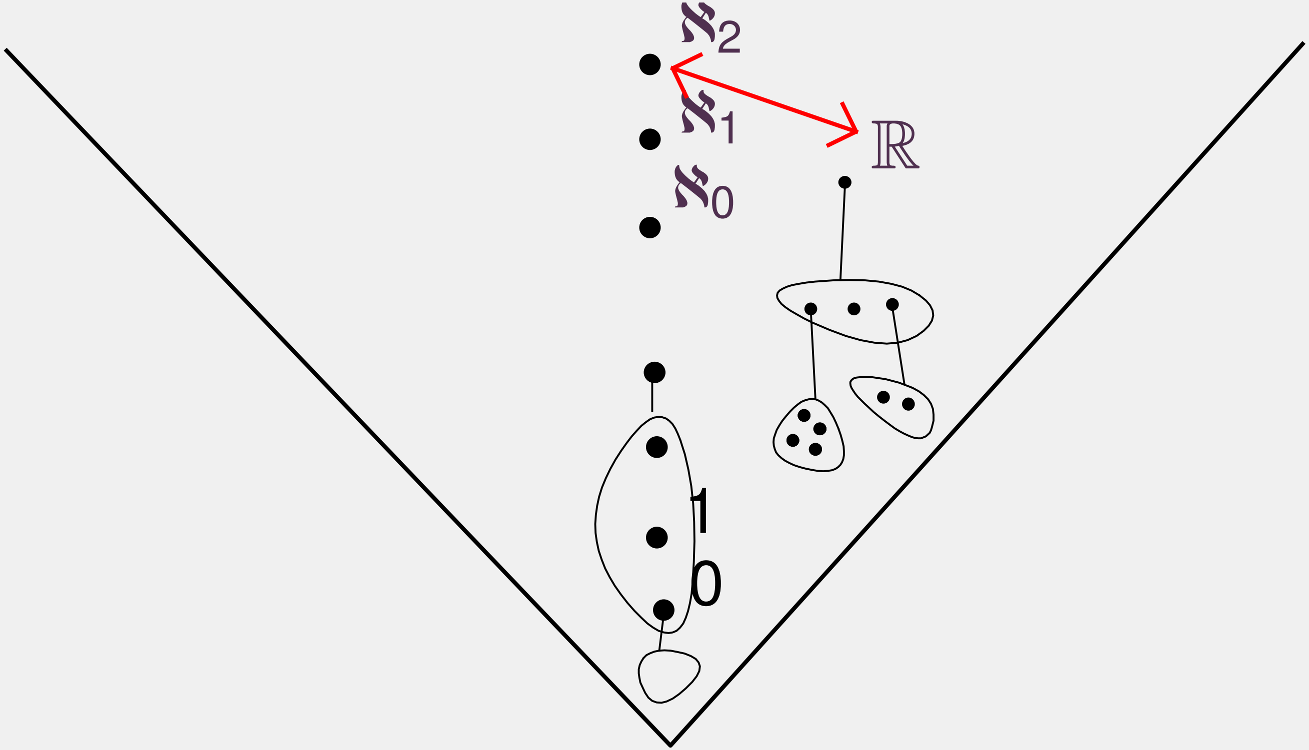
Gödel's *constructible* universe:



Cohen's *forcing* model:



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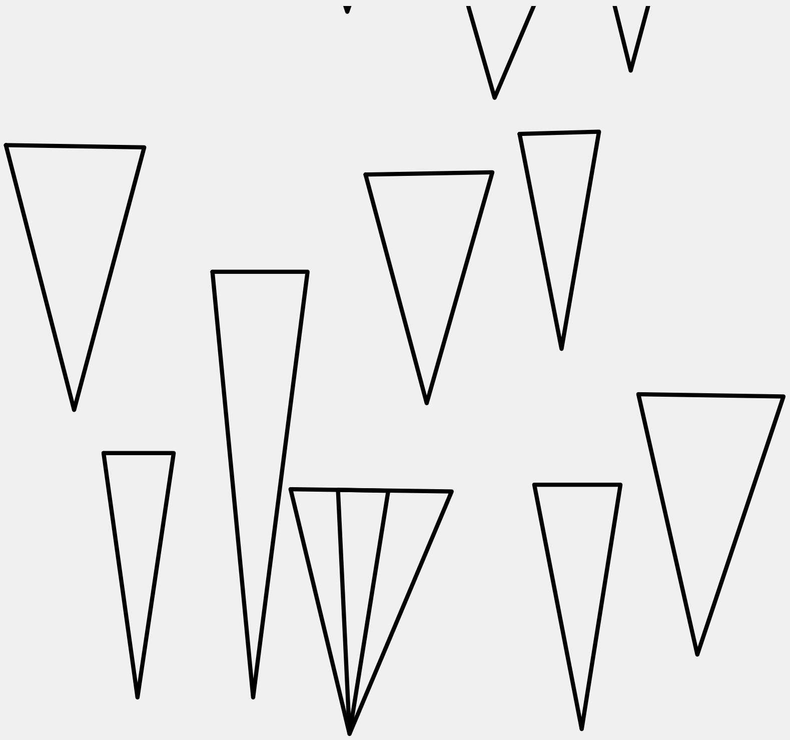
- Forcing models are “minimal” extensions of given models.
- There are many models of set theory.
- Research in axiomatic set theory can be viewed as the exploration of a “multiverse” of models of set theory.

A multiverse of set theoretic universes



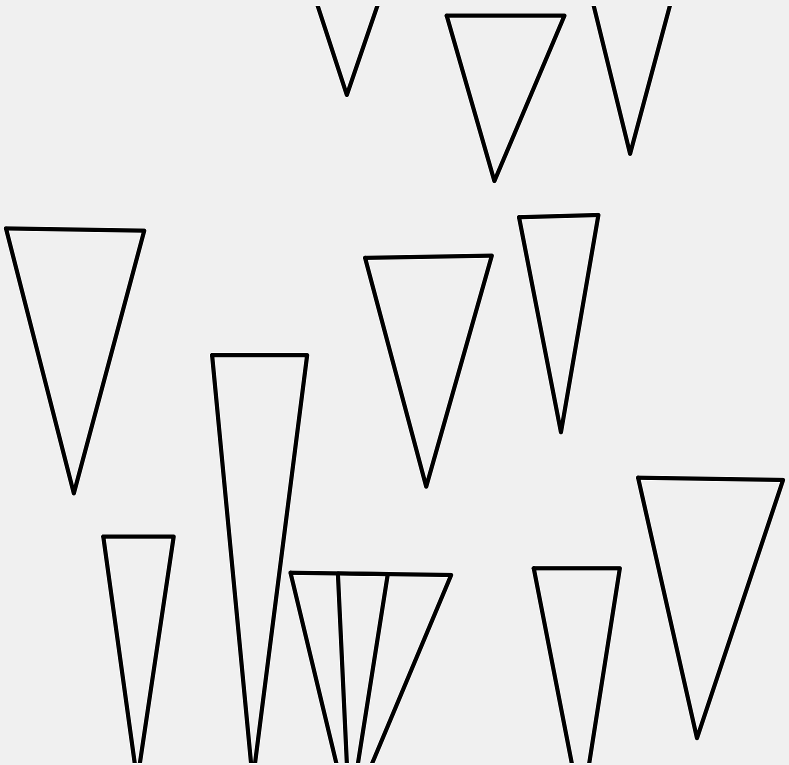
Keith Devlin

Ronald Jensen

Robert Solovay

In my Diploma / Master / PhD / Habilitation-Theses I have studied the constructible models $L^{\#}$/ L^{μ}/ K^{short}/ the core model for one strong cardinal.

The multiverse



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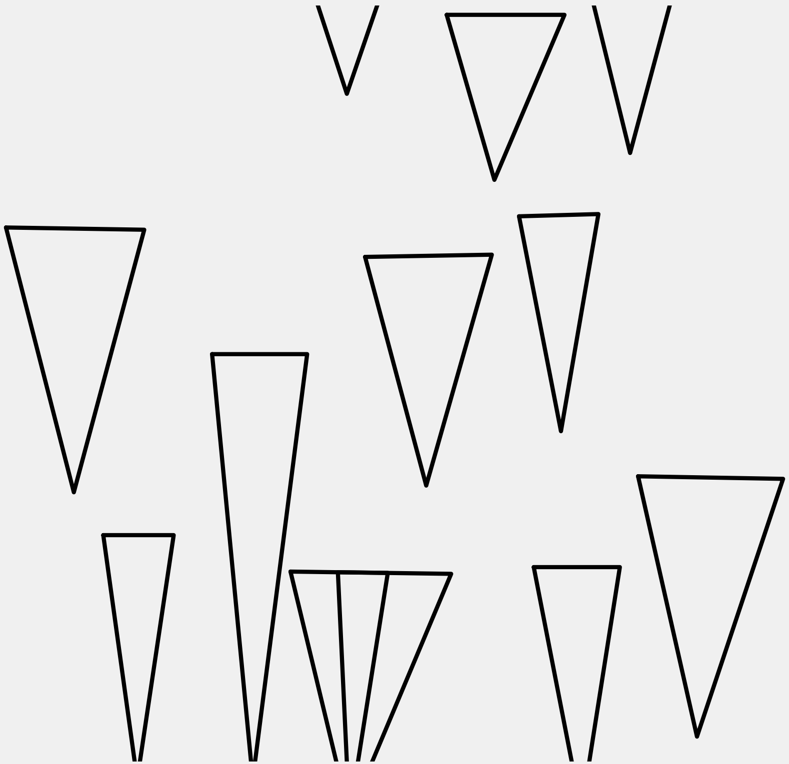

Anne Fernengel (2020)

An Easton-like Theorem for all Cardinals in ZF

Werner Müller

Can one *really* carry out mathematics completely formal?

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- Complexity
- Finite numbers can be **very large**.
- A microprocessor in a modern laptop combines > 100 000 000 basic Boolean functions.
- An adult human consists of $\sim 10^{29}$ atoms.
- A common representation of the natural number n in set theory requires $> 2^n$ symbols.

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- 4-colour theorem, Kepler conjecture, …

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- with Patrick Braselmann and Julian Schlöder: Formalization of the Gödel completeness theorem

```
theorem :: GOEDELCP:38

for Al being QC-alphabet
for X being Subset of (CQC-WFF Al)
for p being Element of CQC-WFF Al
st Al is countable & still_not-bound_in X is finite & X |= p

holds X |- p

proof end;
```

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- Can one use a more natural input language?

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- Controlled natural language (CNL) for mathematics

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- Problems with longer proofs and proof organization

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Proofs from THE BOOK versus Naproche-SAD

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consider the number $n = p_1 p_2 \cdots p_r + 1$.
This n has a prime divisor p.
But p is not one of the p_i;

otherwise

p would be a divisor of n and of the product $p_1 p_2 \cdots p_r$, and thus also of the difference $n - p_1 p_2 \cdots p_r = 1$, which is impossible.

So a finite set $\{p_1, \ldots, p_r\}$ cannot be the collection of *all* prime numbers.

Let A be a finite set of prime numbers. Take a sequence P and a natural number r such that $A = \{P_1, \ldots, P_r\}$.
Take $n = P_1 \cdots P_r + 1$.
Take a prime divisor p of n.
Let us show that p is not an element of A.
Assume the contrary. Take i such that $1 \leqslant i \leqslant r$ and $p = P_i$.
$\{1, \ldots, r\} \subseteq \mathrm{Dom}\, P$ and $\mathrm{Ran}\, P \subseteq \mathbb{N}$. P_i divides $P_1 \cdots P_r$ (by MultProd). Then p divides 1 (by DivMin). Contradiction. qed.

Hence A is not the set of prime numbers.

DEMO

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- With present technology, there seems to be a strong convergence of natural and formal approaches.
- This holds huge promisses and grave dangers.

Dank